ON THE SECOND REALIZATION FOR THE POSITIVE PART OF $U_q(\widehat{sl_2})$ OF EQUITABLE TYPE

PASCAL BASEILHAC

ABSTRACT. The equitable presentation of the quantum algebra $U_q(\widehat{sl_2})$ is considered. This presentation was originally introduced by T. Ito and P. Terwilliger. In this paper, following Terwilliger's recent works the (nonstandard) positive part of $U_q(\widehat{sl_2})$ of equitable type $U_q^{IT,+}$ and its second realization (current algebra) $U_q^{T,+}$ are introduced and studied. A presentation for $U_q^{T,+}$ is given in terms of a K-operator satisfying a Freidel-Maillet type equation and a condition on its quantum determinant. Realizations of the K-operator in terms of Ding-Frenkel L-operators are considered, from which an explicit injective homomorphism from $U_q^{T,+}$ to a subalgebra of Drinfeld's second realization (current algebra) of $U_q(\widehat{sl_2})$ is derived, and the comodule algebra structure of $U_q^{T,+}$ is characterized. The central extension of $U_q^{T,+}$ and its relation with Drinfeld's second realization of $U_q(\widehat{gl_2})$ is also described using the framework of Freidel-Maillet algebras.

MSC: 16T25; 17B37; 81R50.
Keywords: $U_q(\widehat{sl_2})$; Equitable presentation; FRT presentation; Freidel-Maillet algebras

1. Introduction

Originally introduced in [J85, D86], the quantum affine algebra $U_q(\widehat{sl_2})$ admits a presentation in terms of generators $\{E_i, F_i, K_i^{\pm 1} | i = 0, 1\}$ and relations. In the literature, this presentation is usually referred as the *Drinfeld-Jimbo* or *Chevalley* type presentation of $U_q(\widehat{sl_2})$, denoted U_q^{DJ}. V. Drinfeld gave also another presentation [D88], the so-called *Drinfeld's second realization* of $U_q(\widehat{sl_2})$ in terms generators $\{\mathsf{x}_k^{\pm}, \mathsf{h}_\ell, \mathsf{K}^{\pm 1}, C^{\pm 1/2} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$ and relations, denoted U_q^{Dr}. For further analysis, both U_q^{DJ} and U_q^{Dr} are recalled in Appendix A. A third presentation, initiated by Reshetikhin-Semenov-Tian-Shansky in [RS90] and denoted U_q^{RS}, takes the form of a *Faddeev-Reshetikhin-Takhtajan (FRT)* type presentation [FRT89]. In this case, generating functions for the generators of U_q^{Dr} (and more generally Drinfeld's second realization of $U_q(\widehat{gl_2})$) are the entries of the so-called L-operators, see [DF93] for details. In these definitions, note that the derivation generator is ommited (see [CP94, Remark 2, page 393]). In the context of mathematics and physics, the presentation U_q^{DJ} and especially U_q^{Dr}, U_q^{RS}, have played a crucial role in developments of quantum affine algebras, conformal field theory and integrable lattice systems.

In [IT03], T. Ito and P. Terwilliger obtained a fourth presentation of $U_q(\widehat{sl_2})$ called 'equitable', here denoted U_q^{IT}, see Theorem 4.3. It is generated by $\{y_i^{\pm}, k_i^{\pm} | i = 0, 1\}$ subject to the defining relations (2.1)-(2.3). An explicit isomorphism $U_q^{IT} \to U_q^{DJ}$ is known [IT03], see (2.4)-(2.6). To our knowledge, the relationship between U_q^{IT}, U_q^{Dr} and U_q^{RS} has not been investigated. As a starting point, in this paper we consider the subalgebra of U_q^{IT} generated by $\{y_0^+, y_1^+\}$. We denote this subalgebra by $U_q^{IT,+}$ and call it the (nonstandard) positive part of U_q^{IT}. It is known that $U_q^{IT,+}$ has a presentation by generators $\{y_0^+, y_1^+\}$ subject to the q-Serre relations; see (2.3). In a recent work [T19a], P. Terwilliger gave a second realization - called 'alternating' - for an algebra with q-Serre defining relations. Adapting the results and notations of [T19a] to $U_q^{IT,+}$, we introduce *Terwilliger's second realization of* $U_q^{IT,+}$, denoted $U_q^{T,+}$. It has equitable generators $\{y_{-k}^+, y_{k+1}^+, z_{k+1}^+, \tilde{z}_{k+1}^+ | k \in \mathbb{N}\}$ subject to a set of relations displayed in Theorem 2.7. A PBW basis for $U_q^{T,+}$ is given in Proposition 2.9. For completeness, following [T19b] the central extension of $U_q^{T,+}$, denoted $\mathcal{U}_q^{T,+}$, is considered in the last section. See Definitions 4.1, 4.2.

