ON THE SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR CERTAIN UNIVALENT FUNCTIONS

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Abstract. In this paper, we investigate the sharp bounds of the second Hankel determinant of Logarithmic coefficients for the starlike and convex functions with respect to symmetric points in the open unit disk.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Then $\mathcal{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ such that $f(0) = 0$ and $f'(0) = 1$. A function f is said to be *univalent* in a domain $\Omega \subseteq \mathbb{C}$, if it is one-to-one in Ω. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent (*i.e., one-to-one*) in $\mathbb{D}$. If $f \in \mathcal{S}$ then it has the following series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

A function $f \in \mathcal{S}$ belongs to the class $\mathcal{S}^*$, called *starlike function*, if $f(\mathbb{D})$ is a starlike domain with respect to the origin. Moreover, a function $f \in \mathcal{S}$ is called *convex function* if $f(\mathbb{D})$ is a starlike domain with respect to each point. The class of such functions is denoted by $\mathcal{C}$.

In [19], Sakaguchi introduced the class of functions that are starlike with respect to symmetric points. A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points* if for any r close to 1, $r < 1$, and any z_0 on the circle $|z| = r$, the angular velocity of $f(z)$ about the point $f(-z_0)$ is positive at z_0 as z traverses the circle $|z| = r$ in the positive direction, *i.e.*,

$$\operatorname{Re}\left(\frac{z_0 f'(z_0)}{f(z_0) - f(-z_0)}\right) > 0, \quad |z_0| = r.$$

Denote by $\mathcal{S}_S^*$ the class of all functions in $\mathcal{S}$ which are starlike with respect to symmetric points and, functions f in the class $\mathcal{S}_S^*$ is characterized by

$$\operatorname{Re}\left(\frac{z f'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in \mathbb{D}.$$

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It is known that the functions in $\mathcal{S}_S^*$ are close-to-convex and hence are univalent (see [19]). Note that the class of functions starlike with respect to symmetric points obviously includes the classes of convex functions and odd starlike functions with respect to the origin. The notion of starlike functions with respect to N-symmetric points has been studied in [19]. In 2002, Nezhmetdinov and Ponnusamy [14] proved that $\mathcal{S}_S^* \nsubseteq \mathcal{S}^*$ and $\mathcal{S}^* \nsubseteq \mathcal{S}_S^*$.

In 1977, Das and Singh [7] defined the class of convex functions with respect to symmetric points. A function $f \in \mathcal{A}$ is said to be *convex with respect to symmetric points* if, and only if,

$$\operatorname{Re}\left(\frac{(zf'(z))'}{(f(z)-f(-z))'}\right) > 0, \quad z \in \mathbb{D}.$$

The *Logarithmic coefficients* γ_n of $f \in \mathcal{S}$ are defined by,

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

The logarithmic coefficients γ_n play a central role in the theory of univalent functions. A very few exact upper bounds for γ_n seem to have been established. The significance of this problem in the context of Bieberbach conjecture was pointed by Milin [13] in his conjecture. Milin [13] conjectured that for $f \in \mathcal{S}$ and $n \geq 2$,

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0,$$

which led De Branges, by proving this conjecture, to the proof of Bieberbach conjecture [5]. For the Koebe function $k(z) = z/(1-z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function k plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $|\gamma_n| \leq 1/n$ holds for functions in $\mathcal{S}$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \neq O(n^{-0.83})$ (see [8, Theorem 8.4]). By differentiating (1.2) and the equating coefficients we obtain

$$\begin{aligned}
\gamma_1 &= \frac{1}{2}a_2, \\
\gamma_2 &= \frac{1}{2}(a_3 - \frac{1}{2}a_2^2), \\
\gamma_3 &= \frac{1}{2}(a_4 - a_2a_3 + \frac{1}{3}a_2^3).
\end{aligned} \tag{1.3}$$

