ON THE RUIN PROBLEM OF COLLECTIVE RISK THEORY[1]

By N. U. Prabhu

University of Western Australia

0. Summary. The theory of collective risk deals with an insurance business, for which, during a time interval $(0, t)$ (1) the total claim $X(t)$ has a compound Poisson distribution, and (2) the gross risk premium received is λt. The risk reserve $Z(t) = u + \lambda t - X(t)$, with the initial value $Z(0) = u$, is a temporally homogeneous Markov process. Starting with the initial value u, let T be the first subsequent time at which the risk reserve becomes negative, i.e., the business is "ruined". The problem of ruin in collective risk theory is concerned with the distribution of the random variable T; this distribution has not so far been obtained explicitly except in a few particular cases. In this paper, the whole problem is re-examined, and explicit results are obtained in the cases of negative and positive processes. These results are then extended to the case where the total claim $X(t)$ is a general additive process.

1. Introduction. The theory of collective risk, as developed by the Swedish actuary Filip Lundberg, deals with the business of an insurance company. Following a series of papers published by him during the years 1909–1934, a considerable amount of work has been done by Cramér, Segerdahl, Täcklind, Saxén, Arfwedson and many others; a survey of the theory from the point of view of stochastic processes was given by Cramér [2], [3] and an excellent review has recently been given by Arfwedson [1]. Briefly, the mathematical model used in this theory can be described as follows.

(a) The claims occur entirely "at random", that is, during the infinitesimal interval of time $(t, t + dt)$, the probability of a claim occurring is dt and the probability of more than one claim occurring is of a smaller order than dt, these probabilities being independent of the claims which have occurred during $(0, t)$.

(b) If a claim does occur, the amount claimed is a random variable with the probability distribution $dP(x)$ $(-\infty < x < \infty)$, negative claims occurring in the case of ordinary whole-life annuities.

Under the assumptions (a) and (b), it is easily seen that the total amount $X(t)$ of all claims which occur during $(0, t)$ has the compound Poisson distribution given by

$$K(x, t) = \Pr\{X(t) \leqq x\} = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} P_n(x), \tag{1.1}$$

where $P_n(x)$ is the n-fold convolution of $P(x)$ with itself, and $P_0(x) = 0$ if

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$x < 0$ and $= 1$ if $x \geqq 0$. The expected claim during $(0, t)$ is given by $t\alpha$, where

$$(1.2) \qquad \alpha = \int_{-\infty}^{\infty} x \, dP(x);$$

$t\alpha$ is called the net risk premium.

(c) During an interval of length t, the company receives an amount λt from the totality of its policyholders; λt is called the gross risk premium. The difference, $\lambda - \alpha$, is called the "safety loading", which is in practice positive. However, we shall not assume this, but only that λ and α are of the same sign. The ratio $\rho = \lambda\alpha^{-1}(> 0)$ is called Lundberg's security factor, and is of great importance in the theory of collective risk.

The function $Z(t) = u + \lambda t - X(t)$ is called the risk reserve, with the initial value $Z(0) = u$. Clearly, $Z(t)$ is a temporally homogeneous Markov process with the transition distribution function

$$(1.3) \quad P(u; z, t) = \Pr\{Z(t) \leqq z \mid Z(0) = u\} = 1 - K(\lambda t + u - z, t).$$

Starting with the initial value u, let T be the first subsequent time at which the risk reserve becomes negative, i.e., the company is "ruined". We shall call T the "period of prosperity". The ruin problem of collective risk theory is concerned with the distribution of the random variable T. Let us denote by

$$(1.4) \qquad G(t, u) = \Pr\{T \leqq t\} \qquad (0 \leqq t < \infty)$$

the cumulative distribution function (c.d.f.) of T. The function

$$F(t, u) = 1 - G(t, u)$$

gives the probability that ruin occurs only after time t, i.e., the risk reserve $Z(t)$ remains non-negative throughout the interval $(0, t)$. By considering $F(t, u)$ over the consecutive intervals $(0, dt)$ and $(dt, dt + t)$ we obtain the relation

$$(1.5) \qquad \begin{aligned} F(t + dt, u) &= (1 - dt)F(t, u + \lambda\, dt) \\ &+ dt \int_{-\infty}^{u+\lambda dt} F(t, u + \lambda\, dt - x)\, dP(x) + o(dt) \end{aligned}$$

which, on simplification, yields the integro-differential equation

$$(1.6) \qquad \frac{\partial F}{\partial t} - \lambda \frac{\partial F}{\partial u} + F(t, u) = \int_{-\infty}^{u} F(t, u - x)\, dP(x)$$

with the initial condition $F(0, u) = 1$ for $u \geqq 0$. This result is due to Arfwedson [1]. Taking the Fourier transforms (F.T.'s) of both sides of (1.6) with respect to t, we obtain an integro-differential equation for the F.T. of $F(t, u)$, whose solution, on inversion, gives the function $F(t, u)$. However, this procedure has not led to explicit expressions for $F(t, u)$ except in some particular cases.