The purpose of this paper is to study the relationship between $U_q^{T,+}$, its central extension $\mathcal{U}_q^{T,+}$ and certain subalgebras of U_q^{RS}, U_q^{Dr} and $U_q(\widehat{gl_2})$'s counterparts. The main result is a Freidel-Maillet type presentation [FM91] for $U_q^{T,+}$, see Theorem 2.10. In this presentation, the generators of $U_q^{T,+}$ arise as coefficients of generating functions characterizing the entries of a K-operator that satisfies a Freidel-Maillet type equation and a quantum determinant equation. A K-operator that reads as a quadratic combination of L-operators of U_q^{RS} (known in the literature as Sklyanin's dressed operators [Sk88]) is derived, see Lemma 3.5 and (3.37). Using this relation between K and L-operators, the following results are obtained in a straightforward manner: generating functions of equitable generators of $U_q^{T,+}$ and Drinfeld generators of U_q^{Dr} are related, see Proposition 3.6 and Example 3.7; $U_q^{T,+}$ is interpreted as a comodule algebra. See Proposition 3.10 and Lemma 3.11. Relaxing the condition on the quantum determinant, the central extension $\mathcal{U}_q^{T,+}$ is studied along the same lines using a Freidel-Maillet type presentation. See Theorem 4.3, Propositions 4.4, 4.9 and Corollary 4.10.

The text is organized as follows. In Section 2, the equitable presentation U_q^{IT}, its nonstandard positive part $U_q^{IT,+}$ and Terwilliger's second realization $U_q^{T,+}$ are introduced. Then, following recent results [B20] a Freidel-Maillet type presentation for $U_q^{T,+}$ is proposed. In Section 3, the analysis of [B20, Subsection 5.2] is extended: K-operator solutions of a Freidel-Maillet type equation are constructed, from which an injective homomorphism $\nu : U_q^{T,+} \to {U'_q}^{Dr,\triangleright,+}$ is derived, where ${U'_q}^{Dr,\triangleright,+}$ is a subalgebra for U_q^{Dr}. Using the Freidel-Maillet type presentation, it is also shown that $U_q^{T,+}$ admits a (left) comodule algebra structure $\delta : U_q^{T,+} \to {U'_q}^{Dr,\triangleright,+} \otimes U_q^{T,+}$. For the specialization $C = 1$, the image of the equitable generators by the corresponding (left) coaction map is given. In Section 4, for completeness a Freidel-Maillet type presentation for $\mathcal{U}_q^{T,+}$ is given. An injective homomorphism $\mu : \mathcal{U}_q^{T,+} \to U'_q(\widehat{gl_2})^{\triangleright,+}$ is derived, where $U'_q(\widehat{gl_2})^{\triangleright,+}$ is a subalgebra of $U_q(\widehat{gl_2})$. In particular, in terms of the Drinfeld's generators of $U_q(\widehat{gl_2})$ the image of the quantum determinant by μ enjoys a simple factorized structure, see (4.18) or (4.20). In the last section, the results here presented together with [B20] are summarized and some perspectives are given.

