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ON THE SEARCH FOR SINGULARITIES IN INCOMPRESSIBLE FLOWS

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Abstract. In these notes we give some examples of the interaction of mathematics with experiments and numerical simulations on the search for singularities.

Keywords: Navier-Stokes equations, singularities, incompressible flows

MSC 2000: 76D03, 35Q35, 74H35

1. Introduction

The search for singularities in incompressible flows has become a major challenge in the area of non-linear partial differential equations and is relevant in applied mathematics, physics and engineering. The existence of such singularities would have important consequences for the understanding of turbulence. One way to make progress in this direction is to study plausible scenarios for the singularities supported by experiments or numerical analysis. Below we discuss two candidates, one coming from experiments and the other from numerical simulations. The experiments show that for high viscosity fluids, the break-up of a drop is preceded by the formation of long filaments, which (see [28], where experimental data were collected by using a high resolution charge-coupled device sensor) are thin uniformly up to a diameter of the order of a micron. And for low viscosity the pinching occurs at isolated points. At this small scale it is possible that molecular forces, which are not considered in a continuum description, come into play, but it is important to know whether the continuum equations predict break-up or not.

With more sophisticated numerical tools now available, the subject has recently gained considerable momentum. In Section 3 numerical simulations indicate a possible singularity on the boundary of a *patch* which is a weak solution to a family of incompressible equations.

2. Drops

A fluid jet emerging from a faucet is the simplest example where a flow develops a singularity. At a certain distance from the faucet, the jet breaks into drops. In different words, a simply connected mass of incompressible fluid $\Omega(t)$ may evolve in such a way that the domain becomes disconnected. This phenomena were studied at the beginning of the 19th century, with the observations of Savart (1833) [36] who performed experiments to estimate the size of drops resulting from the breaking-up of a jet, the work of Plateau (1863) [31] and the first analytical study done by Rayleigh (1879) [32] who showed the instability of the stationary-jet solutions to the Navier-Stokes equations, explaining, at least partially, Savart's observations. The experiments have shown that viscosity plays a fundamental role in the geometry of breakup. When the flow is highly viscous uniform thin filaments are formed and shortly after they disappear ([37], [28]).

Let us assume we have two immiscible fluids with different parameters (density ϱ and viscosity ν) that occupy all the space and are separated by a surface. Mathematically, it is a free boundary problem in which each fluid evolves according to the Navier-Stokes system

$$\varrho\Big(\frac{\partial u_i}{\partial t}+\sum_{1\leqslant j\leqslant 3} u_j\frac{\partial u_i}{\partial x_j}\Big)=-\frac{\partial p}{\partial x_i}+\nu\Delta u_i+f_i,\quad i=1,\dots 3, \tag{2.1}$$

$$\operatorname{div} u:=\sum_{1\leqslant i\leqslant 3}\frac{\partial u_i}{\partial x_i}=0$$

where the parameters ϱ and ν are the density and the viscosity of the fluid respectively and f_i denotes an external force. These equations describe the evolution of an incompressible flow in a bounded domain $\Omega(t)$ limited by a free surface $\partial\Omega(t)$, together with the boundary condition

$$\Big[-p\delta_{ij}+\nu\Big(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\Big)\Big]n_j=\sigma H n_i \quad \text{in } \partial\Omega(t), \tag{2.2}$$

where $\vec{n}$ is the field of outer normal vectors to $\Omega(t)$ and H is the mean curvature of $\partial\Omega(t)$, and with the kinematic condition for the evolution of $\partial\Omega(t)$

$$V_N=\vec{v}\cdot\vec{n} \tag{2.3}$$

expressing the fact that the particles on the boundary move with a velocity whose normal component V_N equals the normal component of the velocity field defined in it. σ denotes the surface tension coefficient of the interface which depends upon the fluid itself and the surrounding media.

The question we want to address here is whether one can deduce from the equations (Navier-Stokes equations under the action of surface tension), the possible breakup of a drop through the collapse of a fluid filament.

A filament is a thin tube that moves with the flow, and the breakup is due to the collapse of the boundary. We can assume that the filament remains smooth up to the time of the blow-up. More precisely:

Let $Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$ be a closed rectangular box (with I_j a bounded interval), and let $T > 0$ be given. A *regular thin tube* (or a filament) is an open set $\Omega_t \subset Q$ parameterized by time $t \in [0, T)$, having the form $\Omega_t = \{(x_1, x_2, x_3) \in Q\colon \theta(x_1, x_2, x_3, t) < 0\}$, with $\theta \in C^1(Q \times [0, T))$, and satisfying the following conditions

$$|\nabla_{x_1,x_2}\theta| \neq 0 \quad \text{for } (x_1, x_2, x_3, t) \in Q \times [0, T), \quad \theta(x_1, x_2, x_3, t) = 0;$$
$$\Omega_t(x_3) := \{(x_1, x_2) \in I_1 \times I_2\colon (x_1, x_2, x_3) \in \Omega_t\} \quad \text{is non-empty}$$

for all $x_3 \in I_3$, $t \in [0, T)$; the closure $(\Omega_t(x_3)) \subset$ interior $(I_1 \times I_2)$ for all $x_3 \in I_3$, $t \in [0, T)$.

For example, a thin tubular neighborhood of a curve Γ forms a regular tube, provided the tangent vector Γ' stays transverse to the (x_1, x_2) plane.

We say that a regular thin tube Ω_t *moves with the velocity field* u if we have

$$\Big(\frac{\partial}{\partial t} + u \cdot \nabla_x\Big)\theta = 0 \quad \text{whenever } (x, t) \in Q \times [0, T),\ \theta(x, t) = 0.$$

This definition appears for the case $n = 3$ in [18]. For an n-dimensional definition we have (see [19]):

Let $I_i \subset \mathbb{R}$, $i = 1, \ldots n$ be bounded intervals and let $Q = \times_i I_i \subset \mathbb{R}^n$ be a cube. A regular tube is a relatively open set $\mathcal{S} \subset Q$ characterized as $\mathcal{S} = \{x \in Q\colon f(x) < 0\}$ where $f\colon Q \to \mathbb{R}$ is a C^1 function that satisfies

$$f(x) = 0 \Longrightarrow \nabla_{x_1,\ldots,x_{n-1}} f \neq 0.$$

For every $x_n \in I_n$, the set $\mathcal{S}(x_n) = \mathcal{S} \cap I_1 \times \ldots \times I_{n-1} \times \{x_n\}$ is non-empty and its closure is contained in the interior of $I_1 \times \ldots \times I_{n-1} \times \{x_n\}$.

We will also consider the situation when f_t is a family of functions indexed by time, $t \in [0, T)$. We denote the Lebesgue measure of a set A by $|A|$ and the ball centered at $\mathbf{x}^0$ with radius r by $B_r(\mathbf{x}^0)$.

A vector field u experiences a *tube collapse* (*or filament collapse*) singularity at time T when the boundary of the tube evolve with the velocity field u and

$$\liminf_{t \to T} |\mathcal{S}_t| = 0.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. We consider a C^1 time dependent vector field $u\colon \Omega \times [0,T) \to \mathbb{R}^n$. This vector field defines an evolution for trajectories $\Phi_t(x)$, where $\Phi_t(x)$ denotes the position at time t of the trajectory with an initial condition x at time $t = 0$. The trajectory of a particle $\Phi_t(q)$ is obtained by solving

$$\frac{\mathrm{d}\Phi_t(q)}{\mathrm{d}t} = u(\Phi_t(q), t),$$
$$\Phi_0(q) = q;$$

then

$$(\Phi_t(q) - \Phi_t(p))_t \leqslant |\Phi_t(q) - \Phi_t(p)|\,|\nabla u|_{L^\infty},$$
$$|\Phi_t(q) - \Phi_t(p)| \geqslant |\Phi_0(q) - \Phi_0(p)|\mathrm{e}^{-\int_0^t |\nabla u|_{L^\infty}\,\mathrm{d}s}.$$

This simple computation leads to a criterion; in order to have two particles $(\Phi_t(q), \Phi_t(p))$ of the filament collapse at time T the following integral has to diverge:

$$\int_0^T |\nabla u|_{L^\infty}\,\mathrm{d}s = \infty. \tag{2.4}$$

More generally, we denote by $\Phi_{t,a}(x)$ the position at time t of the trajectory which at time $t = a$ is at x. Note that, when both sides of the formulas make sense, $\Phi_t(x) = \Phi_{t,0}(x)$, $\Phi_{t,a}(x) = \Phi_t \circ \Phi_a^{-1}(x)$, $\Phi_{t,a} \circ \Phi_{a,b}(x) = \Phi_{t,b}(x)$.

