Open access • Journal Article • DOI:10.1093/MNRAS/174.1.59

On the Scattering and Absorption of Electromagnetic Radiation within Pulsar Magnetospheres — Source link

Roger Blandford, E. T. Scharlemann

Institutions: University of Cambridge

Published on: 01 Jan 1976 - Monthly Notices of the Royal Astronomical Society (Oxford University Press)

Topics: Pulsar, Synchrotron radiation, Transition radiation, Near and far field and Scattering

Mon. Not. R. astr. Soc. (1976) **174**, 59–85.

ON THE SCATTERING AND ABSORPTION OF ELECTROMAGNETIC RADIATION WITHIN PULSAR MAGNETOSPHERES

R. D. Blandford and E. T. Scharlemann*

Institute of Astronomy, Madingley Road, Cambridge

(Received 1975 July 10; in original form 1975 May 2)

SUMMARY

The scattering and absorption of coherent radio emission by ultra-relativistic particles streaming outwards along the open field lines of a pulsar magnetosphere are investigated for incident frequencies in the guiding-centre frame less than or comparable with the particle gyrofrequency, ω_G. Two scattering regimes are isolated in which the dominant particle oscillations are parallel and perpendicular to the magnetic field, and both spontaneous and induced scattering processes are considered. For the former (longitudinal) regime, a quantum-electrodynamical calculation of the cross-section correct to first order in $(\hbar\omega/mc^2)$ is also presented. In addition, cyclotron absorption of radio photons and their subsequent re-emission are investigated.

On the basis of these calculations, it is concluded that: (i) induced scattering processes can influence the spectrum and polarization of the radio pulses if they are emitted well within the pulsar magnetosphere, (ii) neither spontaneous scattering nor small-pitch-angle synchrotron emission seems to provide a satisfactory explanation for the optical pulses from NP 0532, (iii) the avoidance of cyclotron absorption imposes important constraints on models of the radio emission process, and (iv) resonant scattering of thermal radiation from the neutron star surface is probably not an important effect.

1. INTRODUCTION

There is substantial evidence from studies of pulsars and binary X-ray sources suggesting that neutron stars form with strong surface magnetic fields. In the case of pulsars the field strengths are inferred to be $\sim 10^{12}$ gauss. In these strong fields, the Thomson scattering cross-section of a free electron is significantly modified, and radiative emission and transfer occur in a highly anisotropic medium, thus presumably giving rise to the pulsation by which these objects are detected.

In this paper, we investigate scattering and absorption processes that are likely to influence the spectrum and polarization of the radiation emerging from a pulsar magnetosphere. It is assumed that the pulsed radio emission originates well within the light cylinder from relativistic particles streaming outwards along open field lines, and so the rotation of the magnetosphere can be ignored. If the scattering particles are ultra-relativistic, quite different frequency ranges can be related electro-magnetically, as in inverse Compton radiation.

The radio emission is observed to have a very high brightness temperature, variously estimated to be in the range 10^{20}–10^{30} K, that must be attributed to some

* Also Institute for Advanced Study, Princeton, New Jersey, 08540, USA.

coherent emission process. The photon occupation numbers are therefore very large and so induced scattering processes (in which the rate of scattering from an incident to a scattered state exceeds the spontaneous scattering rate by a factor equal to the occupation number of the scattered state) are expected to play an important role in establishing the emergent spectrum. Induced Compton scattering by free electrons has been widely discussed in the literature particularly in studies of compact high brightness radio sources. (See Blandford & Scharlemann (1975) for references and a discussion of various electrodynamical approaches to this process.)

In most of the following calculations the scattering and absorbing particles are treated as independent and collective effects are ignored. The presence of a plasma can introduce two important modifications to the cross-section. Firstly, dispersion changes the $\omega(\mathbf{k})$ relation and secondly the scattering particles may well display significant spatial correlation. Indeed many theories of pulsar emission (e.g. Ruderman & Sutherland 1975) rely on the establishment of 'bunches' of charged particles to produce coherent radio emission. We also ignore the influence of the radiation on the particle energy distribution.

As an ultra-relativistic particle moves outward from the surface along an open field line, its gyro-frequency will decrease. Near the stellar surface, the cooling time for the radiation of transverse gyrational motion is so short that an electron will be almost permanently in its lowest Landau orbital. At larger radii this need not be true. For a given particle energy, the scattering and absorption depends on the ratio of the incident wave frequency to the gyro frequency. In Sections 2, 3 and 4 we discuss three separate regimes characterized by increasing values of this ratio and also increasing distance from the pulsar surface. We do not discuss the scattering of photons whose frequency (when Doppler-shifted into the guiding-centre frame) exceeds the particle gyro-frequency. Such scatterings are probably unimportant for the pulsed radiation. They have, however, been discussed by Tademaru & Greenstein (1974) in the context of setting upper limits to either the neutron star temperature or the electron Lorentz factor.

In Section 5, the results of the previous three sections are applied directly to pulsars. This is done partly by adopting the basic model of Ruderman & Sutherland (1975), which we regard as the most plausible of current detailed models, for illustrative numerical examples, and partly by setting constraints on a more general class of emission mechanisms.

2. LONGITUDINAL SCATTERING

We first consider in some detail the theory of Compton scattering by relativistic electrons in a very strong magnetic field (considered spatially uniform). Previous mention of this process occurs in Goldreich, Pacini & Rees (1972) and Kaplan & Tsytovich (1973, p. 243). In the extensive literature on scattering by electron beams (e.g. Bunkin, Kazakov & Federov 1973) the non-magnetic cross-section is generally used.

(i) *The cross-section*

The classical, non-relativistic cross-section for Thomson scattering in an arbitrary magnetic field (including the dispersive effects of a plasma) has been given by Canuto, Lodenquai & Ruderman (1971). We designate as A the polarization state in which the static magnetic field, k-vector and electric vector are coplanar, and as B, the orthogonal polarization. In the vacuum limit the cross-section in the electron rest frame then becomes

$$\frac{d\sigma}{d\Omega'}(A \to A') = r_e^2 \left\{ \sin^2\theta \sin^2\theta' + \frac{1}{(1-\omega_G^2/\omega^2)^2} \times \left[\cos^2\theta \cos^2\theta' \left(\cos^2\phi' + \frac{\omega_G^2}{\omega^2} \sin^2\phi' \right) + 2\left(1 - \frac{\omega_G^2}{\omega^2}\right) \sin\theta \sin\theta' \cos\theta \cos\theta' \cos\phi' \right] \right\},$$

$$\frac{d\sigma}{d\Omega'}(A \to B') = \frac{r_e^2}{(1-\omega_G^2/\omega^2)^2} \cos^2\theta \left(\sin^2\phi' + \frac{\omega_G^2}{\omega^2} \cos^2\phi' \right),$$

$$\frac{d\sigma}{d\Omega'}(B \to A') = \frac{r_e^2}{(1-\omega_G^2/\omega^2)^2} \cos^2\theta' \left(\sin^2\phi' + \frac{\omega_G^2}{\omega^2} \cos^2\phi' \right),$$

$$\frac{d\sigma}{d\Omega'}(B \to B') = \frac{r_e^2}{(1-\omega_G^2/\omega^2)^2} \left(\cos^2\phi' + \frac{\omega_G^2}{\omega^2} \sin^2\phi' \right), \tag{1}$$

where r_e is the classical electron radius and the scattering is from a photon state with $\mathbf{k} = (\omega/c, \theta, 0)$ to a state with $\mathbf{k}' = (\omega/c, \theta', \phi')$, in spherical polar coordinates referred to the field direction.

To lowest order in (ω/ω_G), the only non-zero cross-section connects polarization states A, A':

$$\frac{d\sigma}{d\Omega'} = r_e^2 \sin^2\theta \sin^2\theta'. \tag{2}$$

We are here treating the electron as being constrained to one-dimensional motion like a charged bead on a straight wire.

In the Appendix, we outline the quantum mechanical calculation of the cross-section in the limit of small (ω/ω_G) to second order in the fine structure constant and first order in $\hbar\omega/mc^2$. This cross-section bears the same relation to (2) as the Klein–Nishina cross-section does to the Thomson cross-section and includes (to lowest order) the electron recoil.

(ii) *Energy and momentum conservation*

We are interested in scattering by an electron moving ultrarelativistically along the field with Lorentz factor $\gamma = (1-\beta^2)^{-1/2}$ in the observer frame. If we ignore electron recoil, then

$$\omega'\eta' = \omega\eta,$$

where

$$\eta = 1 - \beta\cos\theta,$$
$$\eta' = 1 - \beta\cos\theta', \tag{3}$$

evaluated in the observer frame.

The possible range of the frequency ratio (ω'/ω) is (to lowest order in γ^{-2})

$$\frac{1}{4\gamma^2} \leq \frac{\omega'}{\omega} \leq 4\gamma^2. \tag{4}$$

Momentum is conserved parallel to the magnetic field and we can therefore include the effects of electron recoil in the initial electron rest frame using

$$\omega' = \omega\left[1-\frac{\hbar\omega}{2mc^2}(\cos\theta-\cos\theta')^2+O\left(\frac{\hbar\omega}{mc^2}\right)^2\right]. \tag{5}$$

Momentum perpendicular to the field is not conserved. In the case of a pulsar, the difference is propagated via magnetic stresses either to the neutron star or to infinity.

(iii) *Spontaneous scattering of isotropic monochromatic radiation*

We consider the scattering by an ultra-relativistic electron of an isotropic monochromatic distribution of photons with spectral intensity (power per unit area per steradian per unit frequency range in polarization A) given by

$$I(\mathbf{k}) = \frac{\mathscr{E}_0 c}{4\pi}\delta(\omega-\omega_0). \tag{6}$$

The calculation proceeds analogously to the calculation of the spectral distribution of inverse Compton radiation (e.g. Blumenthal & Gould 1970). The power scattered into frequency interval $d\omega'$ and solid angle $d\Omega'$ in polarization state A' is given by

$$p(\gamma,\omega',\Omega') = \hbar\omega'\int\frac{d\sigma}{d\Omega'}(\gamma;\Omega\to\Omega')\,\eta\left(\frac{I(\mathbf{k})}{\hbar\omega}\right)\delta\left(\omega'-\frac{\eta\omega}{\eta'}\right)d\omega\,d\Omega. \tag{7}$$

Transforming the cross-section (2) into the observer frame and substituting (6), this becomes for $\gamma\gg 1$:

$$p(\gamma,\omega',\Omega') = \frac{\mathscr{E}_0}{2\omega_0}\cdot\frac{cr_e^2}{\gamma^6\eta'^4}(2\eta'-\eta'^2-\gamma^{-2})\left(2\eta'\frac{\omega'}{\omega_0}-\frac{\omega'^2}{\omega_0^2}\eta'^2-\gamma^{-2}\right),$$
$$\frac{\omega_0}{2\gamma^2\eta'}\le\omega'\le\frac{2\omega_0}{\eta'}. \tag{8}$$

This can now be integrated over solid angle to give a total spectral power. It must be realized that the meaning of this quantity is not quite the same as in inverse Compton radiation, where it is usually assumed that the electron momenta are uniformly distributed in a small angle about the observer direction. In a spatially uniform field, the observed power is clearly angle-dependent. However, in applications to pulsar magnetospheres, we will be assuming that the field direction changes through angles $\gtrsim\gamma^{-1}$ on length scales long compared with the incident wavelength but short compared with the size of the emitting region, and so at a given point within a magnetosphere the integrated emitted spectral power does provide an estimate, albeit approximate, of the time-averaged observed power. Performing this integration we find,

$$p(\gamma,\omega') = \frac{16\pi\mathscr{E}_0 cr_e^2}{\gamma^2\omega_0}\{x+2x^2\ln x-x^3\} \tag{9}$$

to lowest order in γ^{-2}, with $x = \omega'/4\gamma^2\omega_0$ and $1\ge x\gg\gamma^{-2}$. The total power radiated by an electron is given by

$$p(\gamma) = \int p(\gamma,\omega')\,d\omega' = \frac{16\pi r_e^2 c\mathscr{E}_0}{9}. \tag{10}$$

This agrees with a calculation of the total power radiated by an ultra-relativistic electron using classical electrodynamics.