Nota bene. In a recent paper [Ter21], the relationship between the positive part of $U_q(\widehat{sl_2})$ denoted U_q^+ (see comments around eq. (2.7)) and its central extension $\mathcal{U}_q^+$ is studied in details using the framework of generating functions. Explicit relations between generating functions in terms of Damiani's root vectors for U_q^+ and generating functions for the alternating generators of $\mathcal{U}_q^+$ are obtained. For the choice $\bar{\epsilon}_\pm = 0$, fixing $\bar{k}_\pm$ and λ according to the normalizations chosen in [Ter21] and using Beck's correspondence between Drinfeld's generators and root vectors [Be94] (see (3.49)-(3.50)), it can be readily checked that the expressions given in Proposition 4.9 with (3.39)-(3.42), and eq. (4.20) match with the expressions given in [Ter21, Propositions 9.1, 9.3] and [Ter21, eq. (65)], respectively.

Notation 1. *Recall the natural numbers* $\mathbb{N} = \{0, 1, 2, \cdots\}$ *and integers* $\mathbb{Z} = \{0, \pm1, \pm2, \cdots\}$. $\mathbb{C}(q)$ *denotes the field of rational functions in an indeterminate* q. *The q-commutator* $\big[X, Y\big]_q = qXY - q^{-1}YX$ *is introduced. We denote* $[x] = (q^x - q^{-x})/(q - q^{-1})$.

2. The equitable subalgebra $U_q^{IT,+}$ and second realization $U_q^{T,+}$

In this section, the equitable presentation of $U_q(\widehat{sl_2})$ introduced in [IT03] is recalled, and an isomorphism $U_q^{IT} \to U_q^{DJ}$ is displayed. Then, the positive part $U_q^{IT,+}$ is considered. For its second realization $U_q^{T,+}$ recently introduced in [T18], some properties are recalled. Following [B20] a Freidel-Maillet type presentation is given for $U_q^{T,+}$.

Theorem 2.1. [IT03] *The quantum affine algebra* $U_q(\widehat{sl_2})$ *is isomorphic to the unital associative* $\mathbb{C}(q)$*-algebra with equitable generators* $\{y_i^\pm, k_i^{\pm1} | i = 0, 1\}$ *and the following relations:*

$$k_i k_i^{-1} = k_i^{-1} k_i \quad = \quad 1, \qquad k_0 k_1 \quad \textit{central}, \tag{2.1}$$

$$\frac{\big[y_i^+, k_i\big]_q}{q - q^{-1}} \quad = \quad 1, \qquad \frac{\big[k_i, y_i^-\big]_q}{q - q^{-1}} = 1, \qquad \frac{\big[y_i^-, y_i^+\big]_q}{q - q^{-1}} = 1, \qquad \frac{\big[y_i^+, y_j^-\big]_q}{q - q^{-1}} = k_0^{-1} k_1^{-1}, \qquad i \neq j, \tag{2.2}$$

$$(y_i^{\pm})^3 y_j^{\pm} - [3]_q (y_i^{\pm})^2 y_j^{\pm} y_i^{\pm} + [3]_q y_i^{\pm} y_j^{\pm} (y_i^{\pm})^2 - y_j^{\pm} (y_i^{\pm})^3 = 0, \qquad i \neq j. \tag{2.3}$$

We call U_q^{IT} the *Ito-Terwilliger* or *equitable* presentation of $U_q(\widehat{sl_2})$.

An isomorphism $U_q^{IT} \to U_q^{DJ}$ is given in [IT03], where U_q^{DJ} is the Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$ recalled in Appendix A. Namely,

$$k_i^{\pm 1} \mapsto K_i^{\pm 1} , \tag{2.4}$$

$$y_i^{-} \mapsto K_i^{-1} + (q-q^{-1})F_i , \tag{2.5}$$

$$y_i^{+} \mapsto K_i^{-1} - q(q-q^{-1})K_i^{-1}E_i . \tag{2.6}$$

In this paper, we focus on the following subalgebra.

