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On the Sandor-Smarandache Function

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Abstract

In the current literature, a new Smarandache-type arithmetic function, involving binomial coefficients, has been proposed by Sandor. The new function, denoted by $SS(n)$, is named the Sandor-Smarandache function. It has been found that, like many Smarandache-type functions, $SS(n)$ is not multiplicative. Sandor found $SS(n)$ when n (≥ 3) is an odd integer. Since then, the determination of $SS(n)$ for even n remains a challenging problem. It has been shown that the function has a simple form even when n is even and not divisible by 3. This paper finds $SS(n)$ in some particular cases of n, and finds an upper bound of $SS(n)$ for some special forms of n. Some equations involving the Sandor-Smarandache function and pseudo-Smarandache function have been studied. A list of values of $SS(n)$ for n = 1(1)480, calculated on a computer, is appended at the end of the paper.

Keywords: Sandor-Smarandache function; Binomial coefficient; Diophantine equation; Divisor function; Smarandache function; Pseudo-Smarandache function.

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1. Introduction

Soon after the development of the Smaranadache notions by the Romanian-American mathematician, F. Smarandache, in the late nineteen seventies, quite a few arithmetic functions have been introduced in literature. Notable among them are the Smarandache function and the pseudo Smarandache function. These functions are different from the traditional arithmetic functions in number theory. Because of their special properties, they caught the attraction of different researchers. Sandor [1] introduced a new Smarandache-type function, denoted by $SS(n)$, which has been called the Sandor-Smarandache function in Majumdar [2]. The function is defined as follows:

$$SS(n) = max\left\{k : 1 \leq k \leq n-2,\ n \text{ divides } \binom{n}{k}\right\},\ n \geq 7, \tag{1.1}$$

where, for any integer n (≥ 1) fixed, the binomial coefficients $C(n, k)$ are given by

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$$C(n, k) \equiv \binom{n}{k} = \frac{n!}{k!\,(n-k)!},\ 0 \le k \le n,$$

and by convention,

$SS(1) = 1,\ SS(2) = 1,\ SS(3) = 1,\ SS(4) = 1,\ SS(6) = 1.$

It may be mentioned that, $C(n, k)$ are integers (see, for example, Theorem 73 in Hardy and Wright [3]), and may conveniently be calculated from the formula below:

$$C(n, k) = \frac{n(n-1)(n-2)\ \dots\ (n-k+1)}{k!},\ 0 \le k \le n.$$

Then, the problem is as follows: Given any integer n (≥ 7), find the minimum integer k such that $k!$ divides $(n-1)(n-2)\dots(n-k+1)$, where $1 \le k \le n-2$. With this minimum k, $SS(n) = n - k$. Note that, for some n, there might be more than one k satisfying the condition that $k!$ divides $(n-1)(n-2)\dots(n-k+1)$. As an example, let $n = 10$. Here, 3! divides 9×8 (so that 10 divides $C(10, 3)$). Again, 4! divides $9\times 8\times 7$ (so that 10 divides $C(10, 4)$ as well). By definition, $SS(10) = 7$.

When n (≥ 7) is an odd integer, $SS(n)$ has a simple form, as has been derived by Sandor [1]. The following lemma proves more.

Lemma 1.1: $SS(n) = n - 2$ if and only if n (≥ 7) is an odd integer.

Proof: Consider the expression

$$C(n, 2) = n\frac{n-1}{2}.$$

Now, since 2 divides $n - 1$ if and only if n is odd, the lemma follows.

Lemma 1.1 has the following consequence.

Corollary 1.1: $SS(n) \le n - 3$ if n (≥ 8) is an even integer.

Though the application of Sandor-Smarandache function is growing gradually, its calculation poses a difficult problem when n is an even number divisible by 3. The function remained in oblivion till Majumdar [2] initiated a rigorous study of it. Some values have been found by Majumdar [4], who confined the study to functions of the form $SS(p + 1)$, where p is a prime. The problem was later considered by Islam and Majumdar [5], who derived expressions of $SS(2mp)$, $SS(6mp)$, $SS(60mp)$ and $SS(420mp)$, where p is an odd prime and m is any (positive) integer. So, a research gap is still there to establish a concrete method of calculating $SS(n)$, and it seems that the values of $SS(n)$ are to be calculated case by case considering all possible forms of n.

This study derives $SS(n)$ for some particular cases. It has been found that, if n is an even integer not divisible by 3, then $SS(n)$ still has a simple form. In addition, the explicit expressions of $SS(6t)$, $SS(12t)$, $SS(18t)$, $SS(42t)$, $SS(30t)$, and $SS(210t)$ are derived for some particular forms of t. Moreover, upper bounds of $SS(n)$ in some particular cases, which depend on n, are found. These are done in Section 3. Some background materials are given in the next section. Some remarks are made in Section 4. The paper concludes with some concluding remarks in Section 5. Tables 1 and 2 containing the values of $SS(n)$ for $n = 1(1)480$, calculated on a computer, are appended at the end of the paper.

2. Background Material

This section gives the background material that would be needed in the following section.

Lemma 2.1: Let m and n be two relatively prime (positive) integers, that is, $(m, n) = 1$. Let the integer N (>0) be such that both m and n divides N. Then, mn divides N.

Proof: Since m divides N,

$N = ma$ for some integer $a \geq 1$.

Now, since n divides N and $(m, n) = 1$, it follows that, a must be divisible by n, that is,

$a = nb$ for some integer $b \geq 1$.

Then, $N = ma = mnb$, which shows that mn divides N.

Lemma 2.1 is the *fundamental theorem of arithmetic*; for an alternative proof, the readers are referred to Olds, Lax and Davidoff [6].

Lemma 2.2: Let M and N be two (positive) integers such that M is divisible by the integer m and N is divisible by the integer n. Then, MN is divisible by mn.