For $\mathcal{S} \subset \Omega$ we denote $\Phi_{t,a}^{\Omega}\mathcal{S} = \{x \in \Omega \mid x = \Phi_{t,a}(y),\, y \in \mathcal{S},\, \Phi_{s,a}(y) \in \Omega,\, a \leqslant s \leqslant t\}$. That is, $\Phi_{t,a}^{\Omega}$ is the evolution of the set $\mathcal{S}$, starting at time a, after we eliminate the trajectories which step out of Ω. Given the fact that u has zero divergence, we have that $|\Phi_{t,a}\mathcal{S}|$ is independent of t and $|\Phi_{t,a}^{\Omega}\mathcal{S}|$ is non-increasing in t.

The physical intuition is that there is a region of positive volume such that the fluid occupying it gets ejected from a slightly bigger region in finite time. In order to generalize this we give the following definition of a squirt singularity.

Definition 1. Let Ω_-, Ω_+ be open and bounded sets, $\overline{\Omega_-} \subset \Omega_+$. (Therefore, $\operatorname{dist}(\Omega_-, \mathbb{R}^d - \Omega_+ \geqslant r > 0$.)

We say that u experiences a squirt singularity in Ω_- at time $T > 0$ when for every $0 \leqslant s < T$ we can find a set $\mathcal{S}_s \subset \Omega_+$ such that

- $\mathcal{S}_s \cap \Omega_-$ has positive measure, $0 \leqslant s < T$,
- $\lim\limits_{t\to T} \left|\Phi_{t,s}^{\Omega_+}\mathcal{S}_s\right| = 0$.

Theorem 2.1. *If u as before has a squirt singularity, then*

$$\int_s^T \sup_x |u(x,t)|\,\mathrm{d}t = \infty \quad \forall\, s \in (0,T). \tag{2.5}$$

Note that in the argument for Theorem 2.1, some of the hypotheses can be somewhat weakened. For example, using the theory of [22], the hypothesis that $u \in C^1$ can be weakened to $u \in H^1$. Also note that strict volume preservation is not needed. It suffices that the volume contraction remains bounded. That is, for some constant $C \geqslant 1$ and all $M \subset \mathbb{R}^n$ measurable, $C^{-1}|M| \leqslant |\Phi_t(M)| \leqslant C|M|$.

From the assumption that $\left|\Phi_{t,s}^{\Omega_+}\mathcal{S}_s\right| \to 0$ we conclude that almost all trajectories starting in $\mathcal{S}_s$ at time s leave the set Ω_+ at a time $\tau \in (s,T)$.

Therefore we conclude that for any trajectory $x(t)$ starting in $\Omega_- \cap \mathcal{S}_s$ at time s we have

$$\left|\int_s^\tau u(\Phi_t(x),t)\,\mathrm{d}t\right| \geqslant r > 0.$$

Therefore,

$$\int_s^T \sup_x |u(x,t)|\,\mathrm{d}t \geqslant r > 0. \tag{2.6}$$

Since (2.6) holds for every $s \in (0,T)$ we conclude that (2.5) holds.

2.1. Uniform collapse

The quantity $\int_0^T \sup\limits_x |u(x,t)|\,\mathrm{d}t$ is bounded when we consider one single fluid in a fixed domain Ω with the appropriate boundary conditions (see [23], [24], [41]), but this is not the case for the free boundary problem under surface tension. Provided we assume that singularities are somewhat more uniform, it is possible to develop more quantitative information about the rate at which they happen.

For a uniform thin filament of a viscous flow we can take coordinates (r,z) where z is the vertical coordinate, and r denotes the distance to the axis of symmetry. Let us denote by $h(z,t)$ the distance of a point of the boundary of the tube to its axis.

By a collapse of a filament at time T we understand the following situation:

$$\lim_{t\to T} h(z,t) = 0 \quad \text{for every } z \in I,$$

where I is an interval that we take to be $[-L,L]$. The collapse will be uniform if

$$\frac{1}{C}\,\bar{h}(t) \leqslant h(z,t) \leqslant C\bar{h}(t) \quad \text{for every } z \in I, \tag{2.7}$$

where C is a constant and $\bar{h}(t)$ is the average of $h(z,t)$ over I. We will also assume

$$|h_z(z,t)| \leqslant C \quad \text{for every } \; z \in I \; \text{ and every } \; t. \tag{2.8}$$

In a tube collapse singularities of the previous section we say that the collapse is uniform when

$$\max |S(x_n)|_{n-1} \leqslant M \min |S(x_n)|$$

where M is a constant independent of time and $|\cdot|_{n-1}$ denotes the $n-1$ dimensional area.

Given a C^1 set S_t, we denote by $\tilde{\partial} S_t$ the portion of the boundary which is not evolving with the fluid.

We note that by the incompressibility of the fluid, the change of volume is the integral of u over $\tilde{\partial} S_t$. Hence, we always have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathcal{S}_t| \geqslant -\|u\|_{L^\infty} |\tilde{\partial}\mathcal{S}_t|_{n-1}.$$

However, the rate of collapse again depends on the quantity $\int_0^T \sup\limits_x |u(x,t)| \,\mathrm{d}t$ of which, in the case of the drop, we do not have any control.

Using a different approach, see [16] for more details, we have the following theorem:

Theorem 2.2. *Under the conditions* (2.7), (2.8) *in a given interval I, the uniform collapse of a filament (in the sense of* (2.1)*) is impossible. Moreover, the volume of fluid enclosed by the filament satisfies*

$$V(t) \geqslant C\mathrm{e}^{-Ct^4}$$

for some positive constant C.

The proof follows by estimating the evolution of the volume V of the filament. Given a mass of fluid in the interval $[-z_0, z_0]$, the variation of its volume with time is given by the equation

$$\begin{aligned}\frac{\mathrm{d}V(t;z_0)}{\mathrm{d}t} = &\int_0^{2\pi}\int_0^{h(z_0,\theta,t)} v_z(z_0,\varrho,\theta,t)\varrho \,\mathrm{d}\varrho \,\mathrm{d}\theta \\ &- \int_0^{2\pi}\int_0^{h(-z_0,\theta,t)} v_z(-z_0,\varrho,\theta,t)\varrho \,\mathrm{d}\varrho \,\mathrm{d}\theta.\end{aligned}$$

Let us define

$$\overline{V}(t) \equiv \frac{1}{L}\int_0^L V(t;z)\,\mathrm{d}z$$

and make the change of variables

$$z' = z, \quad \varrho' = \frac{\bar{h}}{h(z,\theta,t)}\varrho, \quad \theta' = \theta,$$

where $\bar{h}$ denotes the average of h over the interval $z \in [-L, L]$. Then by the hypothesis of a uniform collapse we can obtain the estimate

$$\frac{\mathrm{d}\overline{V}(t)}{\mathrm{d}t} \geqslant -C\Big(\int_{-L}^{L}\int_{0}^{2\pi} \sup_{\varrho'} \frac{|v_z(z',\theta',\varrho',t)|}{|\ln \varrho'|^{1/2}}\,\mathrm{d}\theta'\,\mathrm{d}z'\Big)\overline{V}(t)|\ln \overline{V}(t)|^{1/2}.$$

In the case of a fluid (with viscosity μ_1 and density ϱ_1) surrounded by another fluid (with viscosity μ_2 and density ϱ_2), each fluid satisfies the Navier-Stokes equations and the kinematic condition at the interface is

$$\big[T_{ij}^{(1)} - T_{ij}^{(2)}\big]n_j = \sigma H n_i \quad \text{in } \partial\Omega(t)$$

with

$$T_{ij}^{(k)} = \Big[-p\delta_{ij} + \mu_k\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big)\Big]$$

and σ being the surface tension coefficient for the interface between the two fluids. Also, continuity of the velocity field across the interface has to be assumed (see [40]).