Definition 2.2. *$U_q^{IT,+}$ is the subalgebra of U_q^{IT} generated by $\{y_0^+, y_1^+\}$. We call $U_q^{IT,+}$ the positive part of $U_q(\widehat{sl_2})$ of equitable type.*

By (2.3), this subalgebra has a presentation by generators $\{y_0^+, y_1^+\}$ subject to the q-Serre relations. Let $U_q^{DJ,+}$ (resp. $U_q^{DJ,-}$) denote the subalgebra of U_q^{DJ} generated by E_0, E_1 (resp. F_0, F_1); See Appendix A. In the literature, $U_q^{DJ,+}$ (resp. $U_q^{DJ,-}$) is usually called the positive (resp. negative) part of $U_q(\widehat{sl_2})$. For this reason, the definition above of positive part of $U_q(\widehat{sl_2})$ is nonstandard. The negative part of $U_q(\widehat{sl_2})$ of equitable type - denoted by $U_q^{IT,-}$ - can be introduced similarly. It is generated by $\{y_0^-, y_1^-\}$. Another subalgebra is the 'Cartan part' denoted $U_q^{IT,0}$, generated by $\{k_0^{\pm 1}, k_1^{\pm 1}\}$.

Definition 2.3. *${U'}_q^{DJ,+}$ (resp. ${U'}_q^{DJ,-}$) denotes the subalgebra of U_q^{DJ} generated by $U_q^{DJ,+}$ (resp. $U_q^{DJ,-}$) and $\{K_0^{\pm 1}, K_1^{\pm 1}\}$.*

By (A.1), (2.3), it follows:

Remark 2.4. *An injective homomorphism $U_q^{IT,+} \to {U'}_q^{DJ,+}$ is given by (2.6).*

From the point of view of generators and relations, $U_q^{DJ,+}$ and $U_q^{IT,+}$ are exactly the same, up to isomorphism. However, it is seen that their embeddings into ${U_q}^{DJ}$ essentially differ. To avoid any confusion in further discussions, let us introduce the algebra U_q^+ with fundamental generators A, B and q-Serre defining relations:

$$\big[A, \big[A, \big[A, B\big]_q\big]_{q^{-1}}\big] = 0 \ , \quad \big[B, \big[B, \big[B, A\big]_q\big]_{q^{-1}}\big] = 0 \ . \tag{2.7}$$

According to previous definitions,

Lemma 2.5. *There exists an algebra isomorphism $U_q^+ \to U_q^{DJ,+}$ that sends $A \mapsto E_0$ and $B \mapsto E_1$.*

Lemma 2.6. *There exists an algebra isomorphism $U_q^+ \to U_q^{IT,+}$ that sends $A \mapsto y_0^+$ and $B \mapsto y_1^+$.*

For U_q^+, Terwilliger recently gave a new presentation called *alternating*. The alternating presentation consists of infinitly many countable alternating elements called the alternating generators satisfying certain relations [T18, T19a].

For $A \mapsto E_0$ and $B \mapsto E_1$, the alternating presentation produces a new 'current' realization for $U_q^{DJ,+}$ besides the known one in terms of Drinfeld generators and relations [Be94]. In this case, an explicit isomorphism between Terwilliger's alternating algebra and certain alternating subalgebras of U_q^{Dr} is established in [B20]. For the precise relation between the alternating and Drinfeld's generators, see [B20, Subsection 5.2.3]. Using the correspondence (A.13), in particular one finds $A \mapsto \mathsf{x}_1^- \mathsf{K}^{-1}$ and $B \mapsto \mathsf{x}_0^+$.

For $A \mapsto y_0^+$ and $B \mapsto y_1^+$, the alternating presentation produces similarly a new realization for $U_q^{IT,+}$.

Theorem 2.7. *(see* [T19a]*)* $U_q^{IT,+}$ *is isomorphic to the unital associative* $\mathbb{C}(q)$*-algebra with equitable generators* $\{y^+_{-k}, y^+_{k+1}, z^+_{k+1}, \tilde{z}^+_{k+1} | k \in \mathbb{N}\}$ *subject to the following relations*

$$[y_1^+, y_{-k}^+] = [y_{k+1}^+, y_0^+] = \frac{(z_{k+1}^+ - \tilde{z}_{k+1}^+)}{q+q^{-1}}, \tag{2.8}$$

$$[y_1^+, \tilde{z}_{k+1}^+]_q = [z_{k+1}^+, y_1^+]_q = \bar{\rho} y_{k+2}^+, \tag{2.9}$$

