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ON THE SECOND DERIVATIVE OF THE TOTAL SCALAR CURVATURE

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0. Introduction

Let M be a compact connected C^∞-manifold of dimension $n \geqq 2$. It is a classical result due to D. Hilbert [4] that a metric g on M is an Einstein metric if and only if g is critical for the scalar curvature τ, that is, g is such that

$$\frac{d}{dt}\bigg|_0 \int \tau_{g(t)} v_{g(t)} = 0$$

for any volume preserving deformation $g(t)$ of g, where $\tau_{g(t)}$ is the scalar cuvature of $g(t)$ and $v_{g(t)}$ is the volume element of $g(t)$. As for the derivative of second order of the integral, Y. Muto [8] shows that there exist volume preserving deformations which gives positive derivative and which gives negative derivative.

In this paper we attempt to decide the sign of the derivative for given volume preserving deformations. The results are as follows. Let (M, g) be an Einstein manifold with certain condition (in Theorem 2.5). If (M, g) is not the standard sphere, then any volume prserving deformation is decomposed to a conformal deformation with positive derivative (Theorem 2.4), a trivial deformation with zero derivative and a deformation of constant scalar curvature with negative derivative (Theorem 2.5).

The paper is organized as follows; after some preliminaries in **1**, we prove the above propositions in **2**. Finally, in **3**, we consider the case when M is a complex manifold and $g(t)$ are Kähler metrics.

1. Preliminaries

First, we introduce notation and definitions which will be used throughout this paper. Let M be an n-dimensional, connected and compact C^∞-manifold, and we always assume $n \geqq 2$. For a riemannian manifold (M, g), we consider the riemannian connection and use the following notation;

v_g; the volume element defined by g,

R; the curvature tensor defined by the riemannian connection,

ρ; the Ricci tensor

(For the standard sphere with orthonormal basis, $R_{1212}<0$ and $\rho_{11}>0$.)

τ; the scalar curvature,

$(,)$; the inner product in fibres of a tensor bundle defined by g,

$\langle , \rangle$; the global inner product for sections of a tensor bundle over M, i.e., $\langle , \rangle=\int_M (,)v_g$,

$C_g^\infty(M)$; the vector space of all functions f such that $\int f v_g=0$,

$C^\infty(S^2)$; the vector space of all symmetric covariant 2-tensor fields,

$C_g^\infty(S^2)$; the vector space of all symmetric covariant 2-tensor fields h such that $\langle h, g\rangle=0$,

L; the operator operating on $C^\infty(S^2)$ defined by $(Lh)_{ij}=R_i{}^k{}_j{}^l h_{kl}$,

∇; the covariant derivative defined by the riemannian connection,

δ; the formal adjoint of ∇ with respect to $\langle , \rangle$,

δ^*; the formal adjoint of $\delta | C^\infty(S^2)$,

$\Delta=\delta d$; the Laplacian operating on the space $C^\infty(M)$,

$\bar{\Delta}=\delta\nabla$; the rough Laplacian operating on the vector space of tensor fields,

Hess $= \nabla d$; the Hessian on $C^\infty(M)$.

REMARK 1.1. Let (M, g) be an Einstein manifold. Then,

$$\operatorname{tr} Lh = g^{ij}R_i{}^k{}_j{}^l h_{kl} = -\rho^{kl}h_{kl} = -\frac{\tau}{n} \operatorname{tr} h .$$

Therefore we see that L operates on $\operatorname{tr}^{-1}(0)$.

In this paper, we consider 1-parameter families of riemannian metrics on M. If $g(0)=g$, then we call such a family $g(t)$ *a deformation of* g. The derivative $g'(0)$ of a deformation $g(t)$ is called *an infinitesimal deformation,* or simply *i-deformation.* Total of i-deformations forms the space $C^\infty(S^2)$. Total of volume preserving i-deformations consists with the space $C_g^\infty(S^2)$.

2. The second derivative of the integral $\int \tau v_g$

First, we give a decomposition of $C_g^\infty(S^2)$.