One can deduce the energy identity

$$\int_{\Omega(t)} \frac{1}{2}\varrho_1|\vec{v}|^2\,\mathrm{d}V + \mu_1\int_0^t\int_{\Omega(t)} |\partial_{x_i}v_j + \partial_{x_j}v_i|^2\,\mathrm{d}V\,\mathrm{d}t + \sigma|\partial\Omega(t)|$$
$$+\int_{\mathbb{R}^3\setminus\Omega(t)} \frac{1}{2}\varrho_2|\vec{v}|^2\,\mathrm{d}V + \mu_2\int_0^t\int_{\mathbb{R}^3\setminus\Omega(t)} |\partial_{x_i}v_j + \partial_{x_j}v_i|^2\,\mathrm{d}V\,\mathrm{d}t = C.$$

An immediate consequence of this is the following inequality:

$$\min_i(\varrho_i)\int_{\mathbb{R}^3} \frac{1}{2}|\vec{v}|^2\,\mathrm{d}V + \min_i(\mu_i)\int_0^t\int_{\mathbb{R}^3} |\partial_{x_i}v_j + \partial_{x_j}v_i|^2\,\mathrm{d}V\,\mathrm{d}t + \sigma|\partial\Omega(t)| \leqslant C.$$

Integrating it over t leads to

$$\int_0^t\int_{\Omega(t)} \frac{1}{2}|\vec{v}|^2\,\mathrm{d}V\,\mathrm{d}t \leqslant Ct.$$

Consequently,

$$\int_0^t\int_{-L}^{L} \sup_{\varrho'} \frac{|v_z(z',\varrho',t)|}{|\ln \varrho'|^{1/2}}\,\mathrm{d}z'\,\mathrm{d}t \leqslant C(1+t)$$

and the theorem follows.

3. The 2D quasi-geostrophic equation

The 2D quasi-geostrophic equation (QG equation in short) that we will discuss below, has the following form:

$$\frac{\partial \theta}{\partial t} + \sum_{1 \leqslant j \leqslant 2} u_j \frac{\partial \theta}{\partial x_j} = 0, \tag{3.1}$$
$$u = \nabla^{\perp}\psi \quad \text{where} \quad \theta = -(-\Delta)^{1/2}\psi$$

with $x \in \mathbb{R}^2$, $t \in \mathbb{R}^+$ and $\theta = \theta(x,t)$ is a scalar function that represents the temperature, $u(x,t) = (-\partial\psi/\partial x_2, \partial\psi/\partial x_1)$ is the velocity field and ψ is the stream function. The non-local operator $\Lambda^\gamma = (-\delta)^\gamma$ is defined through the Fourier transform by $\widehat{\Lambda^\gamma f}(\xi) = |\xi|^\gamma \hat{f}(\xi)$, where $\hat{f}$ is the Fourier transform of f.

This equation has applications to meteorology and oceanography, and is a special case of the more general 3D quasi-geostrophic equation (see [25], [30] and [35]).

Due to the incompressibility of the flow, the L^p $(1 \leqslant p \leqslant \infty)$ norms of θ are conserved for all time. That implies that the energy is also conserved because the velocity can be written as

$$u = (-\partial_{x_2}\Lambda^{-1}\theta, \partial_{x_1}\Lambda^{-1}\theta) = (-R_2\theta, R_1\theta)$$

where R_j represent the Riesz transforms:

$$R_j\theta(x,t) = \frac{1}{2\pi}\text{V.P.}\int \frac{y_j \cdot \theta(x+y,t)}{|y|^3}\,\mathrm{d}y.$$

There has been high scientific interest to understand the behavior of the QG equation because it is a possible model to explain the formation of fronts of hot and cold air. In a different direction Constantin, Majda and Tabak [11] proposed this system as a 2D model for the 3D vorticity intensification and showed that there is a geometric and analytic analogy with 3D Euler equations. It is not known at this moment if this equation can produce singularities.

The vorticity is defined by $\omega = \nabla \times u$ and the 3D incompressible Euler equations can be written in terms of the vorticity ω as

$$\omega_t + u \cdot \nabla\omega = (\nabla u)\omega, \tag{3.2}$$
$$\nabla \cdot u = \nabla \cdot \omega = 0.$$

By the Biot-Savart law we recover the velocity from the vorticity by the operator

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y \times \omega(x+y,t)}{|y|^3}\,\mathrm{d}y.$$

Similarly, the QG equation (3.1) can be written as

$$(\partial_t + u\cdot\nabla)\nabla^{\perp}\theta = (\nabla u)\cdot\nabla^{\perp}\theta, \qquad (3.3)$$
$$\nabla\cdot u = 0.$$

The stream function ψ is obtained from θ through the non-local operator

$$\psi(x,t) = -\int_{\mathbb{R}^2}\frac{\theta(x+y,t)}{|y|}\,\mathrm{d}y,$$

therefore

$$u(x,t) = -\int_{\mathbb{R}^2}\frac{\nabla^{\perp}\theta(x+y,t)}{|y|}\,\mathrm{d}y.$$

The two equations (3.2) and (3.3) are similar: the vector $\nabla^{\perp}\theta = (-\theta_{x_2},\theta_{x_1})$ plays the role of the vector $\omega = \nabla\times u$. Furthermore, the operator ∇u, in the case of the QG equation, is a singular integral in dimension two with respect to $\nabla^{\perp}\theta$. In (3.2), the operator ∇u is a singular integral with respect to the vorticity ([38] and [39]).

The vectors $\nabla^{\perp}\theta$ and ω are tangential to the level sets of θ and the vortex lines respectively. The vortex lines and the level sets of θ have the property to move with the flow. See [3] for a more detailed description of the properties and analogies of QG and Euler equations.

The first analytical results for (3.1) appear in [11] (for equivalent results for Euler equations see [10], [1], [4] and [27]):

- If $\theta_0\in H^m(\mathbb{R}^2)$ and $m>2$, then there exists a time $T = T(\kappa,\|\Lambda^m\theta_0\|_{L^2})>0$ such that there is a unique solution to (3.1) in $C^1([0,T),H^m)$. This local existence result is a consequence of the estimates

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^m\theta\|_{L^2}^2 &= \int\Lambda^m\theta\{\Lambda^m(R(\theta)\cdot\nabla^{\perp}\theta) - R(\theta)\cdot\nabla^{\perp}\Lambda^m\theta\}\\ &\leqslant C\|\Lambda^m\theta\|_{L^2}^2(\|\theta\|_{L^2}+\|\Lambda^{2+\varepsilon}\theta\|_{L^2})\end{aligned}$$

  where C is a constant. Therefore, taking $\varepsilon = m-2>0$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^m\theta\|_{L^2}^2 \leqslant C(\|\Lambda^m\theta\|_{L^2}^3 + \|\Lambda^m\theta\|_{L^2}^2\|\theta\|_{L^2}).$$

- If $\theta_0\in H^k(\mathbb{R}^2)$, $k\geqslant 3$, a necessary and sufficient condition to have a singularity at time T is that

$$\int_0^T\|\nabla\theta\|_\infty\,\mathrm{d}t = +\infty. \qquad (3.4)$$

By using micro-local analysis techniques the criterion can be improved by substituting the L^∞ norm by the BMO and the Triebel-Lizorkin spaces ([5]).

- If the direction of the vector

$$\xi(x) = \frac{\nabla^\perp \theta}{|\nabla \theta|} \tag{3.5}$$

is smooth in regions where $|\nabla\theta|$ is high, then there are no singularities ([11]).