A similar expression has been calculated by Kaplan & Tsytovich (1973, p. 243) for the scattering of longitudinal plasmons (rather than transverse photons). They find

$$p(\gamma) = \frac{8\pi}{3} r_e^2 c \mathscr{E}_0. \tag{11}$$

The corresponding power scattered by a free electron is given by

$$p(\gamma) = \frac{32\pi r_e^2 c \gamma^2 \mathscr{E}_0}{9} \tag{12}$$

which exceeds (10) by a factor $\sim\gamma^2$. As noted qualitatively by Goldreich *et al.* (1972) this factor arises because when the electron is constrained to motion in one dimension, only the longitudinal component of the electric field can be scattered. In the electron rest frame, this is typically smaller than the transverse component by a factor $\sim\gamma^{-1}$, thus reducing the effective cross-section by a factor $\sim\gamma^{-2}$. The mean scattered frequency is given by

$$\langle\omega'\rangle = \frac{\int p(\gamma', \omega')\,\omega'\,d\omega'}{\int p(\gamma', \omega')\,d\omega'} = \tfrac{2}{3}\gamma^2\omega_0 \tag{13}$$

which is half the mean frequency when scattering is by a free electron.

We can also integrate (8) in the limit $1/4\gamma^2 \leq x \ll 1$ to obtain

$$p(\gamma, \omega', \Omega') = \frac{\mathscr{E}_0}{2\omega_0}\frac{c r_e^2}{\gamma^6 \eta'^4}\left\{\frac{4\omega}{\omega_0}\eta'^2 - \frac{2}{\gamma^2}\eta'\left(1+\frac{\omega'}{\omega_0}\right) + \frac{1}{\gamma^4}\right\} \tag{14}$$

and

$$p(\gamma, \omega') = \frac{4\pi\mathscr{E}_0 c r_e^2}{\gamma^4\omega_0}\left(\frac{\omega}{\omega_0} - \frac{1}{3}\right); \; \gamma^2\omega_0 \gg \omega' \geq \omega_0. \tag{15}$$

(iv) *Spontaneous scattering of anisotropic, polychromatic radiation*

Within a pulsar magnetosphere, the incident photons will be neither isotropic nor monochromatic and so some generalization of (8), (9) and (10) is required. The integration of (9) and (10) over incident frequencies, ω_0, is straightforward. In order to obtain a qualitatively correct result for anisotropic radiation we consider an angular distribution of monochromatic photons that is uniform with spectral intensity $I(\mathbf{k}) = (\mathscr{E}_0 c/\pi\theta_m^2)\,\delta(\omega-\omega_0)$ for $0 \leq \theta \leq \theta_m \ll 1$ and zero for $\theta > \theta_m$. Because only polarization A will scatter, the distribution in angle ϕ' is unimportant.

Equivalent to (9), (10) and (13) we obtain, for $\theta_m \ll 1$,

$$p(\gamma, \omega') = \frac{16\pi\mathscr{E}_0 r_e^2 c}{\gamma^2\omega_0}\left\{y - 2y^2 + y^3 + \frac{\theta_m^2}{2} y^2 \ln y\right\} \quad 1 \geq y \gg \gamma^{-2}, \tag{16}$$

$$p(\gamma) = \tfrac{4}{3}\pi\mathscr{E}_0 r_e^2 c \theta_m^2, \tag{17}$$

$$\langle\omega'\rangle = \tfrac{1}{4}\theta_m^2\gamma^2\omega_0, \tag{18}$$

where

$$y = \frac{\omega'}{\gamma^2\theta_m^2\omega_0}. \tag{19}$$

(v) *Induced scatterings*

We now consider induced scatterings, by which we mean the scatterings from one radiation state $\mathbf{k}$ to another, $\mathbf{k}'$, enhanced by the photon occupation number of the second ' beam '. As with scattering by free electrons the scatterings between the two ' beams ' cancel identically when electron recoil is ignored; i.e. to order unity in $(\hbar\omega/mc^2)$.

It is instructive to treat the effects of induced scatterings by a distribution of electrons in two complementary ways. We are free to choose as independent, two quantities from the electron energy and the photon momentum before and after scattering, the other two being constrained by conservation of energy, and momentum parallel to the field. We first consider scatterings between two specific photon states, including the effect of recoil on the electron energies. This only requires the classical cross-section. We then compare scatterings, fixing one photon state and the initial electron energy before integrating over electron energies. Our purpose is not only to demonstrate the equivalence of the two methods (*cf.* Blandford & Scharlemann 1975), but also to point out that this equivalence can only be achieved if the quantum mechanical cross-section including recoil terms (calculated in the Appendix) is used in the treatment of individual scatterings. This is in contrast to the free scattering case where the Thomson approximation to the scattering cross-section is sufficient.

For scattering in the observer frame between two photon states $\mathbf{k}$, $\mathbf{k}'$ involving electrons with initial energies γmc^2, $\gamma mc^2+\hbar(\omega-\omega')$, the probability per electron per unit time of a spontaneous scattering in polarization A is given by

$$\frac{dP}{dt} = n\frac{d^3\mathbf{k}}{(2\pi)^3}\eta\frac{d\sigma}{d\Omega'}c.c^3\frac{d^3\mathbf{k}'}{\omega'^2}\delta\left(\omega'-\frac{\eta\omega}{\eta'}\right)$$

$$= \frac{nr_e^2c^4}{(2\pi)^3}\frac{\sin^2\theta\sin^2\theta'}{\omega\omega'\gamma^6\eta^2\eta'^2}\delta(\omega'\eta'-\omega\eta)\,d^3\mathbf{k}\,d^3\mathbf{k}' \quad (20)$$

using equation (2). This equation is clearly symmetric between the two directions of scattering and demonstrates detailed balance. We can then calculate the rate of change of photon occupation number for photons of momentum $\hbar\mathbf{k}$ as a result of induced scatterings, from

$$\left(\frac{\partial n}{\partial t}\right)_i\frac{d^3\mathbf{k}}{(2\pi)^3} = \int\{N_\gamma(\gamma+\delta\gamma)-N_\gamma(\gamma)\}\frac{dP}{dt}n'\,d\gamma \quad (21)$$

where $n' = n(\mathbf{k}')$, $\delta\gamma = h(\omega-\omega')/mc^2$, and N_γ is the one-dimensional electron distribution function.

Therefore

$$\left(\frac{\partial n}{\partial t}\right)_i = \int d\gamma\frac{\partial N_\gamma}{\partial\gamma}\int\frac{nn'}{\omega\omega'}r_e^2c^4\sin^2\theta\sin^2\theta'\frac{\delta(\eta\omega-\eta'\omega')}{\gamma^6\eta^2\eta'^2}\frac{\hbar(\omega-\omega')}{mc^2}d^3\mathbf{k}'. \quad (22)$$

Integrating this by parts we obtain

$$\left(\frac{\partial n}{\partial t}\right)_i = \int d\gamma N_\gamma\frac{\partial}{\partial\gamma}\left\{\int\frac{nn'r_e^2c^4\sin^2\theta\sin^2\theta'\,\delta(\eta\omega-\eta'\omega')}{\omega\omega'\gamma^6\eta^2\eta'^2}\frac{\hbar(\omega'-\omega)}{mc^2}d^3\mathbf{k}'\right\}. \quad (23)$$

An integration of (23) over $d^3\mathbf{k}$ confirms that the number of photons is conserved, as it must be in a scattering process.

In a spatially homogeneous medium we can replace $\partial/\partial t$ by $c(\partial/\partial x)$ and obtain an equation for the non-linear 'absorption' coefficient $\kappa(\mathbf{k})$ defined by

$$\frac{\partial I(\mathbf{k})}{\partial x} = -\kappa I(\mathbf{k}). \tag{24}$$

Then

$$\kappa = \int d\gamma N_\gamma \frac{\partial}{\partial \gamma}\left\{\int (2\pi)^3 r_e^2 c^3 \frac{I(\mathbf{k}')}{m\omega'^4} \frac{\sin^2\theta \sin^2\theta'}{\gamma^6\eta^2\eta'^2}\left(\frac{\omega-\omega'}{\omega}\right)\delta(\eta\omega-\eta'\omega')\, d^3\mathbf{k}'\right\}. \tag{25}$$

In addition we can integrate (24) over $d^3\mathbf{k}$ to calculate the total rate of loss (or gain) of energy by the radiation field,

$$\frac{\partial \mathscr{E}_{\text{rad}}}{\partial t} = -\tfrac{1}{2}\int d\gamma N_\gamma \frac{\partial}{\partial\gamma}\left[\iint \frac{(2\pi)^3 r_e^2 c^6 \sin^2\theta \sin^2\theta'(\omega-\omega')^2}{m\omega^4\omega'^4\gamma^6\eta^2\eta'^2} \times I(\mathbf{k})\, I(\mathbf{k}')\,\delta(\eta\omega-\eta'\omega')\, d^3\mathbf{k}\, d^3\mathbf{k}'\right] \tag{26}$$

where $\mathscr{E}_{\text{rad}}$ is the radiation energy density. The factor $\frac{1}{2}$ is included in order to count each scattering only once. (The partial derivative with respect to γ in (26) is best left outside the integrals over $d^3\mathbf{k}\, d^3\mathbf{k}'$ because of the dependence on γ of the limits to the integrals.)

In the second approach, we consider the scatterings by a single electron of Lorentz factor, γ, in the observer frame within a normalization volume V introduced in order to make the equations dimensionally homogeneous, between photons in states $\mathbf{k}$, $\mathbf{k}'$.