Proof: By assumption,

$M = m\alpha$ for some integer α (>0), $N = n\beta$ for some integer β (>0).

Therefore, $MN = (\alpha\beta)mn$, which proves the statement of the lemma.

Lemma 2.3: For any integer $n \geq 1$ fixed, $n(n+1)\ldots(n+r)$ is divisible by $(r+1)!$ for any integer $r \geq 1$.

Proof: See Theorem 74 in Hardy and Wright [3], which states that the product of any m consecutive (positive) integers is divisible by $m!$.

Corollary 2.1: For any integer $n \geq 1$ fixed, let $P(n, r) \equiv n(n+1)\ldots(n+r)$ for any integer $r \geq 1$. Then, $r+1$ divides $(n+1)(n+2)\ldots(n+r)$ if and only if $r+1$ does not divide n.

Proof: Note that, by Lemma 2.3, $r+1$ divides $P(n, r)$.

The next section gives the main results of the paper, where we would frequently encounter linear Diophantine equations. In this context, the following result is relevant.

Lemma 2.4: The Diophantine equation $ax + by = c$ has an (integer) solution if and only if c is divisible by $d \equiv (a, b)$. Moreover, if (x_0, y_0) is a solution, then there are infinite number of solutions, given parametrically $x = x_0 + (\frac{b}{d})t$, $y = y_0 + (\frac{-a}{d})t$ by for any integer t.

Proof: See, for example, Gioia [7].

Another result that would be needed later is given in Lemma 2.5 (see, for example, Hardy and Wright [3], for a proof).

Lemma 2.5: (Dirichlet Theorem) If a and b are two integers with $a > 0$ and $(a, b) = 1$, then there are infinitely many primes of the form $ax + b$, x (>0) being an integer.

3. Main Results

First the following results are proved.

Lemma 3.1: If n is an integer not of the form $4m$ (for some integer $m \geq 1$), then n divides

$n[\frac{(n-1)(n-2)(n-3)}{2\times3\times4}]$.

Proof: By Lemma 2.3, 3! divides $(n-1)(n-2)(n-3)$. Now by virtue of Corollary 2.1, 4 divides $(n-1)(n-2)(n-3)$ if and only if n is not of the form $4m$.

Corollary 3.1, which follows readily from Lemma 3.1 above, gives an upper bound for $SS(n)$.

Corollary 3.1: $SS(n) \le n-4$ for any even n (≥ 1) not of the form $n = 4m$.

Lemma 3.2: If n is an integer not of the form $10m$ (for some integer $m \ge 1$), then n divides

$$n\left[\frac{(n-1)(n-2)(n-3)(n-4)}{2\times 3\times 4\times 5}\right].$$

Proof: By Lemma 2.3, 4! divides $(n-1)(n-2)(n-3)(n-4)$. Now, if 5 does not divide n, then $(n-1)(n-2)(n-3)(n-4)$ is divisible by 5 as well. Thus, the result is established by virtue of Lemma 2.1.

Corollary 3.2: $SS(n) \le n-5$ for any even n (≥ 1) not of the form $n = 10m$.

Proof: The proof follows immediately from Lemma 3.1.

Lemma 3.3: If n is an integer not of the form $42m$ (for any integer $m \ge 1$), then n divides

$$n\left[\frac{(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{2\times 3\times 4\times 5\times 6\times 7}\right].$$

Proof: By virtue of Lemma 2.3, $(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$ is divisible by 6!. Now, $(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)$ is divisible by 7 as well, since n is not a multiple of 7.

Corollary 3.3, giving an upper bound for $SS(n)$, follows from Lemma 3.3. However, as would be noticed later, for many cases, this bound is rather loose.

Corollary 3.3: $SS(n) \le n-7$ for all n (≥ 1) not of the form $n = 42m$.

Lemma 3.4: Let n be an integer of the form $n = 2t$, where t (≥ 4) is an integer. Then,

$$SS(n) = n-3$$

if and only if t (and hence, n) is not a multiple of 3.

Proof: With $n = 2t$ ($t \ge 4$),

$$C(n, n-3) \equiv \frac{2t(2t-1)(2t-2)}{2\times 3} = 2t\left[\frac{(2t-1)(t-1)}{3}\right].$$

Now, by Lemma 2.3, $(2t-1)(t-1)$ is divisible by 3 if and only if t is not a multiple of 3.

Lemma 3.4 has the following consequences.

Corollary 3.4: For any even integer m (≥ 8), not divisible by 3, $SS(m^n) = m^n - 3$ for any integer $n \ge 1$.

Proof: The proof follows readily from Lemma 3.4.

Corollary 3.5: Let $p \ge 5$ be a prime. Then, $SS(p+3) = p$.

Proof: Since p +3 is an even integer not divisible by 3, the result follows from Lemma 3.4.

Corollary 3.6: $SS(n) \le n-4$ if n is an even integer divisible by 3.

Lemma 3.5: For any prime $p \ge 5$, $SS(2tp) = 2tp-3$ for any integer t not a multiple of 3.