$$[\tilde{z}_{k+1}^+, y_0^+]_q = [y_0^+, z_{k+1}^+]_q = \bar{\rho} y_{-k-1}^+, \tag{2.10}$$

$$[y_{k+1}^+, y_{\ell+1}^+] = 0, \qquad [y_{-k}^+, y_{-\ell}^+] = 0, \tag{2.11}$$

$$[y_{k+1}^+, y_{-\ell}^+] + [y_{-k}^+, y_{\ell+1}^+] = 0, \tag{2.12}$$

$$[y_{k+1}^+, \tilde{z}_{\ell+1}^+] + [\tilde{z}_{k+1}^+, y_{\ell+1}^+] = 0, \tag{2.13}$$

$$[y_{k+1}^+, z_{\ell+1}^+] + [z_{k+1}^+, y_{\ell+1}^+] = 0, \tag{2.14}$$

$$[y_{-k}^+, \tilde{z}_{\ell+1}^+] + [\tilde{z}_{k+1}^+, y_{-\ell}^+] = 0, \tag{2.15}$$

$$[y_{-k}^+, z_{\ell+1}^+] + [z_{k+1}^+, y_{-\ell}^+] = 0, \tag{2.16}$$

$$[\tilde{z}_{k+1}^+, \tilde{z}_{\ell+1}^+] = 0, \qquad [z_{k+1}^+, z_{\ell+1}^+] = 0, \tag{2.17}$$

$$[z_{k+1}^+, \tilde{z}_{\ell+1}^+] + [\tilde{z}_{k+1}^+, z_{\ell+1}^+] = 0 \ , \tag{2.18}$$

and the condition $(z_0^+ = \tilde{z}_0^+ = \bar{\rho}/(q-q^{-1}))$*:*

$$\bar{\rho}(q+q^{-1}) \sum_{k=0}^{n} q^{-n+2k} y_{k+1}^+ y_{-n+k}^+ - \sum_{k=0}^{n+1} q^{2k-n-1} z_k^+ \tilde{z}_{n+1-k}^+ = 0 \ , \quad n \geq 0 \ , \tag{2.19}$$

with

$$\bar{\rho} = q^{-1}(q^2 - q^{-2})^2 \ . \tag{2.20}$$

This algebra is denoted $U_q^{T,+}$. We call $U_q^{T,+}$ *Terwilliger's second realization* of $U_q^{IT,+}$. For a proof of the above Theorem, we refer the reader to [T19a, T19b] for all details. Compared with the conventions in [T19a, T19b], the following substitutions are considered:

$$\begin{aligned} &y_{k+1}^+ \to W_{-k} \ , \quad y_{-k}^+ \to W_{k+1} \ , \\ &z_{k+1}^+ \to q^{-1}(q^2-q^{-2}) G_{k+1} \ , \quad \tilde{z}_{k+1}^+ \to q^{-1}(q^2-q^{-2}) \tilde{G}_{k+1} \ , \\ &\bar{\rho} \to q^{-1}(q^2-q^{-2})(q-q^{-1}) \ . \end{aligned}$$

Remark 2.8. *The relations (2.8)-(2.18) coincide with the defining relations for the alternating central extension of* U_q^+*, denoted* $\mathcal{U}_q^+$*, see* [T19b, Definition 3.1]*. To get* $U_q^{T,+}$ *from* $\mathcal{U}_q^+$*, the additional relation (2.19) is asserted, see* [T19b, Lemma 2.8].

Note that there exists an automorphism σ and an antiautomorphism S (see [T19a, Proposition 5.3]) such that:

$$\sigma: \quad y_{-k}^+ \mapsto y_{k+1}^+ \ , \quad y_{k+1}^+ \mapsto y_{-k}^+ \ , \quad z_{k+1}^+ \mapsto \tilde{z}_{k+1}^+ \ , \quad \tilde{z}_{k+1}^+ \mapsto z_{k+1}^+ \ , \tag{2.21}$$

$$S: \quad y_{-k}^+ \mapsto y_{-k}^+ \ , \quad y_{k+1}^+ \mapsto y_{k+1}^+ \ , \quad z_{k+1}^+ \mapsto \tilde{z}_{k+1}^+ \ , \quad \tilde{z}_{k+1}^+ \mapsto z_{k+1}^+ \ . \tag{2.22}$$

The following proposition is a straightforward adaptation of [T19b, Theorem 10.2].