The QG equation has an extra property: that all L^p $(1 < p < \infty)$ norms of the velocity remain bounded by the L^p of the initial data. This does not imply an improvement with respect to the results known for Euler equations.

Below we first discuss several singular scenarios and propose a possible scenario of a blow-up for a weak solution to QG equations. Finally, we study the role of viscosity in preventing the formation of singularities.

3.1. Hyperbolic saddle scenario

The initial data

$$\theta(x,0) = \sin(x_1)\sin(x_2) + \cos(x_2)$$

was proposed as a possible candidate to develop a singularity in finite time. Numerical simulations indicate the formation of a front when the topology of the level sets contains a hyperbolic saddle (an X-point configuration). The level sets tend to collapse over a single curve and become a two Y-point configuration (see Fig. 1) where the function θ is constant along the two time-dependent arcs $\Gamma_-(t)$ and $\Gamma_+(t)$:

$$\Gamma_\pm = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = f_\pm(x,t),\ x_1 \in [a,b]\} \quad \text{for } \ 0 \leqslant t < T. \tag{3.6}$$

The direction field $\xi(x) = \nabla^\perp\theta/|\nabla\theta|$, in a neighborhood of the saddle, changes abruptly as the angle of the saddle closes in time. Therefore the criterion (3.5) does not apply.

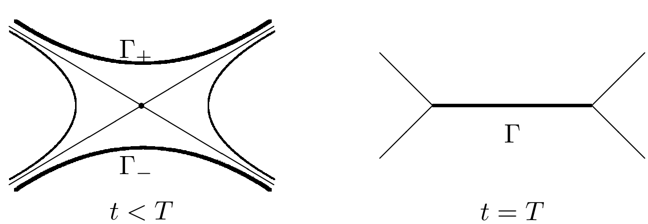


Figure 1. Level sets of θ.

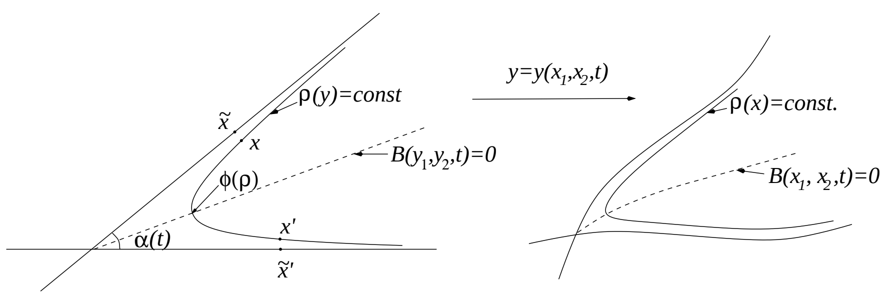
Ohkitani and Yamada [29], with higher numerical resolution, suggested that instead of a singularity there was a double exponential growth of the derivatives of θ. Constantin, Nie and Schorghofer [12] confirmed the results of [29] numerically:

$$
\begin{aligned}
&\text{1994 in [11]} \quad \sup_x |\nabla_x \theta(x,t)| \sim \frac{1}{(8.25-t)^{1.66}},\\
&\text{1997 in [29]} \quad \sup_x |\nabla_x \theta(x,t)| \sim \mathrm{e}^{\mathrm{e}^{[b(t-t_0)]}},\\
&\text{1998 in [12]} \quad \sup_x |\nabla_x \theta(x,t)| \sim \mathrm{e}^{\mathrm{e}^{[0.038(t-4.1)]}}.
\end{aligned}
$$

In order to approach the problem analytically we assume that the level sets are "simple" hyperbolas, which are defined by the set of curves $\varrho = \text{constant}$ that satisfy

$$
\varrho = (y_1\beta(t) + y_2)(y_1\delta(t) - y_2),
$$

and there is a non-linear change of variables depending on time $y_1 = F_1(x_1, x_2, t)$, $y_2 = F_2(x_1, x_2, t)$ in a neighborhood U of the origin with $\beta(t), \delta(t) \in C^1([0, T_*))$, $F_i \in C^2(\overline{U} \times [0, T_*])$, $|\beta|, |\delta| \leqslant C$, $\beta(t) + \delta(t) \geqslant 0$, $|\det \partial F_i/\partial x_j| \geqslant c > 0$ for $x \in U$, $t \in [0, T_*]$.



Non-linear change of variables depending on time.

Level sets move with the flow, therefore we assume below that the temperature is constant along ϱ. With these hypotheses we show in Theorem 3.1 that the angle $\alpha(t) = \beta(t) + \delta(t)$ can not close faster than a double exponential in time and that $|\nabla\theta|$ is bounded by a quadruple exponential.

Theorem 3.1. *Let $\theta(x_1, x_2, t)$ be a solution of QG (3.1). Assume that θ is constant along the curves $\varrho = \text{const}$ defined above with $T_* = \infty$. Assume also, for each fixed t, that θ is not constant on any disc in U. Then*

$$
\left| \log\log \frac{1}{\alpha(t)} \right| \leqslant C_1 \cdot t + C_2.
$$

The proof is divided in two parts. First we write the stream function in terms of the variables (ϱ,σ). The variable ϱ represents the level sets and the variable σ satisfies the equations

$$\frac{\partial x_1}{\partial\sigma}=-\frac{\partial\varrho}{\partial x_2},\qquad \frac{\partial x_2}{\partial\sigma}=\frac{\partial\varrho}{\partial x_1}.$$

Making the change of variables in the original equations, with the hypothesis that $\partial\theta/\partial\varrho\neq 0$ in U and with the condition $\theta(x_1,x_2,t)=\tilde{\theta}(\varrho,t)$ we obtain

$$u\cdot\nabla_x\theta=\frac{\partial\tilde{\theta}}{\partial\varrho}(u\cdot\nabla_x\varrho)=-\frac{\partial\tilde{\theta}}{\partial\varrho}\Big(\frac{\partial\psi}{\partial x_2}\frac{\partial x_2}{\partial\sigma}+\frac{\partial\psi}{\partial x_1}\frac{\partial x_1}{\partial\sigma}\Big),$$

therefore

$$\frac{\partial\tilde{\theta}}{\partial t}+\frac{\partial\tilde{\theta}}{\partial\varrho}\Big(\frac{\partial\varrho}{\partial t}-\frac{\partial\psi}{\partial\sigma}\Big)=0,$$

where the velocity satisfies $u=\nabla^{\perp}\psi$. Then

$$\frac{\partial\psi}{\partial\sigma}=\frac{\partial\varrho}{\partial t}+E_1(\varrho,t).$$

Integrating with respect to σ we obtain the desired expression

$$\psi(\varrho,\sigma,t)=E_1(\varrho,t)\cdot\sigma+\int_0^{\sigma}\frac{\partial\varrho}{\partial t}\,\mathrm{d}\sigma+E_2(\varrho,t). \tag{3.7}$$

For the second part of the proof we choose two points (x,x') in the same level set but in different branches of the hyperbola. We evaluate the stream function at both points, subtract one value from the other and take the limit as ϱ tends to zero. Then from the expression (3.7) we get

$$\lim_{\varrho\to 0}[\psi(x)-\psi(x')]=C\frac{\mathrm{d}\alpha}{\mathrm{d}t}+O(\alpha).$$

Since $\theta=-(-\Delta)^{1/2}\psi$ the stream function can be obtained by setting

$$\psi(x,t)=-\int_{\mathbb{R}^2}\frac{\theta(x+y,t)}{|y|}\,\mathrm{d}y$$

and we get the following estimate:

$$\lim_{\varrho\to 0}|\psi(x)-\psi(x')|\leqslant K|\log\alpha|\,|\alpha|.$$

Putting together the two limits we have

$$\alpha(t)\geqslant c_1\mathrm{e}^{-\mathrm{e}^{t}},$$

which is a local estimate. In order to obtain bounds on the derivatives of θ we need to take into account the nonlocal behavior. Let $\xi(x) = \nabla^{\perp}\theta/|\nabla^{\perp}\theta|$ satisfy $|\nabla\xi| \leqslant \Phi(t)$ in $(\mathbb{R}^2/U) \times [0,\infty)$. Then by estimating the evolution of $|\nabla\theta|$ along the trajectories we obtain

$$|\nabla\theta| \leqslant \exp\biggl(\exp\biggl(c\int_0^t \bigl(\mathrm{e}^{\mathrm{e}^s} + \Phi(s)\bigr)\,\mathrm{d}s\biggr)\biggr)$$

in $\mathbb{R}^2 \times [0,\infty)$. For details of the proof see [17].