Proof: Consider

$$C(2tp, 2tp-3) \equiv \frac{2tp(2tp-1)(2tp-2)}{2\times 3} = 2tp\left[\frac{(2tp-1)(tp-1)}{3}\right].$$

Now, noting that, 3 divides $(2tp-1)(tp-1)$ for any prime $p \geq 5$, the result follows.
It may be mentioned here that, the result of Lemma 3.5 holds true when $t=2$, $p=2$.
Corollary 3.7: $SS(10t)=10t-3$ for any integer t not a multiple of 3.
Proof: With $p=5$ in Lemma 3.5, the result follows.
From Lemma 1.1 and Lemma 3.4, it is found that $SS(n)$ has simple forms when the integer n is odd, or when n is even and not divisible by 3. In such cases,
$n-3 \leq SS(n) \leq n-2$.
In other cases, $SS(n) \leq n-4$.
Lemma 3.6 below deals with the case when n is a multiple of 6. The expression

$$C(6t,\ 6t-3) \equiv \frac{6t(6t-1)(6t-2)}{2\times 3} = 6t\Big[\frac{(6t-1)(3t-1)}{3}\Big],$$

shows that $SS(6t) \neq 6t-3$ (since $(6t-1)(3t-1)$ is not divisible by 3). Thus, $SS(6\text{t}) \leq 6t-4$ for any integer $t \geq 4$. However, the following result holds true.
Lemma 3.6: For any integer $t \geq 2$,

$$SS(6t) = \begin{cases} 6t-4, & if \quad t = 4s+3,\ s \geq 0 \\ 6t-5, & if \quad t\ (\neq 4s+3) \text{ is not divisible by } 5 \end{cases}$$

Proof: Consider the expression below.

$$C(6t,\ 6t-4) \equiv 6t\Big[\frac{(6t-1)(6t-2)(6t-3)}{2\times 3\times 4}\Big] = 6t\Big[\frac{(6t-1)(3t-1)(2t-1)}{4}\Big].$$

Now, the term inside the square bracket on the R.H.S. is an integer if and only if 4 divides $3t-1$. This leads to the Diophantine equation
$3t-1=4a$ for some integer a (≥ 1),
with the solution $t=4s+3$, $s \geq 0$. Thus, in this case, $SS(n)=n-4$.
To prove the remaining part, let $t \neq 4s+3$ for any integer $s \geq 0$, so that, by virtue of the above proof, in this case, $SS(n) \leq n-5$. So, consider the following expression, obtained after some algebraic simplification:

$$C(6t,\ 6t-5) \equiv 6t\Big[\frac{(6t-1)(3t-1)(2t-1)(3t-2)}{2\times 5}\Big].$$

Now, one of the two numbers $3t-1$ and $3t-2$ is even (depending on whether t is odd or even respectively). Also, 5 divides $(6t-1)(6t-2)(6t-3)(6t-4)$ if t is not a multiple of 5. Therefore, if t is not divisible by 5 and is not of the form $4s+3$ (so that t is either of the form $2s$, $s \geq 1$, or of the form $4s+1$, $s \geq 2$), then $SS(n)=n-5$.
Lemma 3.6 proves that, if $t=4s+3$ for any integer $s \geq 0$, then $SS(6t)=6t-4$, even if t is a multiple of 5. Thus, this result is valid when $t=15, 35, 55, \ldots$, with
$SS(90)=86$, $SS(210)=206$, $SS(330)=326$, $SS(450)=446$, $SS(570)=566$.
On the other hand, $SS(30)=23$, $SS(60)=53$, $SS(120)=113$, $SS(150)=143$.
Corollary 3.8: For any prime p (≥ 3),

$$SS(6p) = \begin{cases} 6p-4, & if \quad p = 4s+3,\ s \geq 0 \\ 6p-5, & if \quad p = 4s+1,\ s \geq 3 \end{cases}$$

Proof: The proof follows immediately from Lemma 3.6.

Since any (odd) prime p is either of the form $p = 4s + 1$ ($s \geq 1$) or $p = 4s + 3$ ($s \geq 0$), it follows that

$6p-5 \leq SS(6p) \leq 6p-4$,

where $p \neq 5$ for the L.H.S. inequality.

Corollary 3.9: For any integer $t \geq 1$,

$$SS(18t) = \begin{cases} 18t-4, & \text{if} \quad t = 4s+1,\ s \geq 0 \\ 18t-5, & \text{if} \quad t\ (\neq 4s+1) \text{ is not divisible by } 5 \end{cases}$$

Proof: Replacing t by $3t$ in Lemma 3.6, the first condition reads as $3t = 4s + 3$, whose solution is $t = 4x + 1$ for any integer $x \geq 0$.

In Corollary 3.9, if $t = 4s + 1$ for any integer $s \geq 0$, then $SS(18t) = 18t - 4$, even if t is a multiple of 5. Thus, this result is valid when $t = 5, 25, 45, \ldots$, with

$SS(90) = 86$, $SS(450) = 446$, $SS(810) = 806$, $SS(1170) = 1164$, $SS(1530) = 1526$,

while $SS(180) = 173$, $SS(270) = 263$, $SS(540) = 533$.

From the expression

$$C(12t, 12t-4) \equiv 12t\left[\frac{(12t-1)(6t-1)(4t-1)}{4}\right],$$

it is seen that, $SS(12t) \neq 12t-4$ (since the numerator of the term inside the square bracket is not divisible by 4), and hence, $SS(12t) \leq 12t - 5$ for any $t \geq 1$. The following lemma assets that, the inequality holds with equality sign if and only if t is not divisible by 5.

Lemma 3.7: $SS(12t) = 12t-5$ for any integer t not divisible by 5.

Proof: The result follows from the simplified expression below:

$$C(12t, 12t-5) \equiv 12t\left[\frac{(12t-1)(6t-1)(4t-1)(3t-1)}{5}\right],$$

since one of the four terms in the numerator inside the square bracket must be divisible by 5, if t is not a multiple of 5.

Lemma 3.8: $SS(12t) = 12t-6$ if $t = 5(6s+5)$, $s \geq 0$.

Proof: Consider the following simplified expression:

$$C(12t, 12t-6) \equiv 12t\left[\frac{(12t-1)(6t-1)(4t-1)(3t-1)(12t-5)}{5\times 6}\right].$$

Now, the term inside the square bracket is an integer if t is a multiple of 5, 3 divides $4t-1$ and 2 divides $3t-1$. Thus, the following three Diophantine equations result.