If $f \in \mathcal{S}$, it is easy to see that $|\gamma_1| \leq 1$, because $|a_2| \leq 2$. Using the Fekete-Szegö inequality [8, Theorem 3.8] for functions in $\mathcal{S}$ in (1.4), we obtain the sharp estimate

$$|\gamma_2| \leq \frac{1}{2}\left(1 + 2e^{-2}\right) = 0.635\ldots.$$

For $n \geq 3$, the problem seems much harder, and no significant bound for $|\gamma_n|$ when $f \in \mathcal{S}$ appear to be known. In 2017, Ali and Allu [1] obtained the initial logarithmic coefficients bounds for close-to-convex functions. In 2020, Ponnusamy *et al.* [17] computed the sharp estimates for the initial three logarithmic coefficients for a subclass of $\mathcal{S}^*$. The problem of computing the bound of the logarithmic coefficients is also considered in [6, 18, 21] for several subclasses of close to convex functions. In 2021, Zaprawa [22] obtained the sharp bounds of the initial logarithmic coefficients $|\gamma_n|$ for functions in the classes $\mathcal{S}^*_S$ and $\mathcal{K}_S$.

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of Taylor's coefficients of function $f \in \mathcal{A}$ of the form (1.1) is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

The Hankel determinant for various order is also studied recently by several authors in different contexts; for instance see [3, 15, 16, 20]. One can easily observe that the Fekete-Szegö functional is the second Hankel determinant $H_{2,1}(f)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real for $f \in \mathcal{S}$ [8, Theorem 3.8].

Identifying the widespread applications of logarithmic coefficients, recently, Kowalczyk and Lecko [12] together proposed the study of the Hankel determinant whose entries are logarithmic coefficients of $f \in \mathcal{S}$, which is given by

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Kowalczyk and Lecko [12] obtained the sharp bound of second Hankel determinant of $F_f/2$, *i.e.*, $H_{2,1}(F_f/2)$ for starlike and convex functions. The problem of computing the sharp bounds of $H_{2,1}(F_f/2)$ has been considered in [4] for various subclasses of $\mathcal{S}$.

Suppose that $f \in \mathcal{S}$ given by (1.1). Then the second Hankel determinant of $F_f/2$ by using (1.3), is given by

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = a_2a_4 - a_3^2 + \frac{1}{12}a_2^4. \tag{1.4}$$

In this paper, we calculate the sharp bounds for $H_{2,1}(F_f/2)$ for functinos in the classes $\mathcal{S}^*_S$ and $\mathcal{K}_S$. We also provide examples of functions to illustrate these results.

2. Main Results

Let $\mathcal{B}_0$ denote the class of analytic functions $w : \mathbb{D} \to \mathbb{D}$ such that $w(0) = 0$. Functions in $\mathcal{B}_0$ are known as Schwarz functions. A function $w \in \mathcal{B}_0$ can be written as a power series

$$w(z) = \sum_{n=1}^{\infty} c_n z^n.$$

For two functions f and g that are analytic in a domain $\mathbb{D}$, we say that the function f is *subordinate* to g in $\mathbb{D}$ and written as $f(z) \prec g(z)$ if there exists a Schwarz function $w \in \mathcal{B}_0$ such that

$$f(z) = g(w(z)), \quad z \in \mathbb{D}.$$

In particular, if the function g is univalent in $\mathbb{D}$, then $f \prec g$ if, and only if, $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

To prove our results, we need the following lemma for Schwarz functions.

Lemma 2.1. *[9] Let $w(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_1| \leq 1,\ |c_2| \leq 1 - |c_1|^2, \quad \text{and } |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.$$

We obtain the following sharp bound for $H_{2,1}(F_f/2)$ for functions in the class $\mathcal{S}_S^*$.

Theorem 2.2. *Let $f \in \mathcal{S}_S^*$. Then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{4}.$$

The inequality is sharp.