3.2. Patches

Resnick [33] studied the weak formulation of the QG equation in a periodic setting $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$. A function θ is a weak solution if it satisfies

$$\int_{\mathbb{T}^2} \varphi(x)\theta(x,T)\,\mathrm{d}x - \int_{\mathbb{T}^2} \varphi(x)\theta_0(x)\,\mathrm{d}x = \int_0^T\int_{\mathbb{T}^2} \nabla\varphi\theta u\,\mathrm{d}x\,\mathrm{d}t$$

for all functions $\varphi \in C^\infty$ where $u = (-R_2\theta, R_1\theta)$ for almost all $t \in [0,T]$. Resnick proved the existence of weak solutions using Galerkin approximations in the periodic case $\mathbb{T}^2$. This solutions satisfy $\|\theta(t)\|_{L^2} \leqslant \|\theta_0\|_{L^2}$, therefore $\|u(t)\|_{L^2} \leqslant \|u_0\|_{L^2}$. The problem of uniqueness is still open.

A fundamental property of QG equation is that the level sets move with the flow, i.e. that there is no transfer of flow along a level set. Then a natural weak solution, with finite energy, is a connected bounded region $\Omega(t)$ where the function θ satisfies

$$\theta(\vec{x},t) = \begin{cases} 1 & \text{if } \vec{x} \in \Omega(t), \\ 0 & \text{if } \vec{x} \in \mathbb{R}^2/\Omega(t) \end{cases}$$

and evolves with the velocity preserving the initial area, $|\Omega(0)| = |\Omega(t)|$. These solutions start initially with a front on the boundary of $\Omega(t)$ and are called *patches*. This type of solutions was studied for 2D incompressible Euler equation (see [2], [7], and [3]), where the vorticity is conserved along trajectories. In [21] we study the dynamics of the α-*patches* which is a family of equations that interpolate 2D Euler and QG ones.

An α-patch ($0 < \alpha < 1$) consists in a 2D region $\Omega(t)$ (bounded and connected) that moves with the velocity given by

$$u(\vec{x}(\gamma,t),t) = \frac{\theta_0}{2\pi}\int_{C(t)} \frac{\frac{\partial\vec{x}}{\partial\gamma}(\gamma',t)}{|\vec{x}(\gamma,t) - \vec{x}(\gamma',t)|^\alpha}\,\mathrm{d}\gamma' \tag{3.8}$$

where $\vec{x}(\gamma,t)$ is the position of $C(t)$ which is the boundary of the domain $\Omega(t)$, parameterized by γ. The evolution of the boundary satisfies

$$\frac{\mathrm{d}\vec{x}(\gamma,t)}{\mathrm{d}t} = u(\vec{x}(\gamma,t),t), \tag{3.9}$$

and they are weak solutions of the equation

$$
\begin{aligned}
&(\partial_t + u \cdot \nabla)\theta = 0, \\
&u = \nabla^{\perp}\psi \quad \text{and} \quad \theta = -(-\Delta)^{1-\frac{1}{2}\alpha}\psi.
\end{aligned} \tag{3.10}
$$

The limiting case $\alpha = 0$ (2D Euler) was studied analytically with success by Chemin [7] and Bertozzi-Constantin [2], showing global existence. In the case $\alpha = 1$, i.e. for the (QG) equation José Luis Rodrigo [34] proved a local existence result using a Nash-Moser scheme.

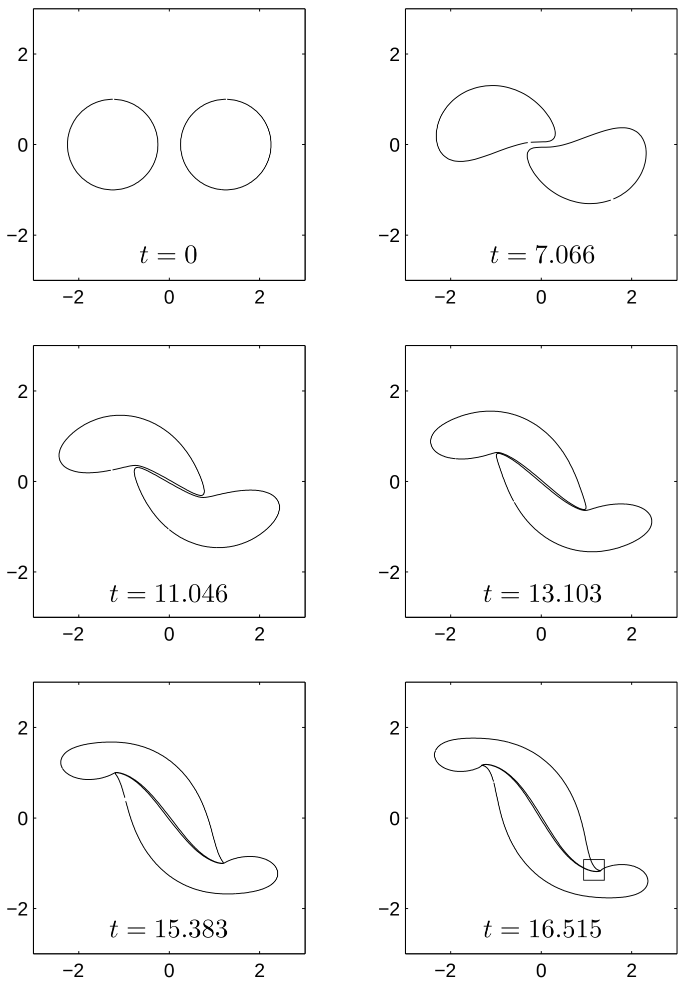


Figure 2. The evolution of two patches with $\alpha = 0.5$. In the plot for $t = 16.515$, the box stands for a magnification of one of the corners displayed in Fig. 4 a).

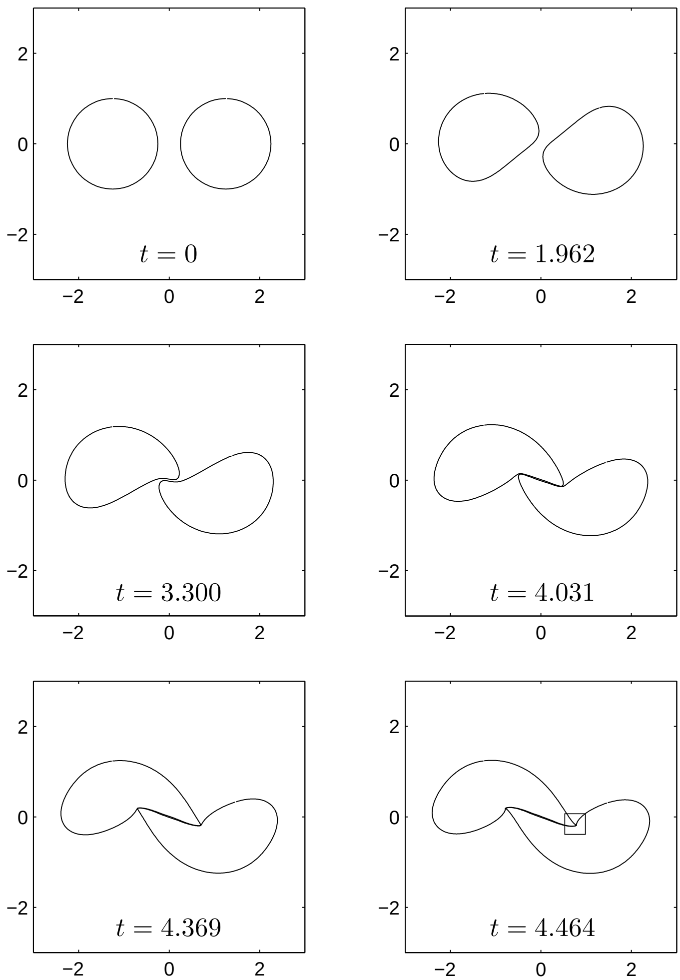


Figure 3. The evolution of two patches with $\alpha = 1$. In the plot for $t = 4.464$, the box stands for a magnification of one of the corners displayed in Fig. 4 b).