$t = 5a$, $4t = 3b+1$, $3t = 2c+1$ for some integers $a \geq 1$, $b \geq 1$ and $c \geq 1$.

The latter two equations have respective solutions

$t = 3\alpha+1$ for some integer $\alpha \geq 0$, $t = 2\beta+1$ for some integer $\beta \geq 0$.

Now, the combined Diophantine equation is $3\alpha = 2\beta$, so that $\alpha = 2x$ for some integer $x \geq 1$.

Thus, $t = 6x+1$, which is to be combined with the equation $t = 5a$, to get

$5a = 6x+1$, whose solution is $a = 6s+5$ for some integer $s \geq 0$.

Hence, finally, $t = 5(6s+5)$, as has been claimed in the lemma.

The result in Lemma 3.8 may be expressed as

$SS(60t) = 60t-6$ if $t = 6s+5$, $s \geq 0$.

This matches with the result found in Majumdar [2], who derived expressions of $SS(60t)$ for more values of t. A consequence of Lemma 3.8 is the following.

Corollary 3.10: $SS(12t) \leq 12t-7$ if t is divisible by 5 and $t \neq 5(6s+5)$ for any $s \geq 0$.

The values below illustrate Corollary 3.10:

$SS(60)=53$, $SS(120)=113$, $SS(180)=173$, $SS(240)=233$, $SS(360)=353$, $SS(420)=412$.

Lemma 3.9: For any integer $t \geq 1$,

$$SS(42t)=\begin{cases} 42t-4, & if \;\; t=4s+1,\; s\geq 0 \\ 42t-5, & if \;\; t \; (\neq 4s+1) \text{ is not divisible by 5} \end{cases}$$

Proof: Consider the expression below:

$C(42t, 42t-4)$

$$\equiv 42t\Big[\frac{(42t-1)(42t-2)(42t-3)}{2\times3\times4}\Big] = 42t\Big[\frac{(42t-1)(21t-1)(14t-1)}{4}\Big].$$

Now, the term inside the square bracket on the R.H.S. is an integer if and only if 4 divides $21t-1$, giving rise to the Diophantine equation

$21t-1=4x$ for some integer $x \geq 1$.

The solution of the above equation is $t=4s+1$, $s \geq 0$. Thus, in this case, $SS(n)=n-4$.

To prove the other part, consider the following simplified expression:

$$C(42t, 42t-5) \equiv 42t\Big[\frac{(42t-1)(21t-1)(14t-1)(21t-2)}{2\times5}\Big].$$

Now, if t is not a multiple of 5, then 5 divides $(42t-1)(21t-1)(14t-1)(21t-2)$. Also, one of the two numbers $21t-1$ and $21t-2$ is even (depending on whether t is odd or even respectively). Therefore, if t is not divisible by 5 and is not of the form $4s+1$ (so that t is either of the form $2s$ or of the form $4s+3$), then $SS(n)=n-5$.

Note that, Lemma 3.9 also follows from Lemma 3.6 by replacing t by $7t$, so that the first condition in Lemma 3.6 is to be replaced by $7t=4s+3$, whose solution is $t=4y+1$, $y \geq 0$. From Lemma 3.9, note that, if $t=4s+1$ for any integer $s \geq 0$, then $SS(42t)=42t-4$, even if t is a multiple of 5. Thus, Lemma 3.9 is valid when $t=5, 25, 45, \ldots$, with

$SS(210)=206$, $SS(1050)=1046$, $SS(1890)=1886$, $SS(2730)=2724$,

while $SS(420)=412$, $SS(630)=622$.

From Lemma 3.6 and Lemma 3.9, it appears that

$SS(n) \leq n-5$ if n is an even integer, divisible by 3 but not divisible by 5.

However, if t (in Lemma 3.6) is a multiple of 5, then the situation would be quite different. For example, though $SS(450) = 446$ (by the first part of Lemma 3.6), $SS(300) = 294$; and it can be easily checked that $SS(150) = 143$. Again, when t (in Lemma 3.6) is a multiple of 5×7, the situation is different. For example, it can be checked that $SS(420) = 412$, but $SS(840)=831$, $SS(22680)=22670$ and $SS(1680)=1669$.

Now attention is given to the function $SS(30t)$, where $t \geq 1$ is an integer, which, as we shall see, is interesting. From the expression

$$C(30t, 30t-3) \equiv 30t\Big[\frac{(30t-1)(30t-2)}{2\times3}\Big] = 30t\Big[\frac{(30t-1)(15t-1)}{3}\Big],$$

it follows that $SS(30t) \neq 30t - 3$ for any integer $t \geq 1$. Again, considering

$$C(30t,\, 30t-5) \equiv 30t\Big[\frac{(30t-1)(15t-1)(10t-1)(15t-2)}{2\times 5}\Big],$$

we see that $SS(30t) \neq 30t - 5$ for any integer $t \geq 1$. However, the following results hold.

Lemma 3.10: For any integer $t \geq 1$,

$$SS(30t) = \begin{cases} 30t-4, & \text{if} \quad t = 4s+3,\ s \geq 0 \\ 30t-6, & \text{if} \quad t = 2(6s+5),\ s \geq 0 \end{cases}$$

Proof: Consider the simplified expression

$$C(30t,\, 30t-4) \equiv 30t\Big[\frac{(30t-1)(15t-1)(10t-1)}{4}\Big].$$

Here, the term inside the square bracket on the R.H.S. is an integer if and only if 4 divides $15t - 1$. This leads to the Diophantine equation

$15t = 4a + 1$ for some integer $a \geq 1$,

whose solution is $t = 4s + 3$ for any integer $s \geq 0$. Thus, in this case, $SS(30t) = 30t - 4$.