Proof. Let $f \in \mathcal{S}_S^*$ be of the form (1.1). Then by the definition of subordination there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 + w(z)}{1 - w(z)}. \tag{2.1}$$

By comparing the coefficients on both sides of (2.1) yields

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= c_2 + c_1^2, \\ a_4 &= \frac{1}{2}\left(c_3 + 3c_1c_2 + 2c_1^3\right). \end{aligned} \tag{2.2}$$

By substituting the above expression for a_2, a_3, and a_4 in (1.4) and then further simplification gives

$$\begin{aligned} H_{2,1}(F_f/2) &= \gamma_1\gamma_3 - \gamma_2^2 \\ &= a_2a_4 - a_3^2 + \frac{1}{12}a_2^4 \\ &= \frac{1}{48}\left(c_1^4 + 6c_1c_3 - 12c_2^2 - 6c_1^2c_2\right). \end{aligned} \tag{2.3}$$

From (2.3) and Lemma 2.1, we obtain

$$48|H_{2,1}(F_f/2)| \leq |c_1|^4 + 6|c_1|\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 6|c_1|^2|c_2| + 12|c_2|^2. \tag{2.4}$$

Now writing $x = |c_1|$ and $y = |c_2|$ in (2.4), we obtain

$$48|H_{2,1}(F_f/2)| \leq F(x,y), \tag{2.5}$$

where

$$F(x,y) = x^4 + 6x\left(1 - x^2 - \frac{y^2}{1+x}\right) + 6x^2y + 12y^2.$$

In view of Lemma 2.1, the region of variability of a pair (x,y) coincides with the set

$$\Omega = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}.$$

Therefore, we need to find the maximum value of $F(x,y)$ over the region Ω. The critical points of F satisfies the conditions

$$\frac{\partial F}{\partial x} = 4x^3 - 18x^2 + 12xy - \frac{6y^2}{(1+x)^2} + 6 = 0$$
$$\frac{\partial F}{\partial y} = x^2 + x^3 + 4y + 2xy = 0,$$

which has no solution in the interior of Ω. Hence the function $F(x,y)$ cannot have a maximum in the interior of Ω. Since F is continuous on a compact set Ω, the maximum of F attains boundary of Ω. On the boundary of Ω, we have

$$F(x,0) = x^4 - 6x^3 + 6x \leq 2.4378 \text{ for } 0 \leq x \leq 1,$$
$$F(0,y) = 12y^2 \leq 12 \text{ for } 0 \leq y \leq 1,$$

and

$$F(x, 1-x^2) = x^4 - 12x^2 + 12 \leq 12 \text{ for } 0 \leq x \leq 1.$$

Thus combining all the above cases we obtain

$$\max_{(x,y)\in\Omega} F(x,y) = 12$$

and hence from (2.5) we have

$$|H_{2,1}(F_f/2)| \leq \frac{1}{4}. \tag{2.6}$$

To prove the equality in (2.6), we consider the function

$$f_1(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots, \quad z \in \mathbb{D}.$$

A simple computation shows that f_1 belongs to the class $\mathcal{S}_S^*$ and $|H_{2,1}(F_{f_1}/2)| = 1/4$ and hence equality holds in (2.6). This completes the proof. □

Here we provide an example that associates to Theorem 2.2.

Example 2.3. Consider the function

$$f_2(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots$$

It is easy to see that the function f belongs to the class $\mathcal{S}_S^*$. It is easy to see that

$$|H_{2,1}(F_{f_2}/2)| = \frac{1}{12} \leq \frac{1}{4}.$$

In the following result, we estimate the sharp bound for $H_{2,1}(F_f/2)$ for functions in the class $\mathcal{K}_S$.