Proposition 2.1. *Let g be a metric of constant scalar curvature such that $\tau_g=0$ or $\tau_g/(n-1)$ is not an eigenvalue of Δ_g. Then $C_g^\infty(S^2)$ is decomposed as follows;*

$$C_g^\infty(S^2) = C_g^\infty(M)\cdot g\oplus \operatorname{Im} \delta^*\oplus\delta^{-1}(0)\cap\alpha^{-1}(0)\cap C_g^\infty(S^2), \tag{2.1}$$

where α is an operator from $C^\infty(S^2)$ to $C^\infty(M)$ which is defined by

$$\alpha(h) = \Delta\{\Delta \operatorname{tr} h+\delta\delta h-(h, \rho)\} .$$

REMARK 2.2. By [7, p. 135], we know that the first positive eigenvalue of $\Delta \geqq \frac{\tau}{n-1}$ on an Einstein manifold. Moreover by [9, Theorem 5], the equality holds if and only if the Einstein manifold is isometric to the standard sphere. Therefore, the condition of the scalar curvature is satisfied if (M, g) is an Einstein manifold but not the standard sphere.

REMARK 2.3. Moreover we can get the following decomposition [6, Corollary 2.9]. Let $g(t)$ be a deformation of g. Then $g(t)$ is decomposed into $f(t)\gamma(t)^*\bar{g}(t)$, where $f(t)$ is a 1-parameter family of positive functions, $\gamma(t)$ is a 1-parameter family of diffeomorphisms and $\bar{g}(t)$ is a volume preservnig deformation of constant scalar curvature such that $\delta\bar{g}'(0)=0$. The decomposition (2.1) is the differential of this decomposition.

Proof. First we show the decomposition

$$C_g^\infty(S^2) = C_g^\infty(M)\cdot g \oplus \alpha^{-1}(0) \cap C_g^\infty(S^2).$$

If $fg \in \alpha^{-1}(0)$, then $\alpha(fg)=0$, which implies

$$0 = \alpha(fg) = (n-1)\Delta^2 f - \tau\Delta f.$$

By the condition of g, f is constant, which implies $f=0$ because of $f \in C_g^\infty(M)$, hence $C_g^\infty(M)\cdot g \cap \alpha^{-1}(0) \cap C_g^\infty(S^2)=0$. If $h \in C_g^\infty(S^2)$ is orthogonal to $C_g^\infty(M)\cdot g + \alpha^{-1}(0) \cap C_g^\infty(S^2)$, then $\operatorname{tr} h=0$ and $\langle h, \alpha^{-1}(0) \cap C_g^\infty(S^2)\rangle=0$. But here, the formal adjoint of α is given by

$$\alpha^*(f) = \Delta^2 f\cdot g + \operatorname{Hess} \Delta f - \Delta f \cdot \rho,$$

and has injective symbol. Therefore, by [3, Corollary 6.9], $C^\infty(S^2) = \operatorname{Im} \alpha^* \oplus \alpha^{-1}(0)$ (orthogonal direct sum), which implies that there are a function f and an element ψ of $\alpha^{-1}(0)$ such that $h=\alpha^* f+\psi$. We easily see that $\alpha^* f \in C_g^\infty(S^2)$, and so $\psi \in \alpha^{-1}(0) \cap C_g^\infty(S^2)$. Therefore

$$0 = \langle \alpha^* f + \psi, \psi\rangle = \langle \psi, \psi\rangle$$

and $\psi=0$. Since $\operatorname{tr} h=0$, we see $(n-1)\Delta^2 f - \tau\Delta f=0$. Therefore f is a constant, which implies $h=\alpha^* f=0$. Thus we get the above decomposition.

By [2, (3.1)], we know $C^\infty(S^2) = \operatorname{Im} \delta^* \oplus \delta^{-1}(0)$. Since $\delta g=0$, $\operatorname{Im} \delta^* \subset C_g^\infty(S^2)$. Therefore we get

$$C_g^\infty(S^2) = \operatorname{Im} \delta^* \oplus \delta^{-1}(0) \cap C_g^\infty(S^2).$$

Moreover, $\alpha\delta^*\xi = \Delta\{\Delta \operatorname{tr}(\delta^*\xi) + \delta\delta(\delta^*\xi) - (\delta^*\xi, \rho)\}$, and

$$\begin{aligned}
&\Delta\,\mathrm{tr}\,(\delta^*\xi)+\delta\delta(\delta^*\xi)-(\delta^*\xi,\ \rho)\\
&\quad= -\nabla^l\nabla_l\nabla^m\xi_m+\frac{1}{2}\nabla^l\nabla^m(\nabla_l\xi_m+\nabla_m\xi_l)-\rho^{lm}\nabla_l\xi_m\\
&\quad= -\nabla^l(\nabla_l\nabla^m-\nabla^m\nabla_l)\xi_m+\frac{1}{2}(\nabla^m\nabla^l-\nabla^l\nabla^m)\nabla_l\xi_m-\rho^{lm}\nabla_l\xi_m\\
&\quad= -\nabla^l(R_l{}^{mk}{}_m\xi_k)+\frac{1}{2}R^{mlk}{}_l\nabla_k\xi_m+\frac{1}{2}R^{mlk}{}_m\nabla_l\xi_k-\rho^{lm}\nabla_l\xi_m\\
&\quad= 0\ .
\end{aligned}$$