To prove the other half, consider the expression

$$C(30t,\, 30t-6) \equiv 30t\Big[\frac{(30t-1)(15t-1)(10t-1)(15t-2)(6t-1)}{3\times 4}\Big].$$

Now, the numerator of the term inside the square bracket on the R.H.S. is divisible by 3 if and only if 3 divides $10t - 1$. Therefore, the term inside the square bracket is an integer if 3 divides $10t - 1$ and 4 divides $15t - 2$. Thus, the following two Diophantine equations result:

$10t = 3b + 1$ for some integer $b \geq 1$; $15t = 4c + 2$ for some integer $c \geq 1$,

with respective solutions

$t = 3x + 1$ for some integer $x \geq 0$; $t = 4y + 2$ for some integer $y \geq 0$.

Next, consider the combined Diophantine equation $3x = 4y + 1$, whose solution is $x = 4s + 3$, $s \geq 0$. Hence, finally

$t = 3(4s + 3) + 1$, $s \geq 0$,

which gives the desired condition after simplification. Thus, the conditions mentioned in the lemma are both necessary and sufficient.

It may be mentioned here that, the first part of Lemma 3.10 may be obtained from the first part of Lemma 3.6 by replacing t by $5t$. The new condition is $5t = 4s + 3$, whose solution is $t = 4z + 3$, $z \geq 0$. The second part of Lemma 3.10 may be expressed as

$SS(60t) = 60t - 6$, if $t = 6s + 5$, $s \geq 0$.

This result is comparable with that found in Majumdar [2]. An interesting consequence of Lemma 3.10 is the following.

Corollary 3.11: $SS(30p) \neq 30p - 6$ for any prime p.

Lemma 3.11: Let t be such that $t \neq 4\alpha + 3$ for any $\alpha \geq 0$, or $t \neq 2(6\beta + 5)$ for any $\beta \geq 0$, such that 7 does not divide t. Then,

$\mathrm{SS}(30t) = 30t - 7$.

Proof: Consider the following simplified expression

$$C(30t,\,30t-7)\equiv 30t\Big[\frac{(30t-1)(15t-1)(10t-1)(15t-2)(6t-1)(5t-1)}{2\times 7}\Big].$$

Now, one of the two numbers $15t-1$ and $15t-2$ is even (depending on whether t is odd or even respectively). Again, since t is not a multiple of 7, the term inside the square bracket on the right-hand side is divisible by 7, and hence, it is divisible by 14.

Corollary 3.12: For any odd prime p,

$$SS(30p)=\begin{cases}30p-4, & if\ p=4s+3,\ s\geq 0\\ 30p-7, & otherwise\end{cases}$$

Proof: The result follows from Lemma 3.10 and Lemma 3.11.

Now, consider the simplified expression below:

$$C(30t,30t-8)\equiv 30t\Big[\frac{(30t-1)(15t-1)(10t-1)(15t-2)(6t-1)(5t-1)(30t-7)}{2\times 7\times 8}\Big].$$

Note that, a necessary condition that the term inside the square bracket on the R.H.S. is an integer is that t must be divisible by 7. The same condition holds for $C(30t,\ 30t-9)$. Also, note that, for any integer $t\geq 1$, $SS(30t)\neq 30t-10$, since 10 does not divide any term in the numerator of the expression $C(30t,\ 30t-10)$ below:

$$30t\Big[\frac{(30t-1)(15t-1)(10t-1)(15t-2)(6t-1)(5t-1)(30t-7)(15t-4)(10t-3)}{3\times 7\times 8\times 10}\Big].$$

From Lemma 3.11, it follows that, if t is a multiple of 7 such that $t\neq 4\alpha+3$ for any $\alpha\geq 0$, or $t\neq 2(6\beta+5)$ for any $\beta\geq 0$, then $SS(30t)\leq 30t-8$. It thus seems necessary to focus attention to the function of the form $SS(210t)$, $t\geq 1$. The following lemma gives the condition on t such that $SS(210t)=210t-4$.

Lemma 3.12: $SS(210t)=210t-4$ if t is an integer of the form $t=4s+1,\ s\geq 0$.

Proof: Consider the expression below:

$$C(210t,210t-4)\equiv 210t\,\frac{(210t-1)(210t-2)(210t-3)}{2\times 3\times 4}$$

$$=210t\Big[\frac{(210t-1)(105t-1)(70t-1)}{4}\Big].$$

Now, the term inside the square bracket above is an integer if and only if 4 divides $105t-1$. The resulting Diophantine equation is

$105t=8x+1$ for some integer $x\geq 1$,

whose solution is $t=4s+1,\ s\geq 0$.

From the expression

$$C(210t,210t-5)\equiv 210t\Big[\frac{(210t-1)(210t-2)(210t-3)(210t-4)}{2\times 3\times 4\times 5}\Big],$$

It follows that $SS(210t)\neq 210t-5$, since 5 does not divide any of the four terms in the numerator inside the square bracket. And, the expression

$$210t\Big[\frac{(210t-1)(210t-2)(210t-3)(210t-4)(210t-5)(210t-6)}{2\times 3\times 4\times 5\times 6\times 7}\Big]$$

shows that $SS(210t) \neq 210t - 7$, since 7 does not divide any of the six terms inside the square bracket. However, the following result can be proved.

Lemma 3.13: $SS(210t) = 210t - 6$ if t is an integer of the form $t = 2(6s+5)$, $s \geq 0$.