Using equations (3), (5) and (20), we obtain

$$V\frac{\partial n}{\partial t} = cn\int n'\, d^3\mathbf{k}'\left\{\frac{\eta'}{k^2}\frac{d\sigma}{d\Omega}(\gamma;\mathbf{k}'\to\mathbf{k})\,\delta\left[k-k'\frac{\eta'}{\eta}(1-\Delta)\right] - \frac{\eta}{k'^2}\frac{d\sigma}{d\Omega'}(\gamma;\mathbf{k}\to\mathbf{k}')\,\delta\left[k'-k\frac{\eta}{\eta'}(1-\Delta)\right]\right\}. \tag{27}$$

Here,

$$\Delta = \frac{\hbar\omega(\eta'-\eta)^2}{2\beta^2\gamma^3 mc^2\eta\eta'^2}. \tag{28}$$

From the Appendix the observer frame cross-sections are

$$\frac{d\sigma}{d\Omega}(\gamma;\mathbf{k}'\to\mathbf{k}) = \frac{r_e^2\sin^2\theta\sin^2\theta'}{\gamma^6\eta^4\eta'^2}(1+\epsilon_1),$$

$$\frac{d\sigma}{d\Omega'}(\gamma;\mathbf{k}\to\mathbf{k}') = \frac{r_e^2\sin^2\theta\sin^2\theta'}{\gamma^6\eta^2\eta'^4}(1-\epsilon_2)$$

$$\simeq \frac{\eta^2}{\eta'^2}\frac{d\sigma}{d\Omega}(\gamma;\mathbf{k}'\to\mathbf{k})(1-\epsilon_1-\epsilon_2)$$

where

$$\epsilon_1 = \frac{\hbar\omega}{\beta^2\gamma mc^2}\left(\frac{\eta'-\eta}{\eta'}\right)\left[3-\frac{1}{2\gamma^2}\left(\frac{5\eta'+\eta}{\eta\eta'}\right)\right],$$

$$\epsilon_2 = \frac{\hbar\omega}{\beta^2\gamma mc^2}\left(\frac{\eta'-\eta}{\eta'}\right)\left[3-\frac{1}{2\gamma^2}\left(\frac{\eta'+5\eta}{\eta\eta'}\right)\right]. \tag{29}$$

(In scattering by free electrons the recoil terms, ϵ_1 and ϵ_2, are related by $\epsilon_1 = -\epsilon_2$, so that recoil corrections to the Thomson cross-section cancel, as mentioned earlier.) The differences in the cross-section for scattering in a very strong magnetic field are not only quantitatively significant but can also change the direction of the effects of induced scattering.

Assuming an isotropic radiation field, inserting the forms for the cross-sections (29) and expanding $k'^2 n'$ in a Taylor series, we find to $O(\hbar\omega/mc^2)$ but to all orders in γ^{-1}:

$$V\frac{\partial n}{\partial t} = \frac{\hbar}{m} n r_e^2 \int \frac{\sin^2\theta \sin^2\theta'}{\gamma^6\eta^3\eta'^3}\left\{\frac{1}{\beta^2\gamma^3}\left(\frac{\eta'-\eta}{\eta'}\right)^2 \frac{\partial}{\partial k}(k'^2 n') + \frac{6k\eta^2(\eta'-\eta)\,n'}{\beta^2\gamma\eta'^2}\left[1-\frac{1}{2\gamma^2}\left(\frac{\eta'+\eta}{\eta\eta'}\right)\right]\right\} d\Omega' \quad (30)$$

where n' is evaluated at $k' = \eta k/\eta'$.

The non-relativistic limit of this (which can be compared with the induced scattering terms in the non-magnetic photon kinetic equation due to Weymann, 1965) is

$$\frac{\partial n}{\partial t} = \frac{\hbar n}{m} r_e^2 \int \sin^2\theta \sin^2\theta' \left\{(\cos\theta - \cos\theta')^2 \frac{\partial}{\partial k'}(k'^2 n') - 3(\cos^2\theta - \cos^2\theta')\,kn'\right\} d\Omega', \quad (31)$$

with n' now evaluated at $k' = k$.

Of more usefulness in the relativistic limit is a second procedure in which n' is set equal to

$$n' = 2\pi^2 n_0' \frac{\delta(k'-k_0)}{k'^2}, \quad (32)$$

i.e. we evaluate the integrals of constant $k' = k_0$. In this case the manipulation of the δ-functions is somewhat more complex, but we find

$$\frac{\partial n}{\partial t} = 4\pi^3 c \frac{n}{k^3} n_0' r_e^2 \frac{\sin^2\theta}{\beta^3\gamma^6\eta^4}\left[\left(2\frac{k}{k_0}\eta - \frac{k^2}{k_0^2}\eta^2 - \frac{1}{\gamma^2}\right) \times \left(2\eta'\frac{\partial\Delta}{\partial\eta'} + \epsilon_1 + \epsilon_2\right) + 4\Delta\left(\frac{1}{\gamma^2} - \frac{k}{k_0}\eta\right)\right] \quad (33)$$

with

$$\eta'\frac{\partial\Delta}{\partial\eta'} = \frac{\hbar\omega}{2\beta^2\gamma^3 mc^2}\left(\frac{\eta'^2-\eta^2}{\eta\eta'^2}\right).$$

Equation (23) can then be shown to be equivalent to equation (33).

To make further progress with equations (33) or (25) we introduce in the next two sections approximations appropriate for relativistic electrons.

(vi) *Induced scattering when $\omega' \sim \gamma^2\omega$*

We first consider the scattering of a distribution of photons with frequency $\sim\omega_0$ to frequencies $\omega' \sim \gamma^2\omega_0$. We can then ignore the energy and momentum of the photon at frequency ω_0 in comparison with that at ω'. Comparison of (25) (interchanging primed and unprimed quantities) with (7) leads to the equation

$$\kappa = (2\pi)^3 \int d\gamma N_\gamma \frac{\partial}{\partial\gamma}\left\{\frac{p(\gamma, \omega', \Omega')}{m\omega'^2}\right\}. \quad (34)$$

This equation can also be derived from equation (33) by careful application of the relation obtained from detailed balance:

$$\frac{d\sigma}{d\Omega'}(\gamma, \mathbf{k} \to \mathbf{k}') = \frac{k'^2}{k^2}\frac{d\sigma}{d\Omega}(\gamma+\delta\gamma, \mathbf{k}' \to \mathbf{k}), \tag{35}$$

a formula that also follows from the cross-sections (29). If the high frequency photons have a spectral intensity $I(\mathbf{k}')$,

$$\frac{\partial \mathscr{E}_{\mathrm{rad}}}{\partial t} = -(2\pi)^3 \int d\omega'\, d\Omega' I(\mathbf{k}') \int d\gamma N_\gamma \frac{\partial}{\partial \gamma}\left\{\frac{p(\gamma, \omega', \Omega')}{m\omega'^2}\right\}. \tag{36}$$

If in addition $I(\mathbf{k}')$ is uniform for angles within a cone of semi-angle $\sim\gamma^{-1}$ about the direction of the electron momentum,

$$\frac{\partial \mathscr{E}_{\mathrm{rad}}}{\partial t} = -(2\pi)^3 \int d\omega' I(\mathbf{k}') \int d\gamma N_\gamma \frac{\partial}{\partial \gamma}\left\{\frac{p(\gamma, \omega')}{m\omega'^2}\right\}. \tag{37}$$

From the γ-dependence of the spectral power given by equations (8), (9) and (16) we see that it is quite possible for the absorption coefficient to be negative and for the electrons to lose rather than gain energy from the radiation, i.e. for a scattering maser to operate. We remark at this stage that this almost certainly does not constitute the basis of a viable model for the production of coherent radio emission from pulsars. The difference between a scattering and an emission maser is that the former does not create photons, and so for any plausible magnetospheric plasma frequency there would still remain the problem of emitting photons with a brightness temperature much larger than the single particle kinetic temperatures.

To illustrate the negative absorption coefficient, consider a monoenergetic electron distribution function, $N_\gamma = n_e \delta(\gamma - \gamma_0)$, scattering isotropic monochromatic, low-frequency photons, with an intensity given by equation (6), into high-frequency photons of intensity $I(\mathbf{k}')$. Substituting equation (9) into equation (37) then gives

$$\frac{\partial \mathscr{E}_{\mathrm{rad}}}{\partial t} = \frac{128\pi^4 n_e \mathscr{E}_0 r_e^2 c}{m\omega_0^2 \gamma_0^5} \int^{4\gamma^2\omega_0} \{1 + x + 3x \ln x - 2x^2\} \frac{I(\mathbf{k}')}{\omega'}\, d\omega'. \tag{38}$$

Thus, in this example, we see that after integration over angle, radiation with frequency such that the expression in braces in (38) is positive, i.e. with $\omega' < 1{\cdot}42\gamma^2\omega_0$, will be amplified whereas higher frequency radiation will be attenuated.

(vii) *Induced scattering when* $\omega \sim \omega_0$

In this limit the evolution of the radiation spectrum clearly depends in a complex fashion on the electron and photon distributions and the magnetic field geometry, and it does not seem possible to produce any further useful general discussion. We therefore anticipate Section 5 and give an approximate relation that is directly relevant to a pulsar magnetosphere.

Within a magnetosphere, the magnetic field lines will not be straight but will be curved with radius of curvature $\sim\rho$. From equation (25) it can be seen that the opacity increases as the angle θ decreases. However, the curvature of the field lines sets an effective lower limit on θ in that the cross-section (2) is only applicable as long as much more than 2π of phase can pass the scattering particle before θ has changed significantly. This requires

$$\theta \lesssim \theta_{\min} \sim \left(\frac{\omega\rho}{c}\right)^{-1/3}; \quad \omega \lesssim \gamma^3 \frac{c}{\rho}$$
$$\sim \left(\frac{\omega\rho}{c}\right)^{-1} \gamma^2; \quad \omega \gtrsim \gamma^3 \frac{c}{\rho}. \tag{39}$$

(Alternative but equivalent definitions of $\theta_{\min}$ are possible.) For $\theta \lesssim \theta_{\min}$ the electrodynamics of curvature radiation is appropriate (*cf.* Blandford 1975). We note that if ω is to be associated with the peak frequency of curvature radiation, then $\theta_{\min} \sim \gamma^{-1}$, and photons will be able to scatter as soon as they have been created. With $\omega \lesssim \gamma^3(c/\rho)$, we can approximate η by $\theta^2/2$. If in addition the electrons have $\gamma \sim \gamma_0$, then from (25) we obtain

$$\kappa \sim -96(2\pi)^3 \frac{n_e r_e^2}{\gamma_0^7} \iint \frac{I(\mathbf{k}')}{m\omega'^2} \frac{(\omega-\omega')}{\theta^2\theta'^2\omega} \delta\left(\frac{\theta^2\omega}{2} - \frac{\theta'^2\omega'}{2}\right) d\omega' \pi \, d(\theta'^2). \tag{40}$$

Now if the radiation is broadband, we can define a radiation brightness temperature T_b by

$$I(\mathbf{k}') = \frac{k_B T_b \omega'^2}{(2\pi)^3 c^2}. \tag{41}$$

The dominant contribution to the integral in equation (40) occurs when $\theta \sim \theta_{\min}$ and so, to order of magnitude, the angle-averaged absorption coefficient is given by

$$|\kappa| \sim 96\pi \frac{k_B T_b}{mc^2} \frac{n_e r_e^2}{\gamma_0^7 \theta_{\min}^4}. \tag{42}$$

More detailed calculations making differing assumptions about the angular and frequency range of the radiation yield essentially similar results.

