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ON THE SASAKI-HSU CONTACT STRUCTURE OF THE BRIESKORN MANIFOLDS*

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Abstract. An odd dimensional real submanifold of a complex analytic manifold which is an embedded submanifold locally defined by some fixed number of analytic equations and one differentiable equation is called a quasi-analytic submanifold. We show that, whenever the ambient manifold is Kähler and with exact fundamental form, the quasi-analytic submanifolds have a contact structure which, under some supplementary conditions is Sasakian. The application of this result to the Brieskorn manifolds gives the contact structure of Sasaki and Hsu [5].

In [5], S. Sasaki and C. J. Hsu constructed a differential 1-form on the so-called Brieskorn manifolds and proved, by a laborious computation, that this form defines a contact structure. In the present note, we shall give a simpler proof of this fact. Moreover, our proof holds for a larger class of manifolds. Also, we shall see that, for some of the Brieskorn manifolds, the respective structure is metric and normal [1]. This is in accordance with the last remark of [5].

1. Contact structure on some quasi-analytic submanifolds. Let X be a complex analytic manifold and V an embedded real submanifold. Suppose that the real dimensions of X and V are respectively $2n$ and $2h+1$ ($h \leqq n-1$). We shall say that V is a *quasi-analytic submanifold* if every point $x \in V$ has an open neighborhood U in X, endowed with local complex cordinates z^i ($i = 1, \cdots, n$), such that $U \cap V$ is the subset of U, characterized by a system of independent equations of the form

$$F_1(z^i) = 0, \cdots, F_{n-h-1}(z^i) = 0, \qquad F(z^i, \bar{z}^i) = 0, \tag{1.1}$$

where $F_1, \cdots, F_{n-h-1}$ are complex analytic functions and F is a real valued differentiable function on U.

It is easy to see that such a submanifold is a *Cauchy-Rieman* (*C-R*)-*manifold* (see, for instance, [1]), whence, by a theorem of S. Ianus ([4] or [1, p. 63]) it carries an induced f-structure ($f^3 + f = 0$), where $\operatorname{rank} f = 2h$. Moreover, if V is orientable, it is simple to construct almost contact metric structures on it, related to f.

Next, suppose that X is a (necessarily non-compact) Kähler manifold,

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with metric g, complex structure J ($J^2 = -\mathrm{Id.} = -I$) and with the exact fundamental form

$$\Omega = d\omega \ , \tag{1.2}$$

where ω is some 1-form on X.

It is known that $\operatorname{rank} \Omega = 2n$ at every point of X, and using this fact, we are able to derive

THEOREM 1.1. *Let V be a quasi-analytic submanifold of a Kähler manifold X which satisfies* (1.2), *and denote by $\iota\colon V \subseteqq X$ the canonical embedding. Then, if either* $\dim V = 1$ *or* $\dim V \geqq 5$, *and* $\zeta = \iota^*\omega \neq 0$ *at every point of V, ζ defines a contact structure on V.*

PROOF. Consider an arbitrary point $x \in V$ and its open neighborhood U such that $U \cap V$ be defined by (1.1). Next, consider on U the 2-form

$$\Lambda = \Omega|_U \quad (\mathrm{mod.}\ dF_1 = \cdots = dF_{n-h-1} = dF = 0) \ . \tag{1.3}$$

Let us remark that $dF_1, \cdots, dF_{n-h-1}, dF$ are complex linearly independent. In fact, if we had a relation

$$\mu_1 dF_1 + \cdots + \mu_{n-h-1} dF_{n-h-1} + \mu dF = 0 \ ,$$

where μ_i and μ are not all zero complex numbers, then we must have $\mu \neq 0$ and some $\mu_i \neq 0$ since the contrary assumptions lead to a contradiction. But then, the real form dF is a linear combination of forms of the type (1, 0), which is impossible.

In this case, we know, by a classical theorem of exterior algebra [3, I §IV. 17], that

$$\Omega|_U = \Lambda + dF_1 \wedge \theta_1 + \cdots + dF_{n-h-1} \wedge \theta_{n-h-1} + dF \wedge \theta \ , \tag{1.4}$$

valid at every point of U, and where $\theta_1, \cdots, \theta_{n-h-1}, \theta$ are some complex valued 1-forms on U.

It is now obvious that $\operatorname{rank} \Omega = 2n$ implies $\operatorname{rank} \Lambda \geqq 2h$ at every point of U and, particularly,

$$\operatorname{rank}(\iota^*\Omega) = \operatorname{rank}(\iota^* d\omega) = \operatorname{rank}(d\zeta) \geqq 2h$$

at every point of $V \cap U$. On the other hand, because $\dim V = 2h + 1$, we must have $\operatorname{rank}(d\zeta) \leqq 2h$.