Proof: Since

$$C(120t, 120t-6) \equiv 210t\Big[\frac{(210t-1)(210t-2)(210t-3)(210t-4)(210t-5)}{2\times3\times4\times5\times6}\Big]$$

$$= 210t\Big[\frac{(210t-1)(105t-1)(70t-1)(105t-2)(42t-1)}{3\times4}\Big],$$

the term inside the square bracket is an integer if and only if 3 divides $70t - 1$ and 4 divides $105t - 2$. Thus,

$70t = 3x + 1$ for some integer $x \geq 1$, $105t = 4y + 2$ for some integer $y \geq 1$,

with respective solutions

$t = 3\alpha + 1$ for any integer $\alpha \geq 0$, $t = 4\beta + 2$ for any integer $\beta \geq 0$.

Now, the combined Diophantine equation is $3\alpha = 4\beta + 1$, whose solution is $\alpha = 4s + 3$, $s \geq 0$. Therefore, finally, $t = 3(4s + 3) + 2$, which gives the desired condition after simplification.

From Lemma 3.12, Lemma 3.13 and the discussion following Lemma 3.12, it follows that, if $t \neq 4s + 1$ for some $s \geq 0$, or $t \neq 2(6s + 5)$ for some $s \geq 0$, then

$SS(210t) \leq 210t - 8$.

However, if the condition (on t) given in Lemma 3.12 is satisfied then the condition given in Lemma 3.13 cannot be satisfied, and conversely.

Lemma 3.14: For any integer $t \geq 1$,

$SS(210t) = 210t - 8$ if $t = 8\alpha + 3$, $\alpha \geq 0$, or if $t = 2(8\beta + 1)$, $\beta \geq 0$.

Proof: Consider the following expression for $C(210t, 210t-8)$:

$$210t\,\frac{(210t-1)(210t-2)(210t-3)(210t-4)(210t-5)(210t-6)(210t-7)}{8!}$$

$$= 210t\Big[\frac{(210t-1)(105t-1)(70t-1)(105t-2)(42t-1)(35t-1)(30t-1)}{2\times8}\Big].$$

Clearly, one of $105t - 1$ and $105t - 2$ is even. Now, we want to find the condition such that the term inside the square bracket is an integer. To do so, first, consider the case when 8 divides $35t - 1$. Then,

$35t = 8x + 1$ for some integer $x \geq 1$,

whose solution is $t = 8\alpha + 3$, $\alpha \geq 0$. Now, the second possibility is that 16 divides $105t - 2$, which leads to the Diophantine equation

$105t = 16y + 2$ for some integer $y \geq 1$,

whose solution is $t = 2(8\beta + 1)$, $\beta \geq 0$.

Lemma 3.15: $SS(210t) = 210t - 9$ if $t = 4(9\alpha + 1)$, $\alpha \geq 0$, or if $t = 4(9\beta + 2)$, $\beta \geq 0$.

Proof: Consider the simplified expression below for $C(210t, 210t-9)$:

$$210t\Big[\frac{(210t-1)(105t-1)(70t-1)(105t-2)(42t-1)(35t-1)(30t-1)(105t-4)}{8\times9}\Big].$$

Now, one of $105t-1$ and $105t-2$ is even. Thus, the term inside the square bracket is an integer if t is a multiple of 4 and 9 divides $70t-1$. The resulting Diophantine equations are $t=4x$ for some integer $x\geq 1$, $70t=9y+1$ for some integer $y\geq 1$.

The solution of the second equation is $t = 9a + 4$, $a \geq 0$, which, together with the first Diophantine equation above gives $4x=9a+4$, with the solution $x=9\alpha+10$. Hence, $t=4(9\alpha+10)$, $\alpha\geq 0$.

Next, considering the second possibility that 9 divides $35t-1$, the Diophantine equation below is obtained:

$35t=9z+1$ for some integer $z\geq 1$.

The solution of the above equation is $t=9b+8$, $b\geq 0$, which coupled with the equation $t=4x$, gives $4x=9b+8$ with the solution $x=9\beta+11$, which, in turn, gives $t=4(9\beta+11)$, $\beta \geq 0$. It now remains to show that $SS(840) = 831$, $SS(1680) = 1671$, which can easily be verified.

4. Some Remarks

Based on the values of $SS(n)$, derived in this paper, we can prove the following results, involving different Diophantine equations satisfied by $SS(n)$.

Lemma 4.1: $n=1$ is the only solution of the equation $SS(n)=n$.

Proof: The proof follows from the definition.

Lemma 4.1: The equation $SS(n+1)=SS(n)$ has an infinite number of solutions.

Proof: Let n be a prime of the form $n = 3m + 1$ for some integer $m\geq 1$. Then, by Lemma 1.1, $SS(n)=n-2$. Now, $n+1=3m+2$ is even, and is not divisible by 3. Thus, by Lemma 3.4, $SS(n+1)=n-2$. Thus,

$SS(n+1)=n-2=SS(n)$.

But, by Lemma 2.5, there is an infinite number of primes of the form $3m+1$.

Lemma 4.2: The equation $SS(n+1)=SS(n)+1$ has no solution.

Proof: First, let n be odd, so that by Lemma 1.1, $SS(n)=n-2$. Now, since $n+1$ is even, by Corollary 1.1,

$SS(n+1)\leq n-2<n-1=SS(n)+1$.

Next, let n be even. Then, $SS(n+1)=n-1>n-2\geq SS(n)+1$.

However, the following result can be proved.

Lemma 4.3: The equation $SS(n+1)=SS(n)-1$ has an infinite number of solutions.

Proof: Let the integer n be such that

$n+1=6t$, $t=4s+3$, $s\geq 0$.

By Lemma 3.6, $SS(n + 1) = 6t - 4$, and by Lemma 1.1, $SS(n) = 6t - 3$. Since there is an infinite number of integers n of the above form, the lemma is established.

Lemma 4.4: The necessary and sufficient condition that $SS(n+1)=SS(n)-1$ is that n is of the form $n=6(4s+3)-1$, $s\geq 0$.