In [21] we find numerically possible candidates that lead to a singularity for the family of equations (3.10). For the particular cases $\alpha = 0.5$ and 1 (see Figs. 2 and 3) we observe the formation of a corner that develops a high increase of the curvature at the same point where it reaches the minimum distance between the two patches. For a more detail scenario of the collapse of the two patches see Fig. 3.

Furthermore, by re-scaling the spatial variable in the form

$$\vec{x} = (t_0 - t)^{\delta} \vec{y}$$

where $\delta = \alpha^{-1}$ and introducing a new variable $\tau = -\log(t_0 - t)$ the contour dynamic equations (3.8) and (3.9) become

$$\frac{\partial \vec{y}}{\partial \tau} - \delta \vec{y} = \frac{\theta_0}{2\pi} \int_{C(t)} \frac{\frac{\partial \vec{y}}{\partial \gamma}(\gamma', t)}{|\vec{y}(\gamma, t) - \vec{y}(\gamma', t)|^{\alpha}} \, \mathrm{d}\gamma'. \tag{3.11}$$

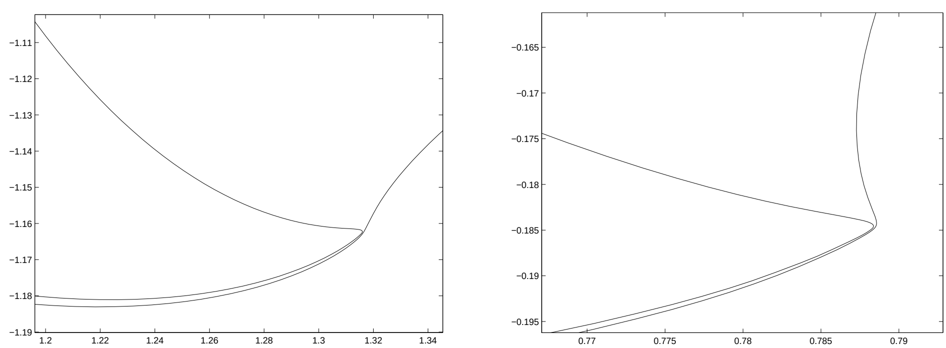
Solutions of (3.11) independent of τ represent solutions of (3.10) with the property that the maximum curvature grows as

$$\kappa = \frac{1}{R} \sim \frac{C}{(t_0 - t)^{1/\alpha}} \quad \text{when } t \to t_0$$

and the minimum distance of the two patches satisfies

$$d \sim C(t_0 - t)^{1/\alpha} \quad \text{when } t \to t_0.$$

These singularities are self-similar and stable and occur at one single point where the curvature blows-up at the same time as the two level sets collapse. Therefore they are not of squirt singularity type.



a) b)

Figure 4. Close caption of the corner region at $t = 16.515$ for $\alpha = 0.5$ (a) and $t = 4.464$ for $\alpha = 1$ (b). Observe that the singularity is point-like in both cases.

4. Viscosity role

As we have seen in Section 2 viscosity plays an important role, and dissipation helps to prevent the formation of singularities. In this section we consider the Cauchy problem for the viscous QG equation

$$
\begin{aligned}
(4.1) \qquad & (\partial_t + u\cdot\nabla)\theta = -\kappa(-\Delta)^{\gamma}\theta, \\
& u = \nabla^{\perp}\psi \quad \text{and} \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi
\end{aligned}
$$

which has been studied by different authors in [26], [8], [6], [9], [13], [14], [15] and [33].

From the equation we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^m\theta\|_{L^2}^2 &\leqslant \left|\int \Lambda^m\theta\{\Lambda^m(R(\theta)\cdot\nabla^{\perp}\theta) - R(\theta)\cdot\nabla^{\perp}\Lambda^m\theta\}\right| - \kappa\|\Lambda^{m+\frac{1}{2}\gamma}\theta_0\|_{L^2}^2 \\
&\leqslant C\|\Lambda^m\theta\|_{L^2}^2(\|\theta\|_{L^2} + \|\Lambda^{2+\varepsilon}\theta\|_{L^2}) - \kappa\|\Lambda^{m+\frac{1}{2}\gamma}\theta\|_{L^2}^2
\end{aligned}
$$

for every $\varepsilon > 0$. Taking $\varepsilon = m + \frac{1}{2}\gamma - 2$ we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^m\theta\|_{L^2}^2 \leqslant C\Big(\frac{1}{\kappa}\|\Lambda^m\theta\|_{L^2}^4 + \|\theta\|_{L^2}\|\Lambda^m\theta\|_{L^2}^2\Big),
$$

which yields the following local existence result for (4.1).

Theorem 4.1. *Let $\gamma \geqslant 0$ and $\kappa > 0$ be given and assume that $\theta_0 \in H^m(\mathbb{R}^2)$, $m + \frac{1}{2}\gamma > 2$. Then there exists a time $T = T(\kappa, \|\Lambda^m\theta_0\|_{L^2}) > 0$ such that there is a unique solution to* (4.1) *in $C^1([0,T), H^m(\mathbb{R}^2))$.*

For $\kappa > 0$, Constantin and Wu [13] showed global existence for $\gamma \in (\frac{1}{2}, 1]$. It is an open problem to know the existence of singularities when $\gamma \leqslant \frac{1}{2}$. It is easy to obtain global existence results for small initial data in several functional spaces, for example in the case of $H^m(\mathbb{R}^2)$ we have

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) \leqslant &-\kappa\|\Lambda^{\frac{1}{2}\gamma}\theta\|_{L^2}^2 + C(\|\theta\|_{L^2}\|\Lambda^m\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^3) \\
&-\kappa\|\Lambda^{m+\frac{1}{2}\gamma}\theta\|_{L^2}^2.
\end{aligned}
$$

Since

$$
\|\Lambda^m\theta\|_{L^2}^2 \leqslant \|\Lambda^{\frac{1}{2}\gamma}\theta\|_{L^2}^2 + \|\Lambda^{m+\frac{1}{2}\gamma}\theta\|_{L^2}^2
$$

we obtain the inequality

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2) \leqslant \|\Lambda^m\theta\|_{L^2}^2(C(\|\theta\|_{L^2}^2 + \|\Lambda^m\theta\|_{L^2}^2)^{\frac{1}{2}} - \kappa)
$$

for a fixed constant $C < \infty$. Then (for $\kappa > 0$ and $0 \leqslant \gamma \leqslant 1$), if the initial data satisfies $\|\theta_0\|_{H^m} \leqslant \kappa/C$ (where $m > 2$), we can conclude that there exists a unique solution to (4.1) which belongs to H^m for all time $t > 0$.