Hence, $\operatorname{rank}(d\zeta) = 2h$ at the arbitrary point x of V.

Next, since $\zeta \neq 0$ on V the theorem is proved for $\dim V = 1$.

Generally, $\zeta \wedge d\zeta^h = 0$ would imply $d\zeta = 0(\mathrm{mod.}\ \zeta = 0)$, i.e., $d\zeta = \zeta \wedge \tau$, whence $\operatorname{rank}(d\zeta) \leqq 2$, which is impossible if $\dim V \geqq 5$, i.e., $h \geqq 2$.

This ends the proof of the theorem.

Also we can prove

THEOREM 1.2. *Let V be a quasi-analytic submanifold of a Kähler manifold X which satisfies* (1.2). *Suppose that at every point of V the following conditions hold*:

(a) $||\omega|| = 1$ *with respect to the metric g*; (b) *the contravariant vector field v associated to ω by g is analytic and tangent to the submanifold V*; (c) *the vector field Jv is normal to V. Then ζ of Theorem* 1.1 *is the contact form of a well defined Sasakian structure on V.*

PROOF. We recall that a Sasakian structure on V [1] is a system $(\phi, \xi, \zeta, \gamma)$, where ζ defines a contact structure on V, γ is a Riemann metric, ξ is a vector field, ϕ is a field of endomorphisms of the tangent spaces, and the following relations hold

$$\begin{aligned} &\zeta(\xi) = 1 , \qquad \phi^2 = -I + \zeta \otimes \xi , \\ &\gamma(\phi a, \phi b) = \gamma(a, b) - \zeta(a)\zeta(b) , \\ &d\zeta(a, b) = \gamma(a, \phi b) , \\ &N_\phi + d\zeta \otimes \xi = 0 , \end{aligned} \tag{1.5}$$

where a, b are arbitrary vector fields on V and

$$N_\phi(a, b) = \phi^2[a, b] + [\phi a, \phi b] - \phi[\phi a, b] - \phi[a, \phi b] \tag{1.6}$$

is the Nijenhuis tensor of ϕ.

In our case, we shall define $\zeta = \iota^*\omega$, $\gamma = \iota^*g$, and $\xi = v|_V$. The last definition is possible by hypothesis (b), and we also recall that v is defined by

$$g(v, u) = \omega(u) \tag{1.7}$$

for every vector field u on X.

As for ϕ, we shall take the tensor of the previously mentioned f-structure of S. Ianus, which may be obtained as follows (in the sequel, we shall generally identify the tangent vectors to V with their images by ι_*):

Let b be a tangent vector field on V and suppose that Jb is also tangent to V. Then we shall call b a *distinguished field*. In view of (1.1), b is then characterized by the relations

$$bF_1 = \cdots = bF_{n-h-1} = 0 , \qquad bF = 0 , \qquad (Jb)F = 0 , \tag{1.8}$$

which can be easily seen to define a $2h$-dimensional distribution D on V, called the *distinguished distribution*.

By hypothesis (c), ξ is normal to D with respect to γ, because, for every $b \in D$, we have $\gamma(\xi, b) = g(v, b) = g(Jv, Jb) = 0$.

Hence, every vector field c on V may be uniquely decomposed as

$$c = b + \lambda\xi , \qquad b \in D , \tag{1.9}$$

and we define

$$\phi c = Jb . \tag{1.10}$$

Now, we must verify that the relations (1.5) hold on V.

First, $\zeta(\xi) = \omega(v) = 1$ because of the hypothesis a). This also implies $\zeta \neq 0$, and by Theorem 1.1, ζ defines a contact structure on V with the eventual exception of the case dim $V = 3$.

Next, for an arbitrary vector field c on V, we have

$$\zeta(c) = \omega(\iota^* c) = g(\iota^*\xi, \iota^* c) = \gamma(\xi, c) ,$$

whence we see that the distinguished vector fields b are characterized by $\zeta(b) = 0$ and it follows that λ of (1.9) is given by $\lambda = \zeta(c)$. By this remark, and using (1.9) and (1.10) we get

$$\phi^2 c = -c + \zeta(c)\xi ,$$

which is just the second relation (1.5).