Proposition 2.9. *A PBW basis for* $U_q^{T,+}$ *is obtained by its equitable generators*

$$\{y_{-k}^+\}_{k\in\mathbb{N}} \ , \quad \{z_{n+1}^+\}_{n\in\mathbb{N}} \ , \quad \{y_{\ell+1}^+\}_{\ell\in\mathbb{N}}$$

in any linear order $<$ *that satisfies*

$$y_{-k}^+ < z_{n+1}^+ < y_{\ell+1}^+ \ , \qquad k, \ell, n \in \mathbb{N} \ .$$

Combining σ, S, other examples of PBW bases can be obtained.

The equitable generators of $U_q^{T,+}$ are polynomials in y_0^+, y_1^+. Explicit expressions are obtained recursively adapting [T19a, Lemma 2.9]. For instance, besides y_0^+, y_1^+, the first generators read:

$$z_1^+ = q y_1^+ y_0^+ - q^{-1} y_0^+ y_1^+ \ , \tag{2.23}$$

$$y_{-1}^+ = \frac{1}{\bar\rho}\left((q^2+q^{-2}) y_0^+ y_1^+ y_0^+ - (y_0^+)^2 y_1^+ - y_1^+ (y_0^+)^2\right) \tag{2.24}$$

and $y_2^+ = \sigma(y_{-1}^+)$, $\tilde z_1^+ = \sigma(z_1^+)$. Thus, the algebra $U_q^{T,+}$ has a natural $\mathbb{N}^2$-grading. Define $\deg : U_q^{T,+} \to \mathbb{N}\times\mathbb{N}$. For instance, $\deg(y_0^+) = (1,0)$ and $\deg(y_1^+) = (0,1)$. More generally, $\deg(y_{-k}^+) = (k+1, k)$, $\deg(y_{k+1}^+) = (k, k+1)$, $\deg(z_{k+1}^+) = \deg(\tilde z_{k+1}^+) = (k+1, k+1)$.

The algebra $U_q^{T,+}$ admits a presentation in the form of a quadratic algebra of Freidel-Maillet type [FM91], which can be viewed as a limiting case of a reflection algebra introduced in the context of boundary quantum inverse scattering theory [C84, Sk88]. Let $R(u)$ be the quantum $R-$matrix defined by [Ba82]

$$R(u) = \begin{pmatrix} uq - u^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & u - u^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & u - u^{-1} & 0 \\ 0 & 0 & 0 & uq - u^{-1}q^{-1} \end{pmatrix} \ , \tag{2.25}$$

where u is an indeterminate, called 'spectral parameter' in the literature on integrable systems, and deformation parameter q. It is known that $R(u)$ satisfies the quantum Yang-Baxter equation in the space $\mathcal{V}_1\otimes\mathcal{V}_2\otimes\mathcal{V}_3$, with $\mathcal{V}\equiv\mathbb{C}^2$. Using the standard notation

$$R_{ij}(u) \in \mathrm{End}(\mathcal{V}_i\otimes\mathcal{V}_j), \tag{2.26}$$

the Yang-Baxter equation reads

$$R_{12}(u/v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u/v) \ . \tag{2.27}$$

As usual, intoduce the permutation operator $P = R(1)/(q-q^{-1})$. Here, note that $R_{12}(u) = PR_{12}(u)P = R_{21}(u)$. In addition to (2.25), define:

$$R^{(0)} = diag(1, q^{-1}, q^{-1}, 1) \ . \tag{2.28}$$

Define the generating functions:

$$\mathcal{Y}_+(u) = \sum_{k\in\mathbb{N}} y_{k+1}^+ U^{-k-1} \ , \quad \mathcal{Y}_-(u) = \sum_{k\in\mathbb{N}} y_{-k}^+ U^{-k-1} \ , \tag{2.29}$$