Proof: First note that**,** in order that $SS(n+1)=SS(n)-1$, n must be odd, for otherwise,

$SS(n+1)=n-1$, $SS(n)\leq n-3 \Rightarrow SS(n+1)>n-4\geq SS(n)-1$.

Therefore, n must be odd (with $SS(n) = n - 2$) so that $n + 1$ is even with $SS(n + 1) = n - 3$. But, $SS(n+1) = n - 3$ if and only if $n + 1$ is divisible by 3 and is of the form $n + 1 = 8m + 2$ for some integer m (≥ 1). This leads to the Diophantine equation

$8m + 2 = 3a$ for some integer $a \geq 1$,

whose solution is $m = 3s + 2$, $s \geq 0$, and consequently, $n + 1 = 8(3s + 2) + 2 = 6(4s + 3)$.

The following two lemmas involves $SS(n)$ and the divisor function of n.

Lemma 4.5: The equation $SS(n) + d(n) = n$ has an infinite number of solutions, where $d(n)$ is the divisor function of n.

Proof: Note that, for any prime $p \geq 5$, $SS(p) = p - 2$, $d(p) = 2$, so that $n = p$ satisfies the given equation.

Lemma 4.6: The inequalities $SS(n) + d(n) > n$ and $SS(n) > d(n)$ each has an infinite number of solutions.

Proof: Let $n = 2p$, where $p \geq 11$ is a prime. Then, $SS(n) = n - 3$, $d(n) = 4$, so that each of the two inequalities is satisfied with this n.

The following results involve both $SS(n)$ and the pseudo-Smarandache function $Z(n)$. Recall that, $Z(n)$, introduced by Kashihara [8], is defined as follows:

$$Z(n) = \min \left\{ m : n \text{ divides } \frac{m(m+1)}{2} \right\}.$$

The function has been treated by several researchers, including Majumdar [2, 9], which also provide summaries of other researchers. A more recent survey on $Z(n)$ may be found in Huaning [10].

Lemma 4.7: The equation $SS(n) = 2Z(n) - 1$ has an infinite number of solutions.

Proof: For any prime $p \geq 5$, $SS(2p^2) = 2p^2 - 3$ and $Z(2p^2) = p^2 - 1$ (by Corollary 4.2.2 in Majumdar [9]). Thus, the given equation is satisfied when $n = 2p^2$.

Lemma 4.8: The equation $SS(n) = 3Z(n) - 1$ possesses an infinite number of solutions.

Proof: For any prime $p \geq 5$, $SS(3p^2) = 3p^2 - 2$ and $Z(3p^2) = p^2 - 1$ (by Corollary 4.2.2 in Majumdar [9]). Thus, $n = 2p^2$ satisfies the given equation.

Lemma 4.9: The equation $SS(n) = 4Z(n) + 1$ admits an infinite number of solutions.

Proof: Let p be a prime of the form $p = 8s + 1$, $s \geq 2$. Then, $SS(4p) = 4p - 3$ and $Z(4p) = p - 1$ (by Lemma 4.2.16 in Majumdar [9]). Thus, the given equation is satisfied when $n = 4p$.

5. Conclusion

The newly introduced Sandor-Smarandache function, $SS(n)$, is a Smarandache-type arithmetic function, defined through the equation (1.1). In an earlier study by Islam and Majumdar [5], it has been shown that, like many Smarandache-type functions, $SS(n)$ is not multiplicative. The function is neither increasing nor decreasing. For example, $SS(11) = 9 > 7 = SS(10)$, but $SS(12) = 7 < SS(11)$. From definition, $SS(n)$ depends directly on the divisors of the preceding k integers. By the unique factorization theorem, any even integer n has the following (unique) form:

$$n = 2^{\alpha} p_1^{\beta_1} p_2^{\beta_2} \ldots p_m^{\beta_m},$$

where $p_1, p_2, \ldots, p_m$ are m distinct odd primes. Thus, for any integer n, $(n, n-1) = 1$, $(n, n-2) = 2$, and $(n, n-3) = 3$ if and only if 3 is a prime factor of n. Thus, $SS(n)$ depends indirectly on the prime factors of n.

As has been pointed out by Sandor [1], if $(n, k) = 1$, then n divides $\binom{n}{k}$. For example, when n is an odd integer, the corresponding minimum k is $k = 2$, so that $(n, k) = 1$. Again, when n is an even integer, not divisible by 3, the corresponding minimum k is $k = 3$, so that $(n, k) = 1$. However, note that the condition is necessary but not sufficient. For example, $(10, 3) = 1$, and hence, it follows that 10 must divide $\binom{10}{3}$. However, 10 divides $\binom{10}{4}$ as well, though $(10, 4) \neq 1$.

When n is an odd integer, $SS(n)$ has a simple form. It has been proved that, when n is even, not divisible by 3, $SS(n)$ has still a simple form. This paper derives the expressions of $SS(n)$ for some particular values of n.

By virtue of Lemma 1.1 and Lemma 3.4, it is sufficient to calculate the values of $SS(n)$ when n is an even integer divisible by 3. A computer program was devised to calculate the values of $SS(n)$ for such values of n. The values of $SS(n)$ for n from 1 through 480 are given in Table 1 and 2. To find $SS(n)$ for any even n (not divisible by 3) fixed, the values of $C(n, k)$ are to be calculated for $k = 4, 5, \ldots$. To find $SS(n)$ for large values of n, the problem faced is that $C(n, k)$ grows very fast. To circumvent this difficulty, it might be advisable to use the following recurrence relation of $C(n, k)$, commonly known as the Pascal's identity, a proof of which may be found in Balakrishnan [11] and Islam [12].

Lemma 5.1: $C(n, k) = C(n-1, k) + C(n-1, k-1),\ 1 \leq k \leq n$.