The third and fourth of the relations (1.5) follow by a straightforward computation, based on the previous formulas and on the known relations

$$g(Ja, Jb) = g(a, b) , \qquad \Omega(a, b) = g(a, Jb) .$$

Here we can also note that ζ defines a contact structure in the case dim $V = 3$, too. In fact, since ζ vanishes on the distinguished vector fields only, we have $(\zeta \wedge d\zeta)(\xi, b, Jb) \neq 0$ for every unit distinguished vector b, i.e., $\zeta \wedge d\zeta \neq 0$, which is just the contact condition if dim $V = 3$.

The verification of the last relation (1.5) is a bit more complicated. Namely, let us consider two vector fields a, b on V and let

$$a = a' + \zeta(a)\xi , \qquad b = b' + \zeta(b)\xi \tag{1.11}$$

be the corresponding decomposition (1.9). Then, we must evaluate the terms of the right-hand side of (1.6) by using (1.11).

The only complication comes from $[\phi a, b]$, whose decomposition (1.9) has to be calculated. (For $[a, \phi b]$ it will be similar.) We have

$$[\phi a, b] = [Ja', b] = [Ja', b'] + \zeta(b)[Ja', \xi] + (Ja')(\zeta(b))\xi .$$

Then, $d\zeta(Ja', \xi) = g(Ja', J\xi) = 0$ by hypothesis (c) and, by evaluating the exterior differential, we get $\zeta([Ja', \xi]) = 0$.

Hence, $\zeta([\phi a, b]) = \zeta([Ja', b']) + (Ja')\zeta(b)$ and the necessary decomposition is given by

$$[\phi a, b] = ([\phi a, b] - \zeta([\phi a, b])\xi) + \zeta([\phi a, b])\xi .$$

Now, a technical calculation, which uses $N_J = 0$, gives

$$(1.12)\qquad (N_\phi + d\zeta \otimes \xi)(a, b) = -\zeta(a)J(L_vJ)(b') + \zeta(b)J(L_vJ)(a') + \zeta([Ja', b'] + [a', Jb'])(J\xi) .$$

Here, L_v denotes the Lie derivative, and the first two terms of the right-hand side vanish because v is analytic (i.e., $L_vJ = 0$). To show that the last term vanishes too, we have to show that $c = [Ja', b'] + [a', Jb']$ is distinguished whenever a', b' are such. But, $a', b' \in D$ clearly imply that c is tangent to V. Next, from $N_J(a', b') = 0$ we get

$$Jc = [a', b'] - [Ja', Jb'] ,$$

which implies that Jc is tangent to V. Hence $c \in D$.

The proof of Theorem 1.2 is thus finished.

2. Application to the Brieskorn manifolds. Let us begin with the remark that Theorems 1.1 and 1.2 can be applied for $X = C^{n+1}$ (we take $n + 1$ instead of n for later convenience) with the natural Kähler metric g defined by

$$(2.1)\qquad ds^2 = \sum_{i=0}^{n} dz^i \otimes d\bar{z}^i .$$

Actually, in this case we have

$$(2.2)\qquad \Omega = -\sqrt{-1} \sum_{i=0}^{n} dz^i \wedge d\bar{z}^i ,$$

which is an exact form because $\Omega = d\omega$ with

$$(2.3)\qquad \omega = -(\sqrt{-1}/2) \sum_{i=0}^{n} (z^i d\bar{z}^i - \bar{z}^i dz^i) .$$

It is simple to see that we have

$$(2.4)\qquad ||\omega||^2 = \sum_{i=0}^{n} z^i \bar{z}^i ,$$

and that the contravariant vector field associated to ω is

$$(2.5)\qquad v = -(\sqrt{-1}/2) \sum_{i=0}^{n} [z^i(\partial/\partial z^i) - \bar{z}^i(\partial/\partial \bar{z}^i)] ,$$

which is obviously the real expression of a complex analytic vector field on C^{n+1}.

Clearly, Theorems 1.1 and 1.2 now give

COROLLARY 2.1. *Let ι: $V \to C^{n+1}$ be the natural embedding of a quasi-analytic submanifold V in C^{n+1}. Then, if V does not contain the origin and* $\dim V = 1$ *or* $\dim V \geqq 5$, *the form $\zeta = \iota^*\omega$, with ω of* (2.3), *defines* *a contact structure on V.*

COROLLARY 2.2. *Let V be a quasi-analytic submanifold of C^{n+1}, of an arbitrary odd dimension. If V is contained in the unit sphere $S^{2n+1} \subset C^{n+1}$ and is tangent to the vector field v of* (2.5), *then ζ above is the contact structure of a well defined Sasakian structure on V.*

In fact, we have $||\omega|| = 1$ on V, the hypotheses of the corollary are meaningful because, as it follows easily, v is tangent to S^{2n+1}, and Jv is normal to S^{2n+1}, hence to V as well. The conclusion follows then by Theorem 2.