The expression

$$(1.7) \qquad \psi(u) = G(\infty, u) = \Pr\{T < \infty\}$$

gives the probability that a company with initial capital u will eventually be ruined. Proceeding as in (1.5) we obtain the integro-differential equation

$$-\lambda\psi'(u) + \psi(u) = 1 - P(u) + \int_{-\infty}^{u} \psi(u - x)\, dP(x), \tag{1.8}$$

which is essentially the same as the integral equation obtained by Cramér [1]. Further, let us set $F(u) = F(\infty, u) = 1 - \psi(u)$; $F(u)$ gives the probability that the ruin never occurs.

2. The probability of ruin for a negative process. Let us first consider the case of an insurance company which deals only in ordinary whole-life annuities. Here all the claims are negative, and the process is sometimes referred to as a "negative process". If we put $\bar{X}(t) = -X(t)$, and $B(x) = 1 - P(-x)$, $0 \leqq x < \infty$, then the distribution of $\bar{X}(t)$ is

$$K(x, t) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} B_n(x) \qquad (0 \leqq x < \infty), \tag{2.1}$$

its Laplace transform (L.T.) being given by

$$\int_0^{\infty} e^{-\theta x}\, d_x K(x, t) = \exp\{-t[1 - \phi(\theta)]\}, \tag{2.2}$$

where $\phi(\theta)$ is the L.T. of $dB(x)$. Here $\alpha < 0$, and, since λ and α are of the same sign, $\lambda < 0$; without loss of generality we can take $\lambda = -1$, so that the risk reserve in this case becomes $Z(t) = u - t + \bar{X}(t)$.

Now consider the queueing system M/G/1, in which (a) the inter-arrival times of the customers have the negative exponential distribution $e^{-t}\, dt (0 < t < \infty)$; (b) the queue discipline is "first come, first served"; and (c) there is only one counter, and the service time has the distribution $dB(t) (0 < t < \infty)$. It is easily seen that the total service time of customers joining this system during the time interval $(0, t)$ has the distribution (2.1), and that this total service time is steadily exhausted by the server at a unit rate except when the counter is free (see Prabhu [6]). The busy period of the server initiated by a waiting time u is thus seen to be analogous to the period of prosperity in collective risk theory, the arrival of a customer corresponding to the occurrence of a claim, and the service time of a customer corresponding to the amount claimed. Using this analogy and the results obtained by Prabhu [7] we see that the joint probability distribution of the length of the period of prosperity T and the number of claims settled during this period is given by

$$G_n(t, u) = \int_u^t e^{-\tau} u \frac{\tau^{n-1}}{n!}\, dB_n(\tau - u), \qquad n = 0, 1, 2, \cdots. \tag{2.3}$$

Hence we obtain the expression

$$G(t, u) = \int_u^t \sum_{n=0}^{\infty} e^{-\tau} \frac{u\tau^{n-1}}{n!}\, dB_n(\tau - u) \tag{2.4}$$

for the distribution of T, a result which has not been explicitly obtained before. Further, we obtain

$$\int_0^\infty e^{-\theta t}\, d_t G(t, u) = e^{-u\eta(\theta)} \tag{2.5}$$

as the L.T. of this distribution, where $\eta(\theta)$ satisfies the functional equation

$$\eta(\theta) = \theta + 1 - \phi\{\eta(\theta)\}. \tag{2.6}$$

(See Takács [8].) From (2.5) it follows that the probability of the eventual ruin of the company is given by

$$\psi(u) = G(\infty, u) = \begin{cases} 1 & \text{if } \rho \geqq 1 \\ e^{-Ru} & \text{if } \rho < 1, \end{cases} \tag{2.7}$$

a result due to Lundberg; here $\rho = |\alpha|^{-1}$ is Lundberg's security factor, and R is the largest positive root of the equation

$$R = 1 - \phi(R). \tag{2.8}$$

EXAMPLES.