A glance at Table 1 reveals the following facts:

1. there is no integer n such that $SS(n) = 2$,
2. there is no integer n such that $SS(n) = 4$,
3. there is no integer n such that $SS(n) = 6$.

In this context, note that,

$SS(n) = n - 2 > 6$, if n is odd satisfying the condition $n \geq 9$,

$SS(n) = n - 3 > 6$, if $n \geq 10$ is even, not divisible by 3.

We conclude the paper with the following conjecture.

Conjecture 5.1: There is an infinite number of primes p such that $SS(p+2) = p$.

It is conjectured that, there is an infinite number of twin primes (see, for example, Tenenbaum and France [13]). If this conjecture is true, then Conjecture 5.1 is proved by taking p and $p+2$ as the twin primes.

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Table 1. Values of $SS(n)$ for $n = 1(1)240$.

n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)
1	1	41	39	81	79	121	119	161	159	201	199
2	1	42	38	82	79	122	119	162	158	202	199
3	1	43	41	83	81	123	121	163	161	203	201
4	1	44	41	84	79	124	121	164	161	204	199
5	3	45	43	85	83	125	123	165	163	205	203
6	1	46	43	86	83	126	121	166	163	206	201
7	5	47	45	87	85	127	125	167	165	207	205
8	5	48	43	88	85	128	125	168	163	208	205
9	7	49	47	89	87	129	127	169	167	209	207
10	7	50	47	90	86	130	127	170	167	210	206
11	9	51	49	91	89	131	129	171	169	211	209
12	7	52	49	92	89	132	127	172	169	212	207
13	11	53	51	93	91	133	131	173	171	213	211
14	11	54	49	94	91	134	131	174	169	214	211
15	13	55	53	95	93	135	133	175	173	215	213
16	13	56	53	96	91	136	133	176	173	216	211
17	15	57	55	97	95	137	135	177	175	217	215
18	14	58	55	98	95	138	134	178	175	218	214
19	17	59	57	99	97	139	137	179	177	219	217
20	17	60	53	100	97	140	137	180	173	220	217
21	19	61	59	101	99	141	139	181	179	221	219
22	19	62	59	102	97	142	139	182	179	222	217
23	21	63	61	103	101	143	141	183	181	223	221
24	19	64	61	104	101	144	139	184	181	224	221
25	23	65	63	105	103	145	143	185	183	225	223
26	23	66	62	106	103	146	143	186	182	226	223
27	25	67	65	107	105	147	145	187	185	227	225
28	25	68	65	108	103	148	145	188	185	228	223
29	27	69	67	109	107	149	147	189	187	229	227
30	23	70	67	110	107	150	143	190	187	230	227
31	29	71	69	111	109	151	149	191	189	231	229
32	29	72	67	112	109	152	149	192	187	232	229
33	31	73	71	113	111	153	151	193	191	233	231
34	31	74	71	114	110	154	151	194	191	234	230
35	33	75	73	115	113	155	153	195	193	235	233
36	31	76	73	116	113	156	151	196	193	236	233
37	35	77	75	117	115	157	155	197	195	237	235
38	35	78	73	118	115	158	155	198	193	238	235
39	37	79	77	119	117	159	157	199	197	239	237
40	37	80	77	120	113	160	157	200	193	240	233

Table 2. Values of $SS(n)$ for $n = 241(1)480$.

n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)	n	SS(n)
241	239	281	279	321	319	361	359	401	399	441	439
242	239	282	278	322	319	362	359	402	398	442	439
243	241	283	281	323	321	363	361	403	401	443	441
244	241	284	281	324	319	364	361	404	401	444	439
245	243	285	283	325	323	365	363	405	403	445	443
246	241	286	283	326	323	366	361	406	403	446	443
247	245	287	285	327	325	367	365	407	405	447	445
248	245	288	283	328	325	368	365	408	403	448	445
249	247	289	287	329	327	369	367	409	407	449	447
250	247	290	287	330	326	370	367	410	407	450	446
251	249	291	289	331	329	371	369	411	409	451	449
252	247	292	289	332	329	372	367	412	409	452	449
253	251	293	291	333	331	373	371	413	411	453	451
254	251	294	289	334	331	374	371	414	409	454	451
255	253	295	293	335	333	375	373	415	413	455	453
256	253	296	293	336	331	376	373	416	413	456	451
257	255	297	295	337	335	377	375	417	415	457	455
258	254	298	295	338	335	378	374	418	415	458	455
259	257	299	297	339	337	379	377	419	417	459	457
260	257	300	294	340	337	380	377	420	412	460	457
261	259	301	299	341	339	381	379	421	419	461	459
262	259	302	299	342	337	382	379	422	419	462	439
263	261	303	301	343	341	383	381	423	421	463	461
264	259	304	301	344	341	384	379	424	421	464	461
265	263	305	303	345	343	385	383	425	423	465	463
266	263	306	302	346	343	386	383	426	422	466	463
267	265	307	305	347	345	387	385	427	425	467	465
268	265	308	305	348	343	388	385	428	425	468	463
269	267	309	307	349	347	389	387	429	427	469	467
270	263	310	307	350	347	390	383	430	427	470	467
271	269	311	309	351	349	391	389	431	429	471	469
272	269	312	307	352	349	392	389	432	427	472	469
273	271	313	311	353	351	393	391	433	431	473	471
274	271	314	311	354	350	394	391	434	431	474	470
275	273	315	313	355	353	395	393	435	433	475	473
276	271	316	313	356	353	396	391	436	433	476	473
277	275	317	315	357	355	397	395	437	435	477	475
278	275	318	313	358	355	398	395	438	433	478	475
279	277	319	317	359	357	399	397	439	437	479	477
280	277	320	317	360	353	400	397	440	437	480	473