In the critical case $\gamma = 1$ ($\kappa > 0$), we obtain an improvement. Multiplying the equation by $\Delta^2\theta$ and integrating by parts we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Delta\theta|^2\,\mathrm{d}x = -\int \Delta^2\theta(R(\theta)\cdot\nabla^{\perp}\theta)\,\mathrm{d}x - \kappa\int |(-\Delta)^{\frac{5}{4}}\theta|^2\,\mathrm{d}x.$$

Further integration by parts gives

$$\int \Delta^2\theta(R(\theta)\cdot\nabla^{\perp}\theta)\,\mathrm{d}x = 2\int \nabla u\cdot(\nabla(\nabla\theta))\Delta\theta\,\mathrm{d}x + \int(\Delta\cdot\nabla\theta)\Delta\theta\,\mathrm{d}x.$$

By the Hölder inequality,

$$\left|\int \Delta^2\theta(R(\theta)\cdot\nabla^{\perp}\theta)\,\mathrm{d}x\right| \leqslant C\big[\|\nabla u\|_{L^3}\|\Delta\theta\|_{L^3}^2 + \|\Delta u\|_{L^3}^2\|\nabla\theta\|_{L^3}\|\Delta\theta\|_{L^3}\big].$$

Applying the fact that the Riesz transforms are bounded in L^p spaces together with the Gagliardo-Nirenberg inequalities we obtain

$$\left|\int \Delta^2\theta(R(\theta)\cdot\nabla^{\perp}\theta)\,\mathrm{d}x\right| \leqslant C\|\theta\|_{L^\infty}\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^2.$$

Collecting the above estimates we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Delta\theta|^2\,\mathrm{d}x \leqslant C(\|\theta\|_{L^\infty} - \kappa)\|(-\Delta)^{\frac{5}{4}}\theta\|_{L^2}^2.$$

The relevance of this inequality comes from the maximum principle of (4.1); the L^∞ norm decreases in time. Resnick showed ([33]) that the solutions to the viscous QG equation satisfy

$$\|\theta(\cdot,t)\|_{L^p} \leqslant \|\theta_0\|_{L^p} \quad \text{for } 1 < p \leqslant \infty \text{ and for all } t \geqslant 0,$$

which yields the following theorem

Theorem 4.2 (Global existence for small data)**.** *Let θ be a weak solution of* (4.1) *with an initial data $\theta_0 \in H^2$ satisfying $\|\theta_0\|_{L^\infty} \leqslant \kappa/C$ (where $C < \infty$ is a fixed constant). Then there is a unique global solution θ satisfying*

$$\|\theta(\cdot,t)\|_{H^2} \leqslant \|\theta_0\|_{H^2}$$

for all $t \geqslant 0$.

In order to understand the decay of the L^p norms of θ we obtained the following inequality (for more details see [14] and [15]).

P o i n t w i s e i n e q u a l i t y: Let $0 \leqslant \gamma \leqslant 2$, $x \in \mathbb{R}^2$, $\mathbb{T}^2$ and $\theta \in S$ (the Schwartz class). Then

$$2\theta\Lambda^\gamma\theta(x) \geqslant \Lambda^\gamma\theta^2(x).$$

This local pointwise estimate or the chain rule for fractional derivatives is surprising because of the non-local nature of the operators involved. The inequality is used together with the energy estimates to obtain

$$\|\theta(\cdot,t)\|^p_{L^p} \leqslant \frac{\|\theta_0\|^p_{L^p}}{(1+\varepsilon Ct\|\theta_0\|^{p\varepsilon}_{L^p})^{1/\varepsilon}} \tag{4.2}$$

where $C = C(\kappa,\gamma,p,\|\theta_0\|_1)$ is a positive constant and $\varepsilon = \frac{1}{2}\gamma/(p-1)$.

For the case $p = \infty$ it is not sufficient to take the limit $p \to \infty$ of (4.2). We have to study the evolution of the scalar θ along the trajectory x_t where it reaches its maximum (or minimum), and use the differentiability of Lipschitz functions to justify the existence for almost all time of the derivative $\mathrm{d}x_t/\mathrm{d}t$.

Theorem 4.3. *Let θ and u be smooth functions in $\mathbb{R}^2 \times [0,T)$ (or $\mathbb{T}^2 \times [0,T)$) satisfying $\theta_t + u\cdot\nabla\theta + \kappa\Lambda^\gamma\theta = 0$ with $\kappa > 0$, $0 < \gamma \leqslant 2$, $\theta(\cdot,t) \in H^s(\mathbb{R}^2)$, $0 \leqslant t < T$ (or $H^s(\mathbb{T}^2)$) ($s > 1$) and $\nabla\cdot u = 0$. Then*

$$\|\theta(\cdot,t)\|_{L^\infty} \leqslant \frac{\|\theta_0\|_{L^\infty}}{(1+\gamma Ct\|\theta_0\|^\gamma_{L^\infty})^{1/\gamma}}, \quad 0 \leqslant t < T, \tag{4.3}$$

where $\theta_0 = \theta(\cdot,0)$ and $C = C(\kappa,\theta_0) > 0$. Furthermore, when $\gamma = 0$ there is an exponential decay $\|\theta(\cdot,t)\|_{L^\infty} \leqslant \|\theta_0\|_{L^\infty}\mathrm{e}^{-\kappa t}$.

The previous theorem allows us to analyze the existence of solutions after a time T. For that purpose we study weak solutions of the critical case $\gamma = 1$

$$\theta_t + R(\theta)\cdot\nabla^\perp\theta = -\kappa\Lambda\theta$$

which we define to be a viscosity solution with initial data $\theta_0 \in H^s(\mathbb{R}^2)$ (or $H^s(\mathbb{T}^2)$), $s > 1$, if it is the limit of a sequence of solutions, when $\varepsilon \to 0$, of the system

$$\theta_t^\varepsilon + R(\theta^\varepsilon) \cdot \nabla^\perp \theta^\varepsilon = -\kappa \Lambda \theta^\varepsilon + \varepsilon \Delta \theta^\varepsilon \tag{4.4}$$

with $\theta^\varepsilon(x, 0) = \theta_0$. The result we obtain is

Theorem 4.4. *Let θ be a viscosity solution with initial data $\theta_0 \in H^s(\mathbb{R}^2$ or $\mathbb{T}^2)$, $s > \frac{3}{2}$, of the equation $\theta_t + R(\theta) \cdot \nabla^\perp \theta = -\kappa \Lambda \theta$ $(\kappa > 0)$. Then there exist two times $T_1 \leqslant T_2$ depending only upon κ and the initial data θ_0 such that:*

1) *If $t \leqslant T_1$ then $\theta(\cdot, t) \in C^1([0, T_1); H^s)$ is a classical solution of the equation satisfying*

$$\|\theta(\cdot, t)\|_{H^s} \leqslant C \|\theta_0\|_{H^s}.$$

2) *If $t \geqslant T_2$ then $\theta(\cdot, t) \in C^1([T_2, \infty); H^s)$ is also a classical solution and $\|\theta(\cdot, t)\|_{H^s}$ is monotonically decreasing in t, bounded by $\|\theta_0\|_{H^s}$, and satisfying*

$$\int_{T_2}^{\infty} \|\theta\|_{H^s}^2 \, \mathrm{d}t < \infty.$$

In particular, this implies that

$$\|\theta(\cdot, t)\|_{H^s} = O(t^{-\frac{1}{2}}), \quad t \to \infty.$$

The proof is based on the L^∞ decay and a *bootstrap* mechanism associated with the evolution of several Sobolev norms. An example of this mechanism is the following chain of inequalities:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\theta^\varepsilon\|_{L^2}^2 &= 2 \int \theta^\varepsilon R(\theta^\varepsilon) \cdot \nabla^\perp \theta^\varepsilon - 2\kappa \int \theta^\varepsilon \Lambda \theta^\varepsilon - 2\varepsilon \int |\Lambda \theta^\varepsilon|^2 \\
&= -2\kappa \|\Lambda^{\frac{1}{2}} \theta^\varepsilon\|_{L^2}^2 - 2\varepsilon \|\Lambda \theta^\varepsilon\|_{L^2}^2 \leqslant -2\kappa \|\Lambda^{\frac{1}{2}} \theta^\varepsilon\|_{L^2}^2, \\
\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^{\frac{1}{2}} \theta^\varepsilon\|_{L^2}^2 &= 2 \int \Lambda^{\frac{1}{2}} \theta^\varepsilon \Lambda^{\frac{1}{2}} (R(\theta^\varepsilon) \cdot \nabla^\perp \theta^\varepsilon) - 2\kappa \|\Lambda \theta^\varepsilon\|_{L^2}^2 - 2\varepsilon \|\Lambda^{\frac{3}{2}} \theta^\varepsilon\|_{L^2}^2 \\
&\leqslant (C \|\theta^\varepsilon(\cdot, t)\|_{L^\infty} - 2\kappa) \|\Lambda \theta^\varepsilon\|_{L^2}^2, \\
\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda \theta^\varepsilon\|_{L^2}^2 &\leqslant (C \|\Lambda \theta^\varepsilon\|_{L^2} - \kappa) \|\Lambda^{\frac{3}{2}} \theta^\varepsilon\|_{L^2}^2, \\
\frac{\mathrm{d}}{\mathrm{d}t} \|\Lambda^{\frac{3}{2}} \theta^\varepsilon\|_{L^2}^2 &\leqslant (C \|\theta^\varepsilon\|_{L^\infty} - \kappa) \|\Delta \theta^\varepsilon\|_{L^2}^2,
\end{aligned}$$

where C is a universal constant, uniform with respect to the artificial viscosity ε.