(viii) *Polarization*

The longitudinal cross-section (2) only relates photons with polarization A. Therefore a distant observer whose direction makes a finite angle with a spatially uniform field will observe 100 per cent linearly polarized spontaneously-scattered radiation. However, when the scattered frequency is much larger than the incident frequency, the scattered photons can only be observed at a small angle to the field line. If the field lines are diverging, as they will be in a pulsar magnetosphere, an observed photon can be linearly polarized in any direction dependent on the projection of the magnetic field at the point of emission.

At any given time let $n_\Omega(\theta, \phi)$ be the number of emitting electrons per unit solid angle with energy in some small range whose instantaneous unit vector momenta $\hat{p}$ make a small angle θ with the direction of the observer $\hat{\mathbf{n}}$, so that

$$\mathbf{n} \times \hat{\mathbf{p}} . \hat{\mathbf{x}} = -\sin\theta \sin\phi,$$
$$\mathbf{n} \times \hat{\mathbf{p}} . \hat{\mathbf{y}} = \sin\theta \cos\phi, \tag{43}$$

with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ orthonormal unit vectors perpendicular to $\hat{\mathbf{n}}$. Then the observed degree of polarization along $\hat{\mathbf{x}}$ is given by

$$\Pi_x = \frac{\iint n_\Omega(\theta, \phi)\, p(\gamma, \omega, \Omega) \cos(2\phi)\, \theta \, d\theta \, d\phi}{\iint n_\Omega(\theta, \phi)\, p(\gamma, \omega, \Omega)\, \theta \, d\theta \, d\phi} \tag{44}$$

where we ignore spatial variation in the function $p(\gamma, \omega, \Omega)$. Now we can expand $n_\Omega(\theta, \phi)$ in a Taylor series, using coordinates x, y for small angles measured from $\hat{\mathbf{n}}$ rotating about axes $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, to obtain

$$n_\Omega(\theta, \phi) = n_\Omega(0, 0) + \theta\cos\phi\,\frac{\partial n_\Omega}{\partial x} + \theta\sin\phi\,\frac{\partial n_\Omega}{\partial y} + \tfrac{1}{2}\theta^2\cos^2\phi\,\frac{\partial^2 n_\Omega}{\partial x^2} + \theta^2\cos\phi\sin\phi\,\frac{\partial^2 n_\Omega}{\partial x\,\partial y} + \tfrac{1}{2}\theta^2\sin^2\phi\,\frac{\partial^2 n_\Omega}{\partial y^2}. \tag{45}$$

Therefore

$$\Pi_x = \frac{\pi}{4}\frac{\langle\theta^2\rangle}{n_\Omega}\left\{\frac{\partial^2 n_\Omega}{\partial x^2} - \frac{\partial^2 n_\Omega}{\partial y^2}\right\} \tag{46}$$

where $\langle\theta^2\rangle$ is the mean square angle at which the radiation at that frequency is emitted. Thus, if the distribution function of emitting electrons varies over an angular scale $\sim\xi$, then the observed degree of linear polarization $\Pi \sim \langle\theta^2\rangle\xi^{-2}$. An alternative way of making this plausible is to observe that the integrated spectral power $p(\gamma, \omega)$ emitted by each electron is completely unpolarized and therefore so is the observed polarization. For the case of a stationary dipole field, this result can be demonstrated explicitly.

(ix) *Collective effects*

As mentioned in the Introduction, in a variety of pulsar emission mechanisms the coherence of the radio emission is produced by bunches of charged particles. We can characterize a typical ' bunch ' (more precisely a Fourier component of the density) by a linear size L and a charge excess, $\pm Ne$, where N is the effective number of particles in the bunch. For scattered wavelengths $\ll L$, the constituent particles will spontaneously scatter incoherently. However, for wavelengths $\gtrsim L$, the spontaneous scattering rate will be enhanced by a factor N. In addition the shape of the scattering bunch must be taken into consideration.

For induced scattering, we see that the opacity as defined in equation (25) is not appreciably enhanced by bunching (because of the presence of an extra mass in the denominator). However, any more detailed discussion is particularly sensitive to the plasma parameters assumed.

(x) *Validity of the cross-section*

In a strong magnetic field ($B \gtrsim 10^{14}$ gauss), pair creation can occur and this provides an effective maximum field strength for the validity of the cross-section. The minimum field strength is discussed in the following section.

In deriving the cross-section (2) it has been assumed that the motion in the electron rest frame is non-relativistic. This will be the case as long as $\gamma \gg f$, where $f = e(8\pi\mathscr{E})^{1/2}/mc\omega$ is the strength parameter of the incident radiation (energy density $\mathscr{E}$).

3. TRANSVERSE SCATTERING

In a weaker magnetic field, the cross-section given by equation (2) no longer describes the dominant scattering effects of an ultra-relativistic electron. In effect, the drift velocity of the guiding centre, $(\mathbf{E}\times\mathbf{B})/B^2$, leads to a larger cross-section than the longitudinal oscillation parallel to the field. Transverse scattering is significantly more complex than longitudinal scattering, and we only attempt a semi-quantitative discussion. In particular we specialize immediately by summing over incident polarization states, postponing a qualitative discussion of the polarization of the scattered radiation to (v).

(i) *Cross-section*

Either from the complete cross-section (1) or directly from first principles, we see that the ϕ-averaged transverse cross-section after averaging over incident polarization is given by

$$\frac{d\sigma}{d\Omega'} = \tfrac{1}{4}r_e^2 \frac{\omega^2}{\omega_G^2}(1+\cos^2\theta)(1+\cos^2\theta'), \tag{47}$$

in the electron rest frame to lowest order in ω/ω_G. This assumes that the incident radiation has azimuthal symmetry with respect to the field.

(ii) *Spontaneous scattering of isotropic monochromatic radiation*

Again we assume that the incident radiation has a spectral intensity given by (6), transform the cross-section (47) into the observer frame and substitute in (7) to obtain for $\gamma \gg 1$,

$$p(\gamma, \omega', \Omega') = \frac{1}{2}\frac{\mathscr{E}_0 c r_e^2 \omega'^4}{\omega_0^3 \omega_G^2}\left\{\eta'^2 - \frac{1}{2\gamma^2}(2\eta' - \gamma^{-2})\right\}, \tag{48}$$

$$\frac{\omega_0}{2\gamma^2\eta'} \leq \omega' \leq \frac{2\omega_0}{\eta'}.$$

Integration over solid angle gives a total spectral power,

$$p(\gamma, \omega') = \frac{32\pi}{3}\frac{\mathscr{E}_0 r_e^2 c\omega_0\gamma^2}{\omega_G^2}\{x - 3x^2 + 6x^3 - 4x^4\} \tag{49}$$

where again $x = \omega'/4\gamma^2\omega_0$ and $1 \geq x \geq 1/4\gamma^2$. Note that the single expression (49) suffices for all $\omega' \geq \omega_0$. The total power radiated is given by

$$p(\gamma) = \frac{128}{15}\pi\frac{\mathscr{E}_0 r_e^2}{\omega_G^2}c\omega_0^2\gamma^4 \tag{50}$$

and the mean scattered frequency is

$$\langle\omega'\rangle = \tfrac{8}{5}\gamma^2\omega_0. \tag{51}$$

(iii) *Spontaneous scattering of anisotropic polychromatic radiation*

Again, integration over the incident frequency ω_0 is straightforward. In addition, we can derive expressions directly analogous to (16)–(18) for the scattering of monochromatic radiation uniformly distributed in a cone of semi-angle θ_m about the field direction. We find that for $\theta_m \ll 1$,

$$p(\gamma, \omega') = \frac{\pi r_e^2 c\mathscr{E}_0\gamma^2\theta_m^6\omega_0}{6\omega_G^2}\{y - 3y^2 + 6y^3 - 4y^4\}, \tag{52}$$

$$p(\gamma) = \frac{\pi}{30}\frac{r_e^2 c\mathscr{E}_0\gamma^4\theta_m^8\omega_0^2}{\omega_G^2}, \tag{53}$$

$$\langle\omega'\rangle = \tfrac{2}{5}\gamma^2\theta_m^2\omega_0, \tag{54}$$

with y given by (19).

(iv) *Induced scattering*

The formalism outlined in equations (20)–(42) can be developed analogously, replacing the longitudinal cross-section (2) with the transverse cross-section (47). Two important features emerge. The first is that for the scattering of isotropic radiation, $p(\gamma, \omega', \Omega')$ as given by (48) is a decreasing function of energy only for $\theta' \lesssim \gamma^{-1}$. Thus from equation (34), only when this is satisfied can κ be negative. Furthermore, in a realistic magnetospheric geometry, the net optical depth for $\omega' \gg \omega_0$ along a line of sight taking into account the changing field orientation will be positive.

The second feature is that when $\omega' \ll \gamma^2 \omega_0$, the absolute magnitude of the non-linear optical depth resulting from transverse scattering only exceeds that from longitudinal scattering when $\omega \gtrsim \omega_G/\gamma$. (This follows from an examination of the relation corresponding to (25) incorporating the spectral power given by (48).) As we show below, this is the condition that cyclotron absorption occurs, an effect which will dominate either induced scattering process.

(v) *Polarization*

In contrast to longitudinal scattering, the polarization of spontaneous transversely-scattered radiation can be quite large and will reflect the polarization and anisotropy of the incident radiation. If the (complex) amplitudes of the electric field of an incident photon in the A and B polarizations are proportional to e_A, e_B then the scattered polarization is given by

$$\begin{pmatrix} e_A' \\ e_B' \end{pmatrix} = \begin{pmatrix} \cos\theta \cos\theta' \sin\phi' & -\cos\theta' \cos\phi' \\ \cos\theta \cos\phi' & \sin\phi' \end{pmatrix} \begin{pmatrix} e_A \\ e_B \end{pmatrix}. \tag{55}$$

In the absence of any clear picture of the scattering geometry within a pulsar magnetosphere, we do not attempt a detailed calculation of the polarization of transversely-scattered radiation. Four features are, however, apparent from an examination of (55). First, the degrees of both linear and circular polarization will be partially preserved by the scattering. (For the case of incident circular polarization it is sufficient to observe that, in the electron rest frame, scattered photons of opposite helicity are emitted in opposite hemispheres. On Doppler-shifting to the observer frame, the photons of opposite helicity have frequencies differing by a factor ~ 2. Thus on averaging over incident frequencies and electron energies a net circular polarization will be observed.)

Secondly, if both the incident and the scattered radiation are observed then $\theta = \theta'$, $\phi' = 0$ and from (49) the planes of linear polarization are perpendicular. Of course unobserved incident radiation can contribute to the scattered flux, but if the radiation is fairly well collimated some rotation of the polarization plane might be anticipated at peak intensity. Thirdly, if the incident radiation is unpolarized but anisotropic, appreciable scattered polarization can result. Finally, we note that for frequencies such that the longitudinal and transverse cross-sections are approximately equal, it is possible to scatter unpolarized incident radiation with elliptical polarization.