$$\mathcal{Z}_+(u) = \sum_{k\in\mathbb{N}} \tilde z_{k+1}^+ U^{-k-1} \ , \quad \mathcal{Z}_-(u) = \sum_{k\in\mathbb{N}} z_{k+1}^+ U^{-k-1} \ , \tag{2.30}$$

where the shorthand notation $U = qu^2/(q+q^{-1})$ is used. Let $\bar k_\pm \in \mathbb{C}(q)$ such that

$$\bar\rho = \bar k_+ \bar k_- (q+q^{-1})^2 \ . \tag{2.31}$$

For the alternating central extension of U_q^+, a Freidel-Maillet type presentation has been proposed in [B20, Theorem 3.1]. It is given in terms of a K-operator satisfying a Freidel-Maillet type equation. To get U_q^+, a condition on the quantum determinant of the K-operator is required.

Theorem 2.10. *The algebra $U_q^{T,+}$ has a presentation of Freidel-Maillet type. Let $K(u)$ be a square matrix such that*

$$K(u) = \begin{pmatrix} uq\mathcal{Y}_+(u) & \frac{1}{\bar k_-(q+q^{-1})}\mathcal{Z}_+(u) + \frac{\bar k_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{1}{\bar k_+(q+q^{-1})}\mathcal{Z}_-(u) + \frac{\bar k_-(q+q^{-1})}{(q-q^{-1})} & uq\mathcal{Y}_-(u) \end{pmatrix} \tag{2.32}$$

with (2.29)-(2.30). The defining relations are given by:

$$R(u/v)\ (K(u)\otimes \mathbb{I})\ R^{(0)}\ (\mathbb{I}\otimes K(v))\ =\ (\mathbb{I}\otimes K(v))\ R^{(0)}\ (K(u)\otimes \mathbb{I})\ R(u/v) \tag{2.33}$$

and[1]

$$\mathrm{tr}_{12}\big(P_{12}^{-}(K(u)\otimes \mathbb{I})\ R^{(0)}(\mathbb{I}\otimes K(uq))\big) = -\frac{\bar{\rho}}{(q-q^{-1})^2}\ . \tag{2.34}$$

Proof. By specializing some results of [B20], the proof follows. The first part of the proof concerns the equivalence between (2.8)-(2.18) and (2.33). Recall the defining relations of the alternating central extension of U_q^+ (i.e. $\mathcal{U}_q^+$) given in [B20, Definition 2.1]. Observe that they coincide with the subset of relations (2.8)-(2.18) upon the substitution:

$$\mathsf{W}_{-k}\to y_{k+1}^{+}\ ,\qquad \mathsf{W}_{k+1}\to y_{-k}^{+}\ , \tag{2.35}$$

$$\mathsf{G}_{k+1}\to \tilde{z}_{k+1}^{+},\qquad \tilde{\mathsf{G}}_{k+1}\to z_{k+1}^{+}\ . \tag{2.36}$$

Now, by [B20, Theorem 3.1] it is known that $\mathcal{U}_q^+$ admits a Freidel-Maillet type presentation given by a K-operator satisfying (2.33). So, for the K-operator (2.32), the relations (2.8)-(2.18) are equivalent to (2.33).

The second part of the proof concerns the equivalence between (2.19) and (2.34). By [B20, Proposition 3.3], the l.h.s of (2.34) is the so-called quantum determinant that generates the center of $\mathcal{U}_q^+$. For convenience, define

$$\mathcal{C}(u) = (q-q^{-1})u^2q^2\mathcal{Y}_+(u)\mathcal{Y}_-(uq) - \frac{(q-q^{-1})}{\bar{\rho}}\mathcal{Z}_-(u)\mathcal{Z}_+(uq) - \mathcal{Z}_-(u) - \mathcal{Z}_+(uq)\ . \tag{2.37}$$

Inserting (2.32) into the l.h.s. of (2.34), the quantum determinant reduces to:

$$\mathrm{tr}_{12}\big(P_{12}^{-}(K(u)\otimes \mathbb{I})\ R^{(0)}(\mathbb{I}\otimes K(uq))\big) = \frac{1}{2(q-q^{-1})}\left(\mathcal{C}(u)+\sigma(\mathcal{C}(u)) - \frac{2\bar{\rho}}{(q-q^{-1})}\right)\ . \tag{2.38}$$