References

[1] *J. T. Beale, T. Kato, and A. Majda*: Remarks on the breakdown of smooth solutions for the 3D Euler equations. Commun. Math. Phys. *94* (1984), 61–66. Zbl 0573.76029

[2] *A. L. Bertozzi, P. Constantin*: Global regularity for vortex patches. Commun. Math. Phys. *152* (1993), 19–28. Zbl 0771.76014

[3] *A. L. Bertozzi, A. J. Majda*: Vorticity and the Mathematical Theory of Incompresible Fluid Flow. Cambridge Texts in Applied Mathematics No. 27. Cambridge University Press, Cambridge, 2002.

[4] *D. Chae*: On the Euler equations in the critical Triebel-Lizorkin spaces. Arch. Ration. Mech. Anal. *170* (2003), 185–210. Zbl pre 02021076

[5] *D. Chae*: The quasi-geostrophic equation in the Triebel-Lizorkin spaces. Nonlinearity *16* (2003), 479–495. Zbl 1029.35006

[6] *D. Chae, J. Lee*: Global well-posedness in the super-critical dissipative quasi-geostrophic equations. Commun. Math. Phys. *233* (2003), 297–311. Zbl 1019.86002

[7] *J. Y. Chemin*: Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. Ann. Sci. Ec. Norm. Supér. *26* (1993), 517–542. (In French.) Zbl 0779.76011

[8] *P. Constantin*: Energy spectrum of quasi-geostrophic turbulence. Phys. Rev. Lett. *89* (2002), 1804501–1804504.

[9] *P. Constantin, D. Córdoba, and J. Wu*: On the critical dissipative quasi-geostrophic equation. Indiana Univ. Math. J. *50* (2001), 97–107. Zbl 0989.86004

[10] *P. Constantin, C. Fefferman, and A. J. Majda*: Geometric constraints on potentially singular solutions for the 3-D Euler equations. Commun. Partial Differ. Equation *21* (1996), 559–571. Zbl 0853.35091

[11] *P. Constantin, A. J. Majda, and E. Tabak*: Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. Nonlinearity *7* (1994), 1495–1533. Zbl 0809.35057

[12] *P. Constantin, Q. Nie, and N. Schorghofer*: Nonsingular surface-quasi-geostrophic flow. Phys. Lett. A *241* (1998), 168–172. Zbl 0974.76512

[13] *P. Constantin, J. Wu*: Behavior of solutions of 2D quasi-geostrophic equations. SIAM J. Math. Anal. *30* (1999), 937–948. Zbl 0957.76093

[14] *A. Córdoba, D. Córdoba*: A pointwise estimate for fractionary derivatives with applications to partial differential equations. Proc. Natl. Acad. Sci. USA *100* (2003), 15316–15317. Zbl pre 02187633

[15] *A. Córdoba, D. Córdoba*: A maximum principle applied to quasi-geostrophic equations. Commun. Math. Phys. *249* (2004), 511–528. Zbl pre 02158321

[16] *A. Córdoba, D. Córdoba, C. L. Fefferman, and M. A. Fontelos*: A geometrical constraint for capillary jet breakup. Adv. Math. *187* (2004), 228–239. Zbl 1061.35074

[17] *D. Córdoba*: Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. Ann. Math. *148* (1998), 1135–1152. Zbl 0920.35109

[18] *D. Córdoba, C. Fefferman*: On the collapse of tubes carried by 3D incompressible flows. Commun. Math. Phys. *222* (2001), 293–298. Zbl 0999.76020

[19] *D. Córdoba, C. Fefferman, and R. de la Llave*: On squirt singularities in hydrodynamics. SIAM J. Math. Anal. *36* (2004), 204–213. Zbl 1078.76018

[20] *D. Córdoba, C. Fefferman, and J. L. Rodrigo*: Almost sharp fronts for the surface quasi-geostrophic equations. Proc. Natl. Acad. Sci. USA *101* (2004), 2687–2691. Zbl 1063.76011

[21] *D. Córdoba, M. Fontelos, A. Mancho, and J. L. Rodrigo*: Evidence of singularities for a family of contour dynamics equations. Proc. Natl. Acad. Sci. USA *102* (2005), 5949–5952.

[22] *R. J. Diperna, P. L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547. Zbl 0696.34049
[23] *C. R. Doering, J. D. Gibbon*: Applied analysis of the Navier Stokes equations. Cambridge University Press, Cambridge, 1995. Zbl 0838.76016
[24] *C. Foias, C. Guillopé, and R. Témam*: New a priori estimates for Navier-Stokes equations in dimension 3. Commun. Partial Differ. Equations *6* (1981), 329–359. Zbl 0472.35070
[25] *I. M. Held, R. Pierrehumbert, and S. T. Garner*: Surface quasi-geostrophic dynamics. J. Fluid Mech. *282* (1995), 1–20. Zbl 0832.76012
[26] *N. Ju*: The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations. Commun. Math. Phys. *255* (2005), 161–181. Zbl pre 02214807
[27] *H. Kozono, Y. Taniuchi*: Limiting case of the Sobolev inequality in BMO, with application to the Euler equations. Commun. Math. Phys. *214* (2000), 191–200. Zbl 0985.46015
[28] *T. A. Kowalewski*: On the separation of droplets from a liquid jet. Fluid Dyn. Res. *17* (1996), 121–145.
[29] *K. Ohkitani, M. Yamada*: Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow. Phys. Fluids *9* (1997), 876–882.
[30] *J. Pedlosky*: Geophysical Fluid Dynamics. Springer-Verlag, New York, 1987. Zbl 0713.76005
[31] *M. T. Plateau*: Smithsonian Report 250. 1863.
[32] *Rayleigh, Lord (J. W. Strutt)*: On the instability of jets. Proc. L. M. S. *10* (1879), 4–13. Zbl JFM 11.0685.01
[33] *S. Resnick*: Dynamical problem in nonlinear advective partial differential equations. PhD. Thesis. University of Chicago, 1995.
[34] *J. L. Rodrigo*: On the evolution of sharp fronts for the quasi-geostrophic equation. Commun. Pure Appl. Math. *58* (2005), 821–866. Zbl 1073.35006
[35] *R. Salmon*: Lectures on Geophysical Fluid Dynamics. Oxford University Press, New York, 1998.
[36] *F. Savart*: Mémoire sur la Constitution des veines liquides lancées par des orifices circulaires en mince paroi. Ann. Chim. Phys. *53* (1833), 337–386. (In French.)
[37] *X. D. Shi, M. P. Brenner, and S. R. Nagel*: A cascade of structure in a drop falling from a faucet. Science *265* (1994), 219–222.
[38] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, 1970. Zbl 0207.13501
[39] *E. M. Stein*: Harmonic Analysis. Princeton University Press, Princeton, 1993. Zbl 0821.42001
[40] *M. Sussman, P. Smereka*: Axisymmetric free boundary problems. J. Fluid Mech. *341* (1997), 269–294. Zbl 0892.76090
[41] *L. Tartar*: Topics in Nonlinear Analysis. Publications Mat. D'Orsay, No. 7813. Univ. de Paris-Sud, Orsay, 1978.

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