Therefore we see $\mathrm{Im}\,\delta^*\subset\alpha^{-1}(0)$, which implies

$$\alpha^{-1}(0)\cap C_g^\infty(S^2)=\mathrm{Im}\,\delta^*\oplus\alpha^{-1}(0)\cap\delta^{-1}(0)\cap C_g^\infty(S^2)\,.$$

Now, we decide the sign of the second derivative according to the decomposition (2.1). Recall that any element of $\mathrm{Im}\,\delta^*$ is an i-deformation of a trivial deformation $\gamma(t)^*g$ ([2, Lemma 6.2]). Therefore $\langle\tau,\ 1\rangle''=0$ for any element of $\mathrm{Im}\,\delta^*$.

Theorem 2.4. *Let (M, g) be an Einstein manifold but not be the standard sphere. If $h=fg$ is a conformal and volume preserving non-zero i-deformation, i.e., $\langle h, g\rangle=0$, then any volume preserving deformation $g(t)$ of g such that $g'(0)=h$ satisfies*

$$\frac{d^2}{dt^2}\bigg|_0\int\tau_{g(t)}v_{g(t)}>0\,.$$

Proof. We recall the formula [8, 2];

$$\langle\tau,\ 1\rangle''=\frac{n-2}{2}\{(n-1)\langle df,\ df\rangle-\tau\langle f, f\rangle\}\ .$$

Hence we get $\langle\tau,\ 1\rangle''=\frac{n-2}{2}\langle(n-1)\Delta f-\tau f, f\rangle$. But here, by Remark 2.2, we know that the first eigenvalue of $\Delta>\frac{\tau}{n-1}$. Thus, since $f\in C_g^\infty(M)$, we see $\langle\tau,\ 1\rangle''>0$.

Theorem 2.5. *Let (M, g) be an Einstein manifold. We denote by α_0 the minimum eigenvalue of the operator L: $\mathrm{tr}^{-1}(0)\to\mathrm{tr}^{-1}(0)$. We assume that $\alpha_0>\min\left\{\frac{\tau}{n}, -\frac{\tau}{2n}\right\}$. Then for any volume preserving deformation $g(t)$ of g such that $g'(0)\in\alpha^{-1}(0)\cap\delta^{-1}(0)\cap C_g^\infty(S^2)$ and $g'(0)\neq0$, we get*

$$\frac{d^2}{dt^2}\bigg|_0\int\tau_{g(t)}v_{g(t)}<0\,.$$

Proof. We set $g'(0)=h$. Recall the formula [8, (1.5)];

$$\langle\tau, 1\rangle'' = \langle\nabla_j h_{ik}, \nabla_k h_{ji}\rangle - \frac{1}{2}\langle\nabla_k h_{ji}, \nabla_k h_{ji}\rangle - \langle\nabla_i h_j{}^j, \nabla^k h_{ki}\rangle$$
$$+\frac{1}{2}\langle\nabla_i h_j{}^j, \nabla_i h_k{}^k\rangle + \frac{\tau}{n}\langle h_{ij}, h_{ij}\rangle - \frac{\tau}{2n}\langle h_i{}^i, h_j{}^j\rangle.$$

But here $\delta h=0$, $\alpha h=0$ and $\langle \mathrm{tr}\, h, 1\rangle=0$, which implies $\Delta\, \mathrm{tr}\, h-\frac{\tau}{n}\mathrm{tr}\, h=0$. Therefore

$$-\langle\nabla_i h_j{}^j, \nabla^k h_{ki}\rangle+\frac{1}{2}\langle\nabla_i h_j{}^j, \nabla_i h_k{}^k\rangle-\frac{\tau}{2n}\langle h_i{}^i, h_j{}^j\rangle$$
$$=\langle d(\mathrm{tr}\, h), \delta h\rangle+\frac{1}{2}\langle d(\mathrm{tr}\, h), d(\mathrm{tr}\, h)\rangle-\frac{\tau}{2n}\langle \mathrm{tr}\, h, \mathrm{tr}\, h\rangle$$
$$=\frac{1}{2}\langle\Delta\, \mathrm{tr}\, h, \mathrm{tr}\, h\rangle-\frac{\tau}{2n}\langle \mathrm{tr}\, h, \mathrm{tr}\, h\rangle=0\,.$$