(vi) *Validity of the cross-section*

We can determine the field strength below which the spontaneous transverse scattering dominates longitudinal scattering by comparing the spectral powers given by (49) and (52) with those given by (9) and (16). For isotropic incident radiation, the condition that transverse scattering dominates is that

$$\omega_G \lesssim \gamma^2 \omega_0 \sim \langle \omega' \rangle. \tag{56}$$

For anisotropic radiation this becomes

$$\omega_G \lesssim 0{\cdot}1\, \theta_m{}^3 \gamma^2 \omega_0 \sim 0{\cdot}3\, \theta_m \langle \omega' \rangle. \tag{57}$$

(The latter condition can also be seen by Lorentz-transforming into a frame in which the radiation is isotropic.) A lower limit for the field strength is given below. As with longitudinal scattering, the electron oscillation in the guiding centre frame is non-relativistic as long as $\gamma \gg f$.

4. CYCLOTRON ABSORPTION AND RESONANCE SCATTERING

When the magnetic field becomes even weaker, the incident frequency, after transformation into the electron rest frame ($\sim \gamma\omega_0$), can become equal to the electron gyro-frequency, ω_G (which is unchanged by the transformation). The electron can then absorb the radiation being excited into higher Landau orbitals. At a later time the electron may de-excite and emit a photon of frequency $\omega' \sim \gamma^2 \omega_0$.

(i) *Absorption cross-section*

In the electron rest frame, $\theta \sim \pi$, as long as we exclude photons that have $\theta \lesssim \gamma^{-1}$ in the observer frame. (These cannot be absorbed within the magnetosphere until the field strength decreases, and before this can happen, θ will exceed γ^{-1}.) As the radiation emitted in this direction by gyrating electron is circularly polarized, only one sense of circular polarization can be absorbed. The spectral power radiated in this polarization is given by

$$p(\omega, \Omega) = \frac{e^2 \omega_G}{4\pi c} u_\perp{}^2 \delta(\omega - \omega_G), \tag{58}$$

where $u_\perp$ is the transverse proper velocity in units of c (all quantities being measured in the electron-guiding-centre frame). Two conditions must be satisfied to ensure the validity of (58). First, the electron must be sufficiently excited that the classical radiation formula is applicable. As we shall see, this will very rapidly become true once the conditions for absorption are satisfied. (In fact, according to the calculations of Chiu & Fassio-Canuto (1969) the correct quantum mechanical formula for an electron in its first excited Landau orbital agrees with (58).) Secondly, the electron's motion in its guiding centre frame must be non-relativistic. In the application to a pulsar magnetosphere, this will probably not be the case, because if cyclotron absorption is occurring at all, there will be sufficient radio photons to excite relativistic transverse motion. In this case the formulae for the absorption of small-pitch-angle synchrotron radiation should be used.

Nevertheless, assuming these two conditions, it is then straightforward to calculate the effective classical absorption-cross-section in the electron-guiding-centre frame using equation (58). (In quantum mechanical terms, this cross-section includes the correction for stimulated emission in transitions between two adjacent Landau levels.) As long as $\hbar\omega_G \ll mc^2$, we can ignore the electron recoil parallel to the field and the cross-section is given by

$$\sigma_c = (2\pi)^2 r_e c\, \delta(\omega - \omega_G). \tag{59}$$

This is also the correct quantum mechanical cross-section for stimulated absorption by an electron in its ground state Landau orbital and presumably also a reasonable approximation for electrons in low states of excitation.

We now transform into the observer frame and calculate an effective cyclotron absorption coefficient from

$$\kappa_{\mathrm{C}} = \int d\gamma N_\gamma \eta \sigma_{\mathrm{C}} \tag{60}$$

$$= (2\pi)^2 \left(\frac{r_{\mathrm{e}}c}{\omega}\right) N_\gamma \left(\frac{\omega_{\mathrm{G}}}{\omega\eta}\right). \tag{61}$$

(The last parentheses refer to the argument of the function N_γ.)

(ii) *Relativistic cyclotron emission*

The problem of relativistic cyclotron emission or small-pitch-angle synchrotron radiation has been considered by Epstein (1973). We present here a simplified treatment valid in the regime appropriate for application to a pulsar magnetosphere and in the spirit of the previous two sections. We again restrict ourselves to non-relativistic guiding centre motion. In the guiding centre frame the total energy radiated per unit frequency per period (a Lorentz invariant) is given by

$$\mathscr{E}_{\mathrm{C}}(\omega') = \int \frac{e^2}{4c} u_\perp{}^2 \omega_{\mathrm{G}}(1+\cos^2\theta')\,\delta(\omega'-\omega_{\mathrm{G}})\,d\Omega'. \tag{62}$$

Transforming into quantities measured in the observer frame and integrating over solid angle, this can be converted into a spectral power:

$$p_{\mathrm{C}}(\gamma, \omega') = \frac{e^2\omega_{\mathrm{G}}u_\perp{}^2}{c\gamma} z(1-2z+2z^2) \tag{63}$$

where

$$z = \frac{\omega'}{2\gamma\omega_{\mathrm{G}}},$$

$$z \leq 1,$$

$$u_\perp \ll 1, \quad \gamma \gg 1. \tag{64}$$

As a check, the total power radiated can be calculated from

$$p_{\mathrm{C}}(\gamma) = \int p_{\mathrm{C}}(\gamma, \omega')\,d\omega'$$

$$= \frac{2}{3}\frac{e^2}{c} u_\perp{}^2 \omega_{\mathrm{G}}{}^2 \tag{65}$$

which is readily recognized as the Lorentz invariant power radiated in the guiding centre frame.

The mean frequency radiated is given by

$$\langle\omega'\rangle = \gamma\omega_{\mathrm{G}}. \tag{66}$$

From (65) we can define an effective cooling time, τ_{C}, for the radiation of gyrational energy. Measured in the observer frame, this is

$$\tau_{\mathrm{C}} = \frac{3c\gamma}{4r_{\mathrm{e}}\omega_{\mathrm{G}}{}^2}. \tag{67}$$

(iii) *Polarization*

As mentioned above, an electron will only absorb one sense of circular polarization. (In a pulsar magnetosphere the unaffected polarization has the same helicity as the gyrating electrons.) However, if as proposed in some pulsar models (e.g. Ruderman & Sutherland 1975) electrons and positrons are present in equal numbers, both senses of circular polarization can be absorbed.

The cyclotron radiation emitted by a single electron has a circular polarization that is a straightforward function of frequency (in fact

$$C = \frac{2(\omega/\gamma\omega_G - 1)}{1 + (\omega/\gamma\omega_G - 1)^2} \tag{68}$$

relative to the sense of particle gyration). Integrating over all particle energies will almost certainly leave a net circular polarization unless again there are equal distributions of electrons and positrons.

(iv) *Collective effects*

As discussed in Section 2, the particles may well be spatially bunched. However, unless this coherence extends to the distribution of gyrational phases (and this seems very unlikely) such spatial bunching will not influence the absorption and emission processes discussed in this section.

(v) *Validity of the cross-section*

Cyclotron absorption sets in when

$$\begin{aligned} \omega_G &= \gamma\omega(1 - \beta\cos\theta) \\ &\sim \tfrac{1}{2}\gamma\theta^2\omega, \\ \gamma^{-1} &\ll \theta \ll 1. \end{aligned} \tag{69}$$

For higher field strengths the transverse cross-section (47) will be approximately valid, although for detailed calculations, the full cross-section given by equation (1) should be used.

If, as seems quite possible, the gyrational motion of the particles can be made relativistic then the formulae derived above must be replaced by the corresponding formulae for the absorption and emission of small-pitch-angle synchrotron radiation (see Epstein 1973 for further details). If the strength parameter of the waves, f (a Lorentz invariant quantity) exceeds unity then the motion of the electron in its rest frame must be relativistic. Absorption can still occur, but the cross-section given by equation (59) is no longer precisely valid.

5. APPLICATION TO PULSAR MAGNETOSPHERES

We now turn to the problem that has motivated the preceding calculations—the possible influence of scattering and absorption processes on coherent radio waves propagating through a pulsar magnetosphere. The structure and composition of the magnetosphere is still to a large extent unresolved (see Ruderman 1972, and ter Haar 1972 for reviews). Therefore, we illustrate the various effects by using parameters appropriate to the model of Ruderman & Sutherland (1975). However, whenever feasible, we also indicate how detailed observations of pulsars can be used as a more general diagonstic to set constraints on the properties of a magnetosphere.

(i) *Standard pulsar model*

For convenience, we state here a set of standard assumptions (generally consistent with those used by Ruderman & Sutherland 1975) about the magnetospheric structure.

The pulsar is taken to be a neutron star of radius 10^6 cm rotating with a period P s and possessing an oblique dipole moment for which the surface magnetic field strength is 10^{12} gauss. The radio emission is assumed to be emitted by relativistic electrons (and possibly positrons) of energy γmc^2, moving along curved field lines and radiating with frequency,

$$\omega \sim \gamma^3 c/\rho \tag{70}$$

where ρ is the radius of curvature of the field lines. If we are neither too close to the star (where higher multipole moments may be important), nor too close to the light cylinder (where rotational distortion becomes significant), the field structure should be approximately dipolar. At a perpendicular distance, x, from the magnetic axis and a radius R, ρ is given by $(4/3)(R^2/x)$, for $cP/2\pi \gg R \gg x$. The radio photons will be emitted tangentially (to within an angle $\sim\gamma^{-1}$) to the field lines at (x_0, R_0) and if they are subsequently scattered at a point (x, R), they will make a small angle, θ, with the local direction of magnetic field given by

$$\theta = \frac{3}{2}\left(\frac{x}{R}-\frac{x_0}{R_0}\right) = \frac{1}{2}\frac{x}{R}\frac{(R-R_0)}{(R-R_0/3)}$$

$$\simeq \frac{3}{4}\frac{x_0}{R_0} \simeq \frac{1}{2}\frac{x}{R}, \quad R \gg R_0. \tag{71}$$

For radii beyond that at which significant pair creation can take place, the particle density (electrons and positrons), n_e, on the open field lines is related to the total particle discharge per magnetic pole, $\dot{N}$, by

$$n_e \sim 5\times 10^{16}\, \dot{N}_{36} P R_6^{-3}\ \mathrm{cm}^{-3} \tag{72}$$

where $\dot{N}_{36}$ is in units of $10^{36}\ \mathrm{s}^{-1}$ etc. The corresponding energy flux is

$$L_e = 10^{33}\, \gamma_3 \dot{N}_{36}\ \mathrm{erg\ s}^{-1}. \tag{73}$$

Thus the inequalities,

$$L_{R30} \lesssim 10^3\, \gamma_3 \dot{N}_{36} \lesssim 10^3\, I_{45} P^{-2} \tau_6^{-1}, \tag{74}$$

where $10^{36}\, L_{R30}$ erg s^{-1} is the integrated radio luminosity of the pulsar, $10^{45}\, I_{45}$ g cm^2 is the moment of inertia and $10^6\, \tau_6$ yr the timing age $(P/\dot{P})$. For the Crab pulsar, L_{R30} can be replaced by L_{X30} ($\sim 10^6$), the X-ray pulsed luminosity. An additional constraint on the density is provided by the requirement that the radio frequency exceeds the effective plasma frequency,

$$\omega_9 \gtrsim 0{\cdot}1\, \dot{N}_{36}^{1/2} R_6^{-3/2} P^{1/2} \gamma_3^{-3/2}. \tag{75}$$

The plasma frequency decreases with radius ($\propto R^{-3/2}$) and so, at a given frequency, plasma effects will diminish in importance with radius.