(a) Let $B(x) = 0$ if $x < \mu$, and $= 1$ if $x \geqq \mu$. Then $B_n(x) = 0$ if $x < n\mu$, and $= 1$ if $x \geqq n\mu$, and (2.4) gives

$$G(t, u) = \sum_{n=0}^{\infty} \int_u^t e^{-\tau} \frac{u\tau^{n-1}}{n!}\, dB_n(\tau - u) = \sum_{n=0}^{N} e^{-(u+n\mu)} \frac{u(u + n\mu)^{n-1}}{n!} \tag{2.9}$$

where $N = [(t - u)/\mu]$ is the largest integer contained in $(t - u)/\mu$.

(b) Let $B(x) = 1 - e^{-\mu x}(0 \leqq x < \infty)$. In this case

$$dB_n(x) = e^{-\mu x}[(\mu x)^{n-1}/(n - 1)!]\mu\, dx,$$

and

$$\begin{aligned} G(t, u) &= e^{-u} + \sum_{n=1}^{\infty} \int_u^t e^{-\tau} \frac{u}{n!} \tau^{n-1} e^{-\mu(\tau-u)} \mu^n \frac{(\tau - u)^{n-1}}{(n - 1)!}\, d\tau \\ &= e^{-u} + \mu u e^{\mu u} \int_u^t e^{-(1+\mu)\tau} J'\{\mu(\tau - u)\tau\}\, d\tau, \end{aligned} \tag{2.10}$$

where

$$J(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} \tag{2.11}$$

is a Bessel function (see Arfwedson [1], part II, equation 152).

3. The case of a positive process. We next consider the case where all claims are positive (a "positive process"). Let us change slightly the notation used in Section 1, and denote the distribution of $X(t)$ by (2.1). Here $\alpha > 0$, and we may take $\lambda = 1$. The risk reserve is then given by $Z(t) = u + t - X(t)$, and the required probability is

$$F(t, u) = \Pr\{u + \tau - X(\tau) \geqq 0 (0 \leqq \tau \leqq t)\}. \tag{3.1}$$

Now consider the transition of the process $Z(\tau)$ from the initial value $Z(0) = u$ to a non-negative value $Z(t) \geqq 0$, the probability of which is

$$\Pr\{u + t - X(t) \geqq 0\} = K(t + u, t). \tag{3.2}$$

Such a transition can occur in two mutually exclusive ways: (1) negative values are not assumed throughout the interval $(0, t)$, and (2) negative values are assumed. In the latter case it is clear that a transition from a negative value to a positive value must occur at some point in $(0, t)$; let the last such transition occur at time τ, so that during the remaining interval (τ, t) only non-negative values are assumed. These considerations lead to the relation

$$K(t + u, t) = F(t, u) + \int_0^t F(t - \tau, 0)\, dK(\tau + u, \tau)$$

or

$$F(t, u) = K(t + u, t) - \int_0^t F(t - \tau, 0)\, dK(\tau + u, \tau). \tag{3.3}$$

Here we have set

$$dK(t + u, t) = [d_x K(x, t)]_{x=t+u}\,. \tag{3.4}$$

To obtain $F(t, 0)$ which appears on the right side of (3.3), let us consider the transition of the process $Z(\tau)$ from the initial value $Z(0) = 0$ to the value $Z(t) = x > 0$, such that $Z(\tau) > 0 (0 < \tau \leqq t)$. Let $F(t, 0; x)\, dt$ denote the probability of such a transition. It is obvious that for every such transition of the process $Z(\tau)$ there corresponds a transition of the process $\bar{Z}(\tau) = Z(t - \tau)$ from the initial value $\bar{Z}(0) = x$ to the value $\bar{Z}(t) = 0$ such that $\bar{Z}(\tau) > 0$ $(0 \leqq \tau < t)$. However, $\bar{Z}(\tau) = x + X(\tau) - \tau$ is the process which has already been studied in Section 2, so that

$$F(t, 0; x)\, dt = \sum_{n=0}^{\infty} e^{-t} x \frac{t^{n-1}}{n!}\, dB_n(t - x). \tag{3.5}$$

Hence we obtain

$$F(t, 0)\, dt = dt \int_0^{\infty} F(t, 0; x)\, dx = \sum_{n=0}^{\infty} e^{-t} \frac{t^{n-1}}{n!} \int_{x=0}^{t} x\, dB_n(t - x)\, dx. \tag{3.6}$$

Further, using (2.5) we obtain the L.T. of $F(t, 0)$ as

$$\int_0^{\infty} e^{-\theta t} F(t, 0)\, dt = \int_0^{\infty} e^{-x\eta(\theta)}\, dx = \frac{1}{\eta(\theta)}, \tag{3.7}$$

where $\eta(\theta)$ is given by (2.6) (cf., Arfwedson [1], part I, equation 44).