Moreover we see

$$\langle\nabla_j h_{ik}, \nabla_k h_{ji}\rangle=-\langle\nabla^k\nabla_j h_{ik}, h_{ji}\rangle\,,$$

and

$$\nabla^k\nabla_j h_{ik}=R^k{}_j{}^l{}_i h_{lk}+R^k{}_j{}^l{}_k h_{il}+\nabla_j\nabla^k h_{ik}$$
$$=(Lh)_{ji}+\frac{\tau}{n}h_{ij}\,.$$

Thus we get

$$\langle\tau, 1\rangle''=-\left\langle Lh+\frac{\tau}{n}h, h\right\rangle-\frac{1}{2}\langle\nabla h, \nabla h\rangle+\frac{\tau}{n}\langle h, h\rangle$$
$$=-\frac{1}{2}\langle\bar{\Delta}h+2Lh, h\rangle\,.$$

We remark that the equation $\Delta\, \mathrm{tr}\, h-\frac{\tau}{n}\mathrm{tr}\, h=0$ implies $\mathrm{tr}\, h=0$. In fact, by Remark 2.2, the first positive eigenvalue of $\Delta \geqq \frac{\tau}{n-1}$. Therefore if $\tau\neq 0$ then $\mathrm{tr}\, h=0$. Even if $\tau=0$, $\mathrm{tr}\, h$ is constant. But here h is volume preserving, i.e., $\langle h, g\rangle=0$, and so $\mathrm{tr}\, h=0$. Now we define the operator $\mathcal{S}\nabla\colon C^\infty(S^2)\to C^\infty(T_3^0)$, $S\nabla\colon C^\infty(S^2)\to C^\infty(T_3^0)$ by

$$(\mathcal{S}\nabla\psi)(X, Y, Z)=u(\nabla_X\psi)(Y, Z)+v(\nabla_Y\psi)(Z, X)+w(\nabla_Z\psi)(X, Y)\,,$$
$$(S\nabla\psi)(X, Y, Z)=(\nabla_Y\psi)(Z, X)\,,$$

where $u, v, w\in \boldsymbol{R}$, $u^2+v^2+w^2=1$. Set $p=uv+vw+wu$. Then the minimum and maximum of p is $-\frac{1}{2}$ and 1, respectively. By simple computations we have

$$\langle\mathcal{S}\nabla\psi, \mathcal{S}\nabla\psi\rangle=\langle\nabla\psi, \nabla\psi\rangle+2p\langle S\nabla\psi, \nabla\psi\rangle=\langle\bar{\Delta}\psi, \psi\rangle+2p\langle\delta S\nabla\psi, \psi\rangle\,,$$

and $(\delta S\nabla\psi)_{ij}=-\nabla^k\nabla_i\psi_{jk}=-(L\psi)_{ij}-\rho_i{}^k\psi_{jk}+(\nabla\delta\psi)_{ij}\,.$

Therefore we get $\langle\bar{\Delta}\psi-2pL\psi-\frac{2}{n}\tau p\psi+2p\nabla\delta\psi,\ \psi\rangle\geqq 0$. Thus, since $\delta h=0$, we see

$$\langle\bar{\Delta}h+2Lh,\ h\rangle\geqq 2\left\langle(1+p)Lh+\frac{\tau}{n}ph,\ h\right\rangle$$
$$\geqq 2\left\{(1+p)\alpha_0+\frac{\tau}{n}p\right\}\langle h,\ h\rangle .$$

If $\alpha_0>\frac{\tau}{n}$, we assume $p=-\frac{1}{2}$. Then $(1+p)\alpha_0+\frac{\tau}{n}p=\frac{1}{2}\alpha_0-\frac{\tau}{2n}>0$. If $\alpha_0>-\frac{\tau}{2n}$, we assume $p=1$. Then $(1+p)\alpha_0+\frac{\tau}{n}p=2\alpha_0+\frac{\tau}{n}>0$, which completes the proof.