(ii) *Spontaneous scattering of radio photons*

So far, only the Crab pulsar has been definitely observed at frequencies outside the radio range. This might therefore be used to set limits on the particle densities and energies from the spontaneous cross-sections derived above. De-dispersed pulse profiles typically occupy $\lesssim 6$ per cent of the pulse period. Using (71) we can estimate the maximum value of the angle θ, i.e. θ_m, to be $\sim 0{\cdot}1$. From equation (57), the longitudinal cross-section will be appropriate as long as

$$\omega_9 \lesssim 10^8 \, R_6{}^{-3} \theta_{m-1}{}^{-3} \gamma_3{}^2, \tag{76}$$

with

$$\theta_m = 0{\cdot}1 \, \theta_{m-1}.$$

When

$$10^8 \, R_6{}^{-3} \theta_{m-1}{}^{-3} \gamma_3{}^{-2} \lesssim \omega_9 \lesssim 4 \times 10^9 \, R_6{}^{-3} \theta_{m-1}{}^{-2} \gamma_3{}^{-1}, \tag{77}$$

the transverse cross-section will dominate (see equation (69)). When the second inequality becomes an equality, cyclotron resonance can occur. The mean scattered frequency in these two regimes is given approximately by

$$\langle \omega' \rangle_{12} \sim 3\omega_9 \gamma_3{}^2 \theta_{m-1}{}^2. \tag{78}$$

Thus in order to scatter a radio photon ($\omega_9 \sim 1$) directly into the optical range would require $\gamma_3 \gtrsim 50$. This is much larger than the particle energy expected from (70) to give curvature radio radiation.

If we assume that the spectral intensity $I(\mathbf{k})$ is uniform for angles $\theta \lesssim \theta_m$ the total spontaneously scattered power per magnetic pole is given by

$$\begin{aligned} \mathscr{L}_s &\sim \tfrac{4}{3}\pi r_e{}^2 \theta_m{}^2 n_e R \mathscr{L}_R \\ &\sim 2 \times 10^{26} \, \dot{N}_{36} P R_6{}^{-2} \theta_{m-1}{}^2 \mathscr{L}_{R30} \text{ erg s}^{-1} \end{aligned} \tag{79}$$

for linear scattering, using (17), and

$$\begin{aligned} \mathscr{L}_s &\sim \frac{\pi}{30} r_e{}^2 \frac{\omega_0{}^2}{\omega_G{}^2} \gamma^4 \theta_m{}^8 n_e R \mathscr{L}_R \\ &\sim 10^{10} \, \dot{N}_{36} P R_6{}^4 \gamma_3{}^4 \theta_{m-1}{}^8 \omega_9{}^2 \mathscr{L}_{R30} \text{ erg s}^{-1} \end{aligned} \tag{80}$$

for transverse scattering, using (53). In equation (79), R is to be interpreted as the smallest radius at which linear scattering can occur, presumably the radius of radio emission. For transverse scattering, R is the largest scattering radius, i.e. just less than the radius at which cyclotron absorption can occur. Setting R_6 equal to 1 in (79) and using the upper limit derived from (77) in (80), we see that any spontaneously scattered power by any pulsar (including the Crab) will be too faint to be detected by several orders of magnitude. (This conclusion has been reached independently by M. Elitzur, private communication, and Sturrock, Petrosian & Turk 1975.) The lowest upper limit on pulsar optical luminosity is $\mathscr{L}_0 \lesssim 10^{30}$ erg s^{-1} set by Chiu, Lynds & Maran (1970) for the Vela pulsar and this is four orders of magnitude greater than the maximum power that could be spontaneously scattered directly from radio to optical frequency. Thus the upper limits on the particle densities and energies implied by (79) and (80) are not particularly useful.

In the case of the Crab pulsar, optical pulsation with a luminosity $\sim 10^{33}$ erg s^{-3} is observed. An additional argument can be given against this arising from spontaneous longitudinal scattering, in that the optical pulses are observed to be significantly linearly polarized (Ferguson, Cocke & Gehrels 1974) and, whereas radio polarization could well be imposed after emission, the observed optical polarization almost certainly reflects that actually emitted. From the discussion of Section 2, we see that the predicted longitudinal scattering polarization is very small.

(iii) *Induced scattering effects for* $\omega' \sim \gamma^2\omega$

There are two potentially observable consequences of induced scatterings. First, they can significantly enhance the spontaneous scattering rate to higher frequencies as discussed above and, secondly, they can cause spectral and polarization changes within the radio spectrum. Here we discuss the former possibility.

If there is sufficient optical depth to raise the brightness temperature of the spontaneously scattered high frequency radiation above $\gamma mc^2/k_B$, then induced processes must be included. As we have shown in Section 3, these are probably of little consequence for transverse scatterings. However, from the discussion of Section 2, we see that the longitudinal scattering optical depth can be negative when there is a particle population inversion in energy (as there probably will be if the acceleration is electrostatic or if most of the particles are produced from the pair conversion of gamma rays).

As an adequate approximation, we can then write the condition that radio photons of brightness temperature $T_R = 10^{25}\, T_{R25}$ K be (longitudinally) scattered into photons of temperature $\gamma mc^2/k_B$ and frequency ω' as

$$\omega'^3 \left(\frac{\gamma mc^2}{k_B}\right) \lesssim \tfrac{4}{3}\pi r_e{}^2 \theta_m{}^2 \omega^3 T_R n_e R,$$

or

$$T_{R25} \gtrsim 10\, \gamma_3{}^7 \theta_{m-1}{}^4 \dot{N}_{36}{}^{-1} P^{-1} R_6{}^2 \tag{81}$$

where we have used (78) and (79).

If the inequality (80) is satisfied by more than an order of magnitude, the scattered flux would be able to exponentiate and we might expect that a geometry-dependent but significant fraction of the radio photons would be scattered to frequencies given by (78) with the scattered power exceeding the radio power by a factor $\sim\gamma^2\theta^2$. As shown in Section 2, these arguments are not seriously affected if the particles are bunched.

If radio photons are generated at $R_6 \gtrsim 10^2$ by electrons and positrons with $\gamma_3 \sim 0{\cdot}8$, $\dot{N}_{36} \sim 10^{-3}\, P^{-2}$ as proposed by Ruderman & Sutherland (1975), then it is unlikely that condition (74) could be satisfied except possibly in the 'giant' pulses from the Crab pulsar. However, if the radio emission is generated closer to the neutron star surface and $\gamma_3 \sim 1$, then pulsars might be expected to emit strong pulses at wavelengths $\lesssim 1$ mm, which could be observable.

This once-scattered radiation could then be rescattered (almost certainly spontaneously), with either the longitudinal or the transverse cross-section dominating. In the former case, the twice-scattered photons would be almost unpolarized whereas, in the latter, significant (propably linear) polarization might be anticipated. This can in fact form the basis of a model to explain the optical pulses from the Crab. It turns out to be just possible to account for the optical luminosity if there are as many millimetre or far-infrared photons as radio photons. However, in the absence of positive observation of intense millimetre pulsation, we do not pursue this model at this stage.

(iv) *Induced scattering effects for* $\omega' \sim \omega$

We now consider induced scatterings involving radio photons. For reasons given in Section 2, we ignore the effects of particle bunching, although this is at best a poor approximation. In particular we note that, if the radio photons arise from some spontaneous process involving bunches (e.g. curvature radiation), then

$$k_B T_R \lesssim \gamma N_b mc^2 \tag{82}$$

and so the spontaneous scattering effects from scattering into nearby radio frequencies will exceed the induced scattering effects. Therefore the following estimates, based on induced scattering, are probably underestimates.

From equation (42), we can write the conditions that induced scatterings be able to cause significant distortion of the electromagnetic spectrum as

$$96\pi \left(\frac{k_B T_R}{mc^2}\right) \frac{n_e r_e^2 \rho}{\gamma^7 \theta_{\min}^3} \gtrsim 1. \tag{83}$$

If we substitute equation (72) for n_e we obtain the condition

$$P T_{R25} R_6^{-3} \dot{N}_{36} \rho_6 \gamma_3^{-7} \theta_{\min}^{-3} \gtrsim 10^3. \tag{84}$$

Constraints on $\dot{N}_{36}$ are imposed by equations (85), (86). In addition, if radio emission is curvature radiation, then $\theta_{\min} \sim \gamma^{-1}$. The observations of extreme pulse to pulse variations suggest that the photon and particle densities are highly non-uniform within the emission region and so averaged values of the quantities are not really appropriate. Nevertheless, from the theory of Ruderman & Sutherland (1975), we expect

$$\begin{aligned} \dot{N}_{36} &\sim 10^{-3} P^{-2}, \\ R_6 &\sim 200, \\ \rho &\sim \tfrac{3}{2}\gamma^3 c/\omega, \\ \gamma &\sim \theta_{\min}^{-1} \sim 800; \end{aligned} \tag{85}$$

condition (84) can then be expressed as

$$T_{R25} \gtrsim 100\, P\omega_9. \tag{86}$$

Unfortunately, little reliance can be put on equation (86) and a more precise estimate will be sensitive to the geometry of the magnetosphere. However, we have demonstrated that it is plausible for some of the characteristic sub-pulse variations observed in pulsars to be attributed to induced scattering.

One such effect has been described by Manchester, Taylor & Huguenin (1975). It has been found that, in the majority of pulsars so far studied, there are certain sub-pulse components that appear in either of two characteristic modes, that are similar except that the position angles of the linear polarization are orthogonal.

It is possible to explain these observations by means of a model in which the radio photons are emitted as curvature radiation by bunches of charged particles moving on curved field lines. These photons can then be scattered by particles on adjacent field lines. For frequencies $\omega \lesssim \gamma^3 c/\rho$, the integrated degree of linear polarization from curvature radiation is ~ 0.5 with position angle parallel to the projected curvature of the field lines. As the majority of the scatterings will occur before the field lines have curved through a significant angle, this polarization is effectively the A polarization for longitudinal scattering. Thus, when the effective optical depth to scatterings by particles on adjacent field lines is large enough to removed most of the curvature photons in the A polarization from the observed frequency interval, then only the orthogonal B polarization photons that cannot be scattered will remain. When the effective optical depth is $\ll 1$ (including when it is negative) the sub-pulse will be observed in the A polarization.