The integro-differential equation (1.6) reduces, in the case of the positive process, to

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial u} + F(t, u) = \int_0^u F(t, u - x)\, dB(x). \tag{3.8}$$

This equation is the same as the one satisfied by the c.d.f. of the waiting time in the queueing system M/G/1 (a result due to Takács [8]), and its solution (3.3) has been obtained by Prabhu [6]. In the present context, however, the solution has been obtained by straight-forward arguments. Nevertheless it is interesting to see the connection between the two situations. Following Gani and Pyke [4] it can be proved that the waiting time $W(t)$ of a customer who joins the system M/G/1 at time t is given by

$$W(t) = \sup_{0 \leqq \tau \leqq t} [X(\tau) - \tau], \tag{3.9}$$

where we have taken $W(0) = 0$. Hence

$$\begin{aligned} \Pr\{W(t) \leqq u\} &= \Pr\{\sup_{0 \leqq \tau \leqq t} [X(\tau) - \tau] \leqq u\} \\ &= \Pr\{\inf_{0 \leqq \tau \leqq t} [u + \tau - X(\tau)] \geqq 0\} = F(t, u). \end{aligned}$$

From this it follows that $F(u) = F(\infty, u)$ is the limiting waiting time distribution, which is known to exist if $\alpha < 1$; its L.T. is given by the Pollaczek-Khintchine formula

$$\int_{0-}^{\infty} e^{-\theta u}\, dF(u) = \frac{(1 - \alpha)\theta}{\theta - 1 + \phi(\theta)} \qquad (\alpha < 1, R(\theta) > 0). \tag{3.11}$$

Letting $\theta \to +\infty$ in the above formula we obtain

$$1 - \psi(0) = F(0) = 1 - \alpha; \qquad \psi(0) = \alpha, \tag{3.12}$$

while inversion of (3.11) yields the result

$$\psi(u) = (1 - \alpha) \int_{t=0}^{\infty} \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!}\, dB_n(t + u) \tag{3.13}$$

as obtained by Prabhu [6]. The explicit results (3.3), (3.6) and (3.13) have not been obtained before.

Examples.

(a) Let $B(x) = 0$ if $x < \alpha$ and $= 1$ if $x \geqq \alpha$. Then

$$F(t, 0) = \sum_{n=0}^{N_0} e^{-t} \frac{(t - n\alpha)t^{n-1}}{n!} = e^{-t} \frac{t^{N_0}}{N_0 !} + (1 - \alpha) \sum_{n=0}^{N_0 - 1} e^{-t} \frac{t^n}{n!}, \tag{3.14}$$

and

$$\begin{aligned} F(t, u) = &\sum_{n=0}^{N_2} e^{-t} \frac{t^n}{n!} - e^{-t} \sum_{N_1}^{N_2} \frac{(n\alpha - u)^n}{n!} \\ &\cdot \left[\frac{(t + u - n\alpha)^{N_2 - n}}{(N_2 - n)!} + (1 - \alpha) \sum_{\nu=0}^{N_2 - n - 1} \frac{(t + u - n\alpha)^{\nu}}{\nu!} \right] \end{aligned} \tag{3.15}$$

where $N_0 = [t/\alpha]$, $N_1 = [u/\alpha]$, $N_2 = [(t + u)/\alpha]$, $[x]$ being the largest integer contained in x (cf., Arfwedson [1], part II, equation 135). Further, if $\alpha < 1$,

$$\psi(u) = (1 - \alpha) \sum_{N_1}^{\infty} e^{-(n\alpha - u)} \frac{(n\alpha - u)^n}{n!}. \tag{3.16}$$