REMARK 2.6. We have many examples of Einstein metrics which satisfy the condition of the operator L in Theorem 2.5. (See [5].)

i) ([5, Proposition 3.4]) An Einstein metric which is decomposed locally to the riemannian manifolds with negative sectional curvature.

ii) ([5, Corollary 3.7]) An Einstein metric whose sectional curvature ranges in the interval $\left(\frac{n-2}{2n-1},\ 1\right]$.

iii) ([5, Corollary 3.5]) An Einstein metric which is decomposed locally to the irreducible symmetric spaces of non-comabt type of dimension >2.

iv) ([5, Table 1, Table 2]) The irreducible symmetric spaces of the following types.

AIII	$SU(p+1)/S(U_p\times U_1)$	
BDI	$SO(p+q)/SO(p)\times SO(q)$	$(p\geqq 3,\ q=1)$, $(p\geqq q\geqq p-1,\ p+q\geqq 7)$
CII	$Sp(p+1)/Sp(p)\times Sp(1)$	$(p\neq 2)$
DIII	$SO(2p)/U(p)$	$(p\geqq 6)$
EVII	$E_7/E_6\times T^1$	

3. The case of Kähler deformation

Let (M, J) be a compact complex manifold of dimension $m=\frac{n}{2}$. We consider 1-parameter families of Kähler metrics on (M, J), which we call *Kähler deformations*. For a Kähler metric g on M, we denote by ω and $\tilde{\rho}$ the Kähler form and Ricci form of g, respectively, i.e., $\omega_{ij}=J_{ij}$, $\tilde{\rho}_{ij}=\rho_{ik}J^k{}_j$. If $g(t)$ is a Kähler deformation, then ω' is a closed real 2-form and $\tilde{\rho}'$ is a 0-cohomologous closed real 2-form. First we will show some formulae. For the integral of closed forms, recall that the exterior product of a closed form and 0-cohomologous form is 0-cohomologous, and the integral of a 0-cohomologous $2m$-form vanishes. By easy computation we see

$$\omega^m = m! v_g , \tag{3.1}$$

$$\phi \wedge \omega^{m-1} = \frac{1}{2m}(\phi, \omega)\omega^m , \tag{3.2}$$

$$\phi \wedge \psi \wedge \omega^{m-2} = \frac{1}{4m(m-1)}\{(\phi, \omega)(\psi, \omega) - 2(\phi, \psi)\}\omega^m , \tag{3.3}$$

where ϕ and ψ are 2-forms on M. By intergrating both sides of (3.1), we get

$$\int_M v_g = \frac{1}{m!}\int_M \omega^m , \tag{3.4}$$

and so
$$\left(\int v_g\right)' = \frac{1}{(m-1)!}\int \omega' \wedge \omega^{m-1} , \tag{3.5}$$

$$\left(\int v_g\right)'' = \frac{1}{(m-1)!}\int \omega'' \wedge \omega^{m-1} + \frac{1}{(m-2)!}\int \omega' \wedge \omega' \wedge \omega^{m-2} . \tag{3.6}$$

Since $\tau = (\tilde{\rho}, \omega)$, we see, by the formula (3.2),

$$\int \tau v_g = \frac{2}{(m-1)!}\int \tilde{\rho} \wedge \omega^{m-1} , \tag{3.7}$$

$$\left(\int \tau v_g\right)' = \frac{2}{(m-1)!}\int \tilde{\rho}' \wedge \omega^{m-1} + \frac{2}{(m-2)!}\int \tilde{\rho} \wedge \omega' \wedge \omega^{m-2} .$$

We assume the deformation is a Kähler deformation, hence $\tilde{\rho}'$ is 0-cohomologous, which implies

$$\left(\int \tau v_g\right)' = \frac{2}{(m-2)!}\int \tilde{\rho} \wedge \omega' \wedge \omega^{m-2} , \tag{3.8}$$

$$\left(\int \tau v_g\right)'' = \frac{2}{(m-2)!}\int \tilde{\rho} \wedge \omega'' \wedge \omega^{m-2} + \frac{2}{(m-3)!}\int \tilde{\rho} \wedge \omega' \wedge \omega' \wedge \omega^{m-3} .$$

Moreover if $\tilde{\rho}$ is cohomologous to $\varepsilon\omega$ for some real number ε, then

$$\left(\int v \tau_g\right)'' = \frac{2\varepsilon}{(m-2)!}\int \omega'' \wedge \omega^{m-1} + \frac{2\varepsilon}{(m-3)!}\int \omega' \wedge \omega' \wedge \omega^{m-2} . \tag{3.9}$$

By the formulae (3.4) and (3.7), if g_1 and g_2 are Kähler metrics such that their Kähler forms ω_1 and ω_2 are cohomologous to each other, then $\int_M v_{g_1} = \int_M v_{g_2}$ and $\int_M \tau_{g_1} v_{g_1} = \int_M \tau_{g_2} v_{g_2}$. Therefore we can consider "critical classes" for the scalar curvature.