(v) *Cyclotron absorption of radio emission*

If we assume that the electrons are mono-energetic and uniformly distributed on the open field lines, then we can calculate the optical depth to cyclotron absorption using equations (62) and (72). We find that

$$\tau_c \sim 10^4 \dot{N}_{36} P \omega_9^{-1/3} \theta_{-1}^{4/3} \gamma_3^{-1/3} \tag{87}$$

and that absorption occurs at a radius

$$R_{c6} \sim 2 \times 10^3 \, \omega_9^{-1/3} \theta_{-1}^{-2/3} \gamma_3^{-1/3} \tag{88}$$

as long as this radius is well within the speed of light cylinder (i.e. $R_{c6} \ll 1{\cdot}5 \times 10^4 P$). We can now impose the condition that the absorbing particles are sufficiently numerous to have produced the radio luminosity. Substituting (74) gives the inequality,

$$\tau_c \gtrsim 10 \, L_{R30} P \omega_9^{-1/3} \theta_{-1}^{4/3} \gamma_3^{-4/3}. \tag{89}$$

If only electrons are streaming out along the open field lines then this is the optical depth to one sense of circular polarization. However, if there are equal number densities of electrons and positrons then both senses will be absorbed. Now in general the integrated pulses of pulsars are not observed to be substantially circularly polarized and so in the context of a 'standard' pulsar model, equations (88) and (89) impose important constraints. For the model of Ruderman & Sutherland (1975) in which the positron and electron densities are practically identical, we can substitute equations (85) with $\theta \sim 0{\cdot}1$, $\omega_9 \sim 1$, to obtain

$$\tau_c \sim 10 \, P^{-1} \tag{90}$$

at a radius $\sim 2 \times 10^9$ cm which is ~ 10 times the emission radius. So with these assumptions we conclude that the model is not self-consistent.

One way of avoiding this problem is for the absorbing particles to be non-uniformly distributed on the open field lines, so that a sufficient number of ray paths have $\tau_c \ll 1$. If vacuum breakdown only occurs on selected field lines within the gap, perhaps reflecting permanent surface inhomogeneities, the fact that both the primary and secondary gamma rays can convert within a distance $\sim 10^4$ cm suggests that this inhomogeneity might be preserved in the outer magnetosphere. This raises the interesting possibility that the observed sub-pulse structure is generated not so much by the emitting bunches but by an absorbing screen. (In this interpretation, the rotation of the screen rather than the emitting bunches gives rise to marching sub-pulses.) In general, however, we regard the absence of net circular polarization as being evidence against emission models in which only electrons stream out along the open field lines. (This argument does not of course apply to models where the emission arises close to the light cylinder.)

As discussed in Section 4, the radio photons will probably not be capable of exciting relativistic transverse wave-induced gyrations unless $f \gtrsim 1$. However, even if this should occur before the absorption radius, R_c, is reached then the opacity given by equation (62) is still approximately valid as long as γmc^2 signifies the total particle energy.

The time scale for radiative de-excitation for non-relativistic transverse gyration is given by equation (68) as

$$\tau_D \sim 3 \times 10^{-13} \, R_6^{\,6} \gamma_3 \text{ s}. \tag{91}$$

Thus, for $R_6 \gtrsim 10^2$ (and this is only weakly dependent on γ and P) an electron will not have sufficient time to decay to its ground state before leaving the light cylinder. In fact the fraction α of the total particle energy radiated away is given by

$$\alpha \sim 7 \times 10^7 u_\perp^2 \gamma_3^{-1} R_6^{*-5}, \quad \lesssim 1 \tag{92}$$

at a frequency

$$\omega_9 \sim 2 \times 10^{11} \gamma_3 R_6^{*-3} \tag{93}$$

where R^* is the radius of excitation. If the only source of excitation is the absorption of outgoing radio photons at a radius given by equation (88), then for $\omega_9 \sim \theta_{-1} \sim \gamma_3 \sim 1$ and $u_\perp \lesssim 1$,

$$\alpha \lesssim 2 \times 10^{-9} \tag{94}$$

and so the total luminosity (αL_e) of resonantly scattered radio photons is negligible.

If the transverse gyrations can become relativistic, then small-pitch-angle synchrotron radiation will be emitted. This has been suggested by, e.g. Shklovsky (1970) as the origin of the optical pulsation from NP0532. Some criticisms of this idea are contained in Epstein & Petrosian (1973) and Sturrock *et al.* (1975).

In addition we note that although small-pitch-angle synchrotron radiation can be linearly polarized, the position angle is normal to the projected field direction. As the angular scale ξ over which the emission is beamed is considerably in excess of the postulated pitch angle, ψ (determined by the requirement of self-absorption for frequencies less than 7×10^{14} Hz) it can be concluded that the observed degree of linear polarization $\lesssim (\psi/\xi)^2$, by using arguments similar to those used in Section 2. (There must be equal densities of electrons and positrons in order to prevent the radiation of a large degree of circular polarization.) It is therefore difficult to explain the observation of substantially linearly polarized optical pulses (Ferguson *et al.* 1974) with this mechanism.

We conclude that none of the incoherent processes considered in this paper seem to be capable of forming the basis of a satisfactory model for the optical pulses from NP 0532. The most likely current explanation involves coherent curvature radiation. The most striking differences between the radio and optical pulses, i.e. the absence of extreme pulse to pulse variation, can then be attributed to differences in the number of emitting bunches observed at any one time, as is also concluded by Sturrock *et al.* (1975).

(vi) *Resonant scattering of thermal radiation*

Finally, we consider the scattering of thermal radiation from the neutron star assumed to be a black body of temperature, $T = 10^6 T_6$ K, by ultra-relativistic electrons (or positrons) streaming out along the open field lines. This problem has also been considered by Tademaru & Greenstein (1974) who use the Thomson and Klein-Nishina scattering cross-sections. However, the effective cross-section given by equation (61) is larger than these by a factor $\gtrsim (c\omega_G^{-1} r_e^{-1})$ for photons at the resonant frequency. Near the surface of the neutron star, this frequency will be lower than the peak frequency of the blackbody spectrum, as long as the particle energy is $\gtrsim (\hbar\omega_G/k_B T)\, mc^2$.

We again consider a uniform distribution of monoenergetic electrons of energy γmc^2 streaming outwards along the open field lines with density and energy flux given by equations (72) and (73). A photon of given frequency $\omega = 10^{15} \omega_{15}$ rad s^{-1} and angle θ will only be liable to be absorbed at one field strength, i.e. at one radius.

The optical depth to absorption for a photon that intersects the open field lines at this radius is given by

$$\tau = \int n_e \sigma_c \eta \sec\theta \, dR. \tag{95}$$

This can be straightforwardly calculated as

$$\tau = 2L_{e\,30}\gamma_3^{-4/3} P B_{12}^{-2/3} \omega_{15}^{-1/3} \eta^{2/3} \sec\theta \tag{96}$$

using the cross-section of Chiu & Fassio-Canuto (1969) (identical to equation (60) for excitation from the ground state Landau orbital) in a field of strength $B = 10^{12} B_{12}$ gauss.

If $B_{12} \sim 1$, the radiative lifetime as given by equation (68) is considerably shorter than R/c (although much longer than ω_G^{-1}) and so the mean emitted frequency is $\gamma\omega_G$, evaluated at the radius of excitation.

At a radius $10^6 R_6$ cm the neutron star subtends a semi-angle $\sin^{-1}(R_6^{-1})$ and so, provided that $\theta \leq \sin^{-1}(R_6^{-1})$ and the field lines can still be considered as radial, the number of photons crossing unit area per second in one sense of circular polarization is given by

$$dF = \frac{\omega^2 \sin\theta \cos\theta \, d\omega \, d\theta}{(2\pi c)^2 \{\exp(\hbar\omega/k_B T) - 1\}}. \tag{97}$$

The total power scattered can then be calculated from

$$P_S = \int \gamma \hbar \omega_G (1 - e^{-\tau}) A(R) \, dF \tag{98}$$

where $A(R)$ is the cross-sectional area of the open field lines at radius R. As R is a unique function of ω and θ, P can be regarded as a double integral over ω, θ in the range within which resonant scattering can occur; i.e.

$$\theta \lesssim \left(\frac{2\omega_{G0}}{\gamma\omega}\right)^{1/2}; \quad \omega \gg \omega_{G0}/\gamma, \tag{99}$$

$$\theta \lesssim \left(\frac{\gamma\omega}{2\omega_{G0}}\right)^{1/2}; \quad \omega \ll \omega_{G0}/\gamma \tag{100}$$

where ω_{G0} is the gyro-frequency at the neutron star surface. For photons with frequency at the peak of the blackbody spectrum, $\omega \sim k_B T/\hbar$ and so the former condition is appropriate for $T_6 \gg 0{\cdot}1\, \gamma_3^{-1} B_{12}$. If we assume that condition (99) is satisfied, the main contribution to equation (98) comes from absorption close to the stellar surface and our earlier assumptions are vindicated.

The condition that the optical depth at the peak frequency, $\tau^* = \tau(k_B T/\hbar)$, satisfy $\tau^* \gg 1$ at the neutron star surface can be expressed

$$L_{e\,30} \gg 10\gamma_3^2 T_6 P^{-1}, \tag{101}$$

independent of ω_{G0}. If this is also satisfied, then equation (98) becomes approximately,

$$P_S \sim \frac{\hbar \omega_{G0}^2 A_0}{(2\pi c)^2} \int \frac{\omega \, d\omega}{\{\exp(\hbar\omega/k_B T) - 1\}} \tag{102}$$

where A_0 is the cross-sectional area of the open field lines at the neutron star surface. Thus,

$$P_S \sim 10^{32} B_{12}^2 T_6^2 P^{-1} \text{ erg s}^{-1} \tag{103}$$

independent of γ.

In the alternative limit, $\tau^* \ll 1$,

$$P_S \sim 3 \times 10^{30} B_{12}^2 L_{e\,30} \gamma_3^{-2} \int \frac{d(\hbar\omega/k_B T)}{\{\exp(\hbar\omega/k_B T) - 1\}} \text{ erg s}^{-1} \tag{104}$$

using equation (96). The lower limit to the integral is given approximately by the maximum of τ^{*-1} and $(h\omega_{GO}/\gamma k_B T)$, dependent upon whether the open field lines are optically thick or thin at the lowest frequency that can be resonantly scattered at the surface. In either case, for anticipated pulsar parameters,

$$P_S \sim 10^{31} B_{12}^2 L_{e\,30} T_6 \gamma_3^{-2} \text{ erg s}^{-1}. \tag{105}$$

(If there are equal densities of electrons and positrons, the powers given by equations (103), (105) should be doubled.) For comparison, the power radiated thermally by the neutron star $\sim 6 \times 10^{32} T_6^4$ erg s^{-1}.

Most of this power is scattered at a photon energy $\sim \gamma h\omega_{GO} \sim 10\gamma_3 B_{12}$ MeV. Using the method outlined in Ruderman & Sutherland (1975), the maximum energy that a photon can have and still escape from the surface of a neutron star without creating an electron–positron pair in the static magnetic field is

$$\sim 300 P^{-1/2} B_{12}^{-1} \text{ MeV}.$$

Thus,

$$\gamma_3 \lesssim 30 B_{12}^{-2} P^{-1/2} \tag{106}$$

is a necessary condition for this power to be observable.