(b) Let $B(x) = 1 - e^{-\rho x}(0 \leqq x < \infty)$. Here

$$\begin{aligned} 1 - K(x, t) &= \sum_{n=1}^{\infty} e^{-t} \frac{t^n}{n!} [1 - B_n(x)] = \sum_{n=1}^{\infty} e^{-t} \frac{t^n}{n!} \sum_{\nu=0}^{n-1} e^{-\rho x} \frac{(\rho x)^{\nu}}{\nu!} \\ &= \sum_{\nu=0}^{\infty} e^{-\rho x} \frac{(\rho x)^{\nu}}{\nu!} \sum_{n=\nu+1}^{\infty} e^{-t} \frac{t^n}{n!} = e^{-\rho x} \int_{y=0}^{t} e^{-y} J(\rho xy)\, dy. \end{aligned} \tag{3.17}$$

We have

$$F(t, 0) = e^{-t} + \sum_{n=1}^{\infty} e^{-t} \frac{t^{n-1}}{n!} \int_{x=0}^{t} (t - x) e^{-\rho x} \rho^n \frac{x^{n-1}}{(n-1)!}\, dx,$$

whence we obtain

$$G(t, 0) = e^{-\rho t} \int_0^t e^{-x} J(\rho t x)\, dx + e^{-t} \rho \int_0^t e^{-\rho x} x J'(\rho t x)\, dx. \tag{3.18}$$

The expression for $G(t, u)$ can be simplified similarly (cf., Arfwedson [1], part II, page 85). Also, if $\rho > 1$,

$$\psi(u) = \left(1 - \frac{1}{\rho}\right) e^{-\rho u} \int_0^{\infty} e^{-(1+\rho)t} t J'(\rho t^2 + \rho u t)\, dt. \tag{3.19}$$

4. A generalized model. Recently, certain alternative specifications for the occurrence of claims have been made (see Arfwedson [1]); however, the reasoning used in deriving the various formulae of Sections 2 and 3 indicates that the total claim $X(t)$ can be a general additive process with stationary increments (of which the compound Poisson distribution is a particular case). If then we try to write down the integro-differential equation of the type (1.6) for this generalized model, it will be found necessary to characterize the process $X(t)$ over the infinitesimal interval $(0, dt)$, which may present difficulties of a serious nature. However, such difficulties can be avoided, at least for processes which are purely negative or purely positive, by using the methods of the present paper.

Consider first the case of a negative process. Its risk reserve is

$$Z(t) = u + X(t) - t,$$

where we now take $X(t)$ to be an additive process with stationary increments, with the continuous frequency function $k(x, t)(0 \leqq x < \infty, 0 \leqq t < \infty)$. It is known that the L.T. of $X(t)$ is given by

$$\int_0^{\infty} e^{-\theta x} k(x, t)\, dx = e^{-t\xi(\theta)}, \tag{4.1}$$

where $\xi(\theta)$ is a function of a specified type. The risk reserve can then be compared to the content of a dam which is fed by inputs of water forming an additive process with stationary increments, and from which there is a steady release of water at a unit rate except when the dam is empty. The period of prosperity T of the insurance company is then analogous to the "wet period" of the dam. It follows from the results of Kendall [5] that the frequency function of T is given by

$$g(t, u) = (u/t)k(t - u, t) \tag{4.2}$$

and its L.T. by

$$\int_0^\infty e^{-\theta t} g(t, u)\, dt = e^{-u\eta(\theta)}, \tag{4.3}$$

where $\eta(\theta)$ satisfies the functional equation

$$\eta(\theta) = \theta + \xi\{\eta(\theta)\} \qquad (\theta > 0). \tag{4.4}$$

Further, from (4.3) and (4.4) it follows that the probability of the eventual ruin of the company is

$$\psi(u) = \begin{cases} 1 & \text{if } \rho \geqq 1 \\ e^{-u\zeta} & \text{if } \rho < 1 \end{cases} \tag{4.5}$$

where ζ is the largest positive root of the equation $\tau = \xi(\zeta)$ and ρ is Lundberg's security factor.

Next, consider the case of the positive process, with the risk reserve $Z(t) = u + t - X(t)$ where $X(t)$ is as defined for the negative process. It is clear that the reasoning employed in obtaining $F(t, 0)$ in Section 3 is valid for a general additive process $X(t)$, so that we have

$$F(t, 0) = \int_0^t \frac{x}{t} k(t - x, t)\, dx. \tag{4.6}$$

The equation (3.3) remain valid, so that $F(t, u)$ can be completely determined in the general case.

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