Theorem 3.1. *Let (M, g) be a Kähler manifold. Then*

$$\left.\frac{d}{dt}\right|_0 \int \tau_{g(t)} v_{g(t)} = 0$$

for any volume preserving deformation $g(t)$ of g, if and only if there exists a real number ε such that $\tilde{\rho}$ is cohomologous to $\varepsilon\omega$.

Proof. If $\tilde{\rho}$ is cohomologous to $\varepsilon\omega$, then the formula (3.8) implies $\left(\int \tau v_g\right)' = \frac{2\varepsilon}{(m-2)!}\int \omega' \wedge \omega^{m-1}$. On the other hand, the deformation is volume preserving, hence the formula (3.5) implies $\int \omega' \wedge \omega^{m-1} = 0$. Thus we see $\left(\int \tau v_g\right)' = 0$.

If $\left(\int v\tau_g\right)' = 0$ for any volume preserving Kähler deformation, $\int \tilde{\rho} \wedge \phi \wedge \omega^{m-2} = 0$ for any closed real 2-form ϕ such that $\int \phi \wedge \omega^{m-1} = 0$. Thus if we define 1-forms p and q on the space of all real closed 2-forms by $p(\phi) = \int \tilde{\rho} \wedge \phi \wedge \omega^{m-2}$ and $q(\phi) = \int \phi \wedge \omega^{m-1}$ then there is a real number c such that $p = cq$, i.e.,

$$\int \phi \wedge (\tilde{\rho} - c\omega) \wedge \omega^{m-2} = 0$$

for all real closed 2-forms ϕ. Let ψ be the harmonic part of $\tilde{\rho} - c\omega$. Then $\Delta(\psi, \omega) = (\Delta\psi, \omega) = 0$ and so (ψ, ω) is constant. Moreover, by the formula (3.3),

$$0 = \int \phi \wedge (\tilde{\rho} - c\omega) \wedge \omega^{m-2} = \int \phi \wedge \psi \wedge \omega^{m-2},$$

and

$$\int \{(\phi, \omega)(\psi, \omega) - 2(\phi, \psi)\}\omega^m = 0 .$$

We set $\phi = (\psi, \omega)\omega - 2\psi$, then

$$\int ((\psi, \omega)\omega - 2\psi, (\psi, \omega)\omega - 2\psi)\omega^m = 0,$$

which implies $(\psi, \omega)\omega - 2\psi = 0$ and so $\tilde{\rho}$ is cohomologous to $\left\{c + \frac{1}{2}(\psi, \omega)\right\}\omega$.

Theorem 3.2. *If there exists a positive (resp. negative) number ε such that $\tilde{\rho}$ is cohomologous to $\varepsilon\omega$, then $\left.\frac{d^2}{dt^2}\right|_0 \int \tau_{g(t)} v_{g(t)}$ is positive (resp. negative) for all volume preserving Kähler deformations $g(t)$ such that $\omega'(0)$ is not 0-cohomologous.*

Proof. Since $g(t)$ is volume preserving, the formulae (3.5) and (3.6) implies $\int \omega' \wedge \omega^{m-1} = 0$ and $\int \omega'' \wedge \omega^{m-1} + (m-1)\int \omega' \wedge \omega' \wedge \omega^{m-2} = 0$. Then, by the formula (3.9),

$$\left(\int \tau v_g\right)'' = -\frac{2\varepsilon}{(m-2)!}\int \omega' \wedge \omega' \wedge \omega^{m-2} .$$

Let ψ be the harmonic part of ω'. Then, by the formula (3.3),

$$\left(\int \tau v_g\right)'' = -\frac{2\varepsilon}{(m-2)!}\int \psi\wedge\psi\wedge\omega^{m-2}$$
$$= -\frac{\varepsilon}{2m!}\int\{(\psi,\omega)(\psi,\omega)-2(\psi,\psi)\}\,\omega^m .$$

But here (ψ, ω) is constant and $\int \psi\wedge\omega^{m-1}=0$, hence $(\psi, \omega)=0$. Thus we see

$$\left(\int \tau v_g\right)'' = \frac{\varepsilon}{m!}\int(\psi,\psi)\omega^m .$$

Since ω' is not 0-cohomologous, we see ψ is a non-zero real 2-form, which completes the proof.

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