In the case of the Crab pulsar, equation (103) is probably applicable. If we assume a beaming factor ~ 0.1, the pulsed power radiated into the 10–100 MeV range is reported to be $\sim 10^{35}$ erg s^{-1} (Albats *et al.* 1972). If this is treated as an upper limit to the scattered flux, equation (103) gives the condition

$$T_6 \lesssim 6 B_{12}^{-1}$$

as long as

$$\gamma_3 \lesssim 6 B_{12}^{-2}. \tag{107}$$

In the absence of an accelerating electric field, equations (103), (105) can be used to give the condition that the scatterings be dynamically unimportant for the relativistic electrons.

$$L_{e\,30} \gtrsim 100 B_{12}^2 T_6^2 P^{-1}$$

$$\gamma_3^2 \gtrsim 10 B_{12}^2 T_6, \tag{108}$$

respectively in the two regimes. These limits are somewhat more stringent than those derived by Tademaru & Greenstein (1974), but still do not place any serious constraints on existing models of pulsars.

ACKNOWLEDGMENTS

We thank M. Elitzur, P. Goldreich, J. Katz, M. Rees and M. Ruderman for helpful discussions and correspondence. RB thanks the Institute for Advanced Study, Princeton, for hospitality and support (under National Science Foundation, Grant No. NSF GP-40768X) during the completion of this work.

REFERENCES

Albats, P., Frye, G. M., Zych, A. D., Mace, O. B., Hopper, V. D. & Thomas, J. A., 1972. *Nature*, **240**, 221.
Bjorken, J. D. & Drell, S. D., 1964. *Quantum mechanics*, McGraw-Hill, New York.
Blandford, R. D., 1975. *Mon. Not. R. astr. Soc.*, **170**, 551.
Blandford, R. D. & Scharlemann, E. T., 1975. *Astrophys. Space Sci.*, in press.
Blumenthal, G. R. & Gould, R. J., 1970. *Rev. mod. Phys.*, **42**, 237.
Bunkin, F. B., Kazakov, A. E. & Federov, M. V., 1973. *Sov. Phys.-Uspeckhi*, **15**, 416.
Canuto, V., Lodenquai, J. & Ruderman, M., 1971. *Phys. Rev.*, **D3**, 2303.
Chiu, H.-Y. & Fassio-Canuto, L., 1969. *Phys. Rev.*, **185**, 1614.
Chiu, H.-Y., Lynds, R. & Maran, S. P., 1970. *Astrophys. J. Lett.*, **162**, L99.
De Raad, Jr., L. L., Hari Dass, N. D. & Milton, K. A., 1974. *Phys. Rev.*, **D9**, 1041.
Epstein, R. I., 1973. *Astrophys. J.*, **183**, 593.
Epstein, R. I. & Petrosian, V., 1973. *Astrophys. J.*, **183**, 611.
Ferguson, D. C., Cocke, W. J. & Gehrels, T., 1974. *Astrophys. J.*, **190**, 375.
Goldreich, P., Pacini, F. & Rees, M. J., 1972. *Comments Astrophys. Space Sci.*, **4**, 23.
Kaplan, S. A. & Tsytovich, V. N., 1973. *Plasma astrophysics*, Pergamon Press Ltd, Oxford.
Manchester, R. N., Taylor, J. H. & Huguenin, G. R., 1975. *Astrophys. J.*, **196**, 83.
Ruderman, M., 1972. *A. Rev. Astr. Astrophys.*, **10**, 427.
Ruderman, M. & Sutherland, P. G., 1975. *Astrophys. J.*, **196**, 51.
Shklovsky, I. S., 1970. *Astrophys. J. Lett.*, **159**, L77.
Sturrock, P. A., Petrosian, V. & Turk, J. S., 1975. *Astrophys. J.*, **196**, 73.
Tademaru, E. & Greenstein, G., 1974. *Nature*, **251**, 39.
ter Haar, D., 1972. *Phys. Reports*, **3C**, 57.
Weymann, R., 1965. *Phys. Fluids*, **8**, 2112.
Zhukovskii, V. Ch. & Nikitina, N. S., 1973. *Sov. Phys.-JETP*, **37**, 595.

APPENDIX A

We outline here a quantum electrodynamical calculation of the cross-section for longitudinal scattering by an electron in a strong magnetic field that is correct to $O(\hbar\omega/mc^2)$; i.e. includes the effects of electron recoil. We work in parallel to the calculation of the Klein-Nishina cross-section as presented by Bjorken & Drell (1964) adopting their notation. More sophisticated calculation of associated problems are contained in De Raad, Dass & Milton (1974) and Zhukovskii & Nikitina (1974).

We start from a set of basic states for an electron with momentum p parallel to a static magnetic field, $\mathbf{H}_{\text{ext.}}$ Setting $c = \hbar = 1$, these are given by

$$u^1 = \frac{1}{[2m(\omega_n{}^1+m)]^{1/2}} \begin{bmatrix} (\omega_n{}^1+m)\,\psi_n \\ 0 \\ p\psi_n \\ 2[h(n+1)]^{1/2}\,\psi_{n+1} \end{bmatrix} e^{ipz}$$

$$u^2 = \frac{1}{[2m(\omega_n{}^2+m)]^{1/2}} \begin{bmatrix} 0 \\ (\omega_n{}^2+m)\,\psi_n \\ 2[hn]^{1/2}\,\psi_{n-1} \\ -p\psi_n \end{bmatrix} e^{ipz}$$

$$u^3 \equiv v^1 = \frac{1}{[2m(\omega_n{}^1+m)]^{1/2}} \begin{bmatrix} p\psi_n \\ -2[hn]^{1/2}\,\psi_{n-1} \\ (\omega_n{}^1+m)\,\psi_n \\ 0 \end{bmatrix} e^{-ipz}$$

$$u^4 \equiv v^2 = \frac{1}{[2m(\omega_n{}^2+m)]^{1/2}} \begin{bmatrix} -2[hn]^{1/2}\,\psi_{n-1} \\ -p\psi_n \\ 0 \\ (\omega_n{}^2+m)\,\psi_n \end{bmatrix} e^{-ipz}$$

with Lorentz invariant normalization

$$\int \bar{u}_n{}^r(x)\, u_n{}^s(x)\, d^3x = \delta rs.$$

Here

$$\omega_n{}^1 = [m^2+p^2+4h(n+1)]^{1/2}$$
$$\omega_n{}^2 = [m^2+p^2+4hn]^{1/2}$$
$$h = \frac{e|\mathbf{H}_{\mathrm{ext}}|}{2\hbar c}$$

and

$$\psi_n(x,y) = \left(\frac{i}{\sqrt{2}}\right)^n [\sqrt{\pi}n!]^{-1/2}\,(2h)^{1/4} \exp[ihy(x-2a)] \times \exp[-h(x-a)^2]\, H_n[\sqrt{2h}(x-a)].$$

$H_n(\xi)$ is a Hermite polynomial. We have used states that are eigenfunctions of x (with eigenvalue a) as well as of energy, and momentum along the field, in order to permit the scattering electron to shift position across the field.

In terms of these basis states, the electron propagator in a magnetic field can be written

$$S_F(x',x) = -i\theta(t'-t) \int dp_I \int d(ha_I) \sum_n \sum_{r=1}^{2} \frac{m}{2\pi^2\omega_n{}^r}\, u_n{}^r(\mathbf{x}')\, \bar{u}_n{}^r(\mathbf{x}) \times \exp[-i\omega_n{}^r(t'-t)] + i\theta(t-t') \int dp_I \int d(ha_I) \times \sum_n \sum_{r=1}^{2} \frac{m}{2\pi^2\omega_n{}^r}\, v_n{}^r(\mathbf{x}')\, \bar{v}_n{}^r(\mathbf{x}) \exp[i\omega_n{}^r(t'-t)];$$

where p_I is the parallel momentum in the intermediate state. This equation is analogous to equation (6.48) of Bjorken & Drell. The important mathematical property of this propagator is that

$$(i\not{\nabla} + e\not{A} - m)\, S_F(x',x) = \delta^4(x'-x)$$

where $\not{\nabla} = \gamma^\mu(\partial/\partial x^\mu)$, etc.

The matrix element for photon scattering can be written

$$S_{fi} = \delta_{fi} - ie^2 \int d^4x'\, d^4x \bar{u}^f(\mathbf{x}') \exp[-i\omega_n{}^f t']\, \not{A}(k'^\mu, x'^\mu)\, S_F(x',x) \times \not{A}(k^\lambda, x^\lambda)\, u^i(\mathbf{x}) \exp[-i\omega_n{}^i t] - ie^2 \int d^4x \bar{u}^f(\mathbf{x}') \exp[-i\omega_n{}^f t'] \times \not{A}(k^\mu, x'^\mu)\, S_F(x',x)\, \not{A}(k'^\lambda, x^\lambda)\, u^i(\mathbf{x}) \exp(-i\omega_n{}^i t].$$

We retain only terms O(1) in an expansion in (ω/ω_G) to obtain

$$S_{fi} = \delta_{fi} + \frac{ir_e \sin\theta \sin\theta'}{4m[(1+\gamma)(1+\gamma')]^{1/2}}$$

$$\times \sum_{\alpha=1}^{2} \left\{ \frac{\left[p'p_I^\alpha(1+\gamma) + p'p\left(1+\frac{\omega_I^\alpha}{m}\right) + \frac{(1+\gamma)(1+\gamma')}{(1+\omega_I^\alpha/m)} p_I^{\alpha 2} + pp_I^\alpha(1+\gamma') \right]}{(m\gamma' - \omega_I^\alpha + k')\omega_I^\alpha} \right.$$

$$\left. + \frac{\left[p'p_I^\alpha(1+\gamma) + \frac{p'pp_I^{\alpha 2}}{m(m+\omega_I^\alpha)} + m(1+\gamma')(1+\gamma)(m+\omega_I^\alpha) + pp_I(1+\gamma') \right]}{(m\gamma' + \omega_I^\alpha + k')\omega_I^\alpha} \right\},$$

where

$$p_I^1 = p + k\cos\theta$$

$$p_I^2 = p - k'\cos\theta'$$

$$\omega_I^\alpha = \omega_0^2(p_I) = (m^2 + p_I^{\alpha 2})^{1/2},$$

for scattering of a photon of momentum $\mathbf{k}$ by an electron of energy γm into final states $\mathbf{k}'$ and $\gamma' m$ respectively.

Primed and unprimed quantities refer to final and initial states respectively.

Then in the initial electron rest frame ($p = 0$) and to $O(\hbar\omega/mc^2)$, we find

$$S_{fi} = \delta_{fi} + ir_e \sin\theta \sin\theta' \left\{ 1 + \frac{hk}{2mc} (\cos^2\theta - \cos^2\theta') \right\}.$$

The scattering cross-section is obtained by incorporating the phase-space factors (which involves putting the electron in a one-dimensional phase space) yielding

$$\frac{d\sigma}{d\Omega'} = r_e^2 \sin^2\theta \sin^2\theta' \left\{ 1 + \frac{\hbar\omega}{2mc^2} (\cos^2\theta - 5\cos^2\theta' + 4\cos\theta\cos\theta') \right\}.$$