Note that in the particular case $V = S^{2n+1}$ we get just the standard Sasakian structure on this sphere.

Now [2, 5], the Brieskorn manifold $\sum^{2n-1}(a)$, where $a = (a_0, \cdots, a_n)$ is a sequence of positive integers, is the real $(2n-1)$-dimensional submanifold of C^{n+1} defined by

$$(2.6)\qquad \begin{aligned} &(z^0)^{a_0} + \cdots + (z^n)^{a_n} = 0 , \\ &\sum_{i=0}^{n} z^i \bar{z}^i = 1 . \end{aligned}$$

The importance of this class of manifolds comes from the fact that it contains all the $(2n-1)$-dimensional $(n \geqq 2)$ homotopy spheres which are boundaries of compact orientable parallelizable manifolds and which, generally, are exotic spheres.

We see from (2.6) that every $\sum^{2n-1}(a)$ is a quasi-analytic submanifold of C^{n+1}, which is contained in the unit sphere S^{2n+1}, whence it does not contain the origin. Moreover, if $a_0 = \cdots = a_n$, $\sum^{2n-1}(a)$ is tangent to the vector field v of (2.5).

Hence, from corollaries 2.1 and 2.2, we get

PROPOSITION 2.3. *The form ζ defines a contact structure on every Brieskorn manifold $\sum^{2n-1}(a)$. If, moreover, $a_0 = \cdots = a_n$, this is the contact structure of a well defined Sasakian structure on $\sum^{2n-1}(a)$.*

PROOF. For $n = 1$ or $n \geqq 3$, the first assertion follows from Corollary 2.1 above. For $n = 2$, we proceed as follows:

Suppose that (with the notation already used) $\zeta \wedge d\zeta = 0$ on $\sum^3(a)$. Then $d\zeta = \zeta \wedge \tau$ and, if we extend τ to a form τ' on S^5 by asking τ' to be 0 on vectors orthogonal to $\sum^3(a)$, this means that, at every point of $\sum^3(a)$, $\Omega = \omega \wedge \tau'$ for arguments tangent to $\sum^3(a)$.

Hence, $\Omega = \omega \wedge \tau' + \Psi$ for arguments tangent to S^5, where Ψ is a quadratic exterior form which vanishes on the three-dimensional tangent space of $\sum^3(a)$. It follows that rank $\Psi = 2$ on S^5 and we must have

$\Psi = \theta_1 \wedge \theta_2$ for some independent 1-forms θ_1, θ_2. Moreover, since, clearly, rank $\Omega = 4$ on S^5, the forms $\omega, \tau', \theta_1, \theta_2$ must be linearly independent on S^5.

On the other hand, it follows from (2.2) and (2.5) that $\Omega(v, x) = 0$ for every vector x tangent to S^5 and, since $\omega(v) \neq 0$, this contradicts the established relation

$$\Omega = \omega \wedge \tau' + \theta_1 \wedge \theta_2 \qquad (\text{on } S^5) . \tag{2.7}$$

Indeed, taking x such that $\tau'(x) \neq 0$ while $\omega(x) = \theta_1(x) = \theta_2(x) = 0$, we get

$$\Omega(v, x) = \omega(v)\tau'(x) \neq 0 . \tag{2.8}$$

The contradiction proves $\zeta \wedge d\zeta \neq 0$, whence our assertion.

Finally, the last assertion of Proposition 2.3 follows by Corollary 2.2.

The contact form ζ is, up to the sign, just the contact form of Sasaki and Hsu [5], which proves the assertions of our introduction.

FINAL REMARKS. K. Abe and J. Erbacher (Non-regular contact structures on Brieskorn manifolds, Bull. Amer. Math. Soc., 81 (1975), 407-409) gave the result of Theorem 1.1 for a particular class of manifolds, which are submanifolds of S^{2n+1} and which include the Brieskorn manifolds. We also note that, as a matter of fact, Theorem 1.1 is easily seen to hold for the more general case of the quasi-analytic submanifolds of the Hermitian manifolds whose fundamental form is of the type $\Omega = d\omega - \varpi \wedge \omega$, where ω and ϖ are some 1-forms on the manifold (ω will play the same role as in Theorem 1.1). Such are, for instance the Hopf manifolds as well as all the locally conformal Kähler manifolds which are compact and have vanishing second Betti number. (See our papers, On locally conformal almost Kähler manifolds, Israel J. of Math., 24 (1976), 338-351 and Remarkable operators and commutation formulas on locally conformal Kähler manifolds—to appear—for the corresponding definitions.)

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