Using the exchange relations between the generating functions (2.29)-(2.30) extracted from (2.33), one shows $\sigma(\mathcal{C}(u)) = \mathcal{C}(u)$. Thus, the condition (2.34) is equivalent to:

$$\mathcal{C}(u) = 0\ . \tag{2.39}$$

Extracting the set of constraints on the coefficients of the generating function $\mathcal{C}(u)$, one gets (2.19). □

Note that eqs. (2.33), (2.34), are left invariant under the transformation $(u,v)\mapsto(\lambda u,\lambda v)$ for λ invertible and $[\lambda, U_q^{T,+}]=0$. This property will be used in further analysis.

3. Relating Terwilliger's and Drinfeld's second realizations

It is natural to ask for the precise relationship between the equitable and Drinfeld's generators. As shown in this section, the Freidel-Maillet type presentation of Theorem 2.10 combined with the framework of FRT algebras [FRT89, RS90, DF93] gives a suitable framework for answering this question. In addition, it provides a tool for constructing left or right coaction maps that ensure a comodule algebra structure for $U_q^{T,+}$.

Below, as a preliminary the FRT presentation for U_q^{Dr} is first recalled, and Drinfeld type 'alternating' subalgebras $\{U_q^{Dr,a,\pm}\}$, their extensions $\{U_q'^{Dr,a,\pm}\}$ for $a=\triangleright,\triangleleft$, are introduced. Then, a K-operator satisfying a Freidel-Maillet type equation is constructed, and used to derive an injective homomorphism $\nu:\ U_q^{T,+}\to U_q'^{Dr,\triangleright,+}$. Using the comodule algebra structure of the Freidel-Maillet type presentation, a left coaction map $\delta:\ U_q^{T,+}\to U_q'^{Dr,\triangleright,+}\otimes U_q^{T,+}$ is also derived. For the specialization $\bar{\delta}:\ U_q^{T,+}\to U_q'^{Dr,\triangleright,+}/_{C=1}\otimes U_q^{T,+}$ the image of the generating functions for the equitable generators (2.29), (2.30) is given.

[1] As usual, 'tr_{12}' stands for the trace over $\mathcal{V}_1\otimes\mathcal{V}_2$. Also, we denote $P_{12}^{-}=(1-P)/2$.

3.1. **FRT presentation.** For the quantum affine Lie algebra $U_q(\widehat{gl_2})$, a FRT presentation is known [RS90, DF93]. Define the R-matrix:

$$\tilde{R}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z-1}{zq-q^{-1}} & \frac{z(q-q^{-1})}{zq-q^{-1}} & 0 \\ 0 & \frac{(q-q^{-1})}{zq-q^{-1}} & \frac{z-1}{zq-q^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

where z is an indeterminate. It is known that $\tilde{R}(z)$ satisfies the quantum Yang-Baxter equation

$$\tilde{R}_{12}(z_1/z_2)\tilde{R}_{13}(z_1)\tilde{R}_{23}(z_2) = \tilde{R}_{23}(z_2)\tilde{R}_{13}(z_1)\tilde{R}_{12}(z_1/z_2) \ . \tag{3.2}$$

In terms of $\tilde{R}(z)$, the permutation operator reads $P = \tilde{R}(1)$. Note that $\tilde{R}_{12}(z) = \tilde{R}_{21}^{t_1t_2}(z)$.

Theorem 3.1. [RS90, DF93] *$U_q(\widehat{gl_2})$ admits a FRT presentation given by a unital associative algebra with generators $\{\mathsf{x}^{\pm}_k, \mathsf{k}^{+}_{j,-\ell}, \mathsf{k}^{-}_{j,\ell}, q^{\pm c/2} | k \in \mathbb{Z}, \ell \in \mathbb{N}, j = 1, 2\}$. The generators $q^{\pm c/2}$ are central and mutally inverse. Define:*

$$L^{\pm}(z) = \begin{pmatrix} \mathsf{k}^{\pm}_1(z) & \mathsf{k}^{\pm}_1(z)\mathsf{f}^{\pm}(z) \\