Theorem 2.4. *Let $f \in \mathcal{K}_S$ be of the form* (1.1). *Then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{36}.$$

The inequality is sharp.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in $\mathcal{K}_S$, then there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} c_n z^n$ such that

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = \frac{1+w(z)}{1-w(z)}. \tag{2.7}$$

First note that by equating coefficients in (2.7) we have,

$$\begin{aligned} a_2 &= \frac{1}{2}c_1, \\ a_3 &= \frac{1}{3}\left(c_2 + c_1^2\right), \\ a_4 &= \frac{1}{8}\left(c_3 + 3c_1c_2 + 2c_1^3\right). \end{aligned} \tag{2.8}$$

A simple computation using (1.4) gives,

$$H_{2,1}(F_f/2) = \frac{1}{2304}\left(11c_1^4 + 36c_1\left(1 - c_1^2 - \frac{c_2^2}{1+c_1}\right) + 20c_1^2c_2 + 64c_2^2\right). \tag{2.9}$$

Following the same method as used in the proof of Theorem 2.2, we obtain

$$|H_{2,1}(F_f/2)| \leq \frac{1}{2304}\left(11|c_1|^4 + 36|c_1|\left(1 - |c_1|^2 - \frac{|c_2|^2}{1+|c_1|}\right) + 20|c_1|^2|c_2| + 64|c_2|^2\right), \tag{2.10}$$

where

$$0 \leq |c_1| \leq 1 \text{ and } 0 \leq |c_2| \leq 1 - |c_1|^2.$$

Now by replacing $|c_1|$ by x and $|c_2|$ by y in (2.10) gives

$$2304|H_{2,1}(F_f/2)| \leq G(x,y), \tag{2.11}$$

where

$$G(x,y) = 11x^4 + 36x\left(1 - x^2 - \frac{y^2}{1+x}\right) + 20x^2y + 64y^2.$$

In view of Lemma 2.1, the region of variability of a pair (x, y) coincides with the set

$$\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1 - x^2\}.$$

Thus we need to find the maximum value of $G(x, y)$ over the region Ω. The critical points of G satisfies the conditions

$$\frac{\partial G}{\partial x} = 44x^3 - 108x^2 + 40xy - \frac{36y^2}{(1+x)^2} + 36 = 0,$$

and

$$\frac{\partial G}{\partial y} = 5x^2 + 5x^3 + 32y + 14xy = 0,$$

which has no solution in the interior of Ω. By using the elementary calculus, we can show that the maximum of $G(x, y)$ should exists on the boundary of Ω. It is easy to see that on the boundary line $x = 0$ and $0 \leq y \leq 1$, we have $G(0, y) = 64y^2$ and its maximum on this line is equal to 64. Similarly, on the boundary line $y = 0$ and $0 \leq x \leq 1$, we have $G(x, 0) = 11x^4 - 36x^3 + 36x$ and its maximum on this line is 15.512. Finally, on the boundary curve $y = 1 - x^2$ and $0 \leq x \leq 1$, we have $G(x, 1 - x^2) = 19x^4 - 72x^2 + 64$ and its maximum on this curve is 64. Thus, combining all the above cases yields

$$\max_{(x,y)\in\Omega} G(x, y) = 64$$

and hence from (2.11) we obtain

$$|H_{2,1}(F_f/2)| \leq \frac{1}{36}. \tag{2.12}$$

For the sharpness of the inequality (2.12) we consider the function

$$f_3(z) = \frac{1}{2}\log\frac{1+z}{1-z} = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots$$

which belongs to the class $\mathcal{K}_S$. A simple computation shows that $|H_{2,1}(F_{f_3}/2)| = 1/36$ and hence the inequality in (2.12) is sharp. This completes the proof. □

In the following example we construct a function that agree with Theorem 2.4.

Example 2.5. Consider the function

$$f_4(z) = -\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots .$$

A simple compuation shows that

$$\text{Re}\left(\frac{(zf_4'(z))'}{(f_4(z) - f_4(-z))'}\right) = \frac{1}{2}\text{Re}\left(\frac{1+z}{1-z}\right) > 0.$$

and hence the function $f_4 \in \mathcal{K}_S$. It is easy to see that

$$|H_{2,1}(F_f/2)| = \frac{11}{576} \leq \frac{1}{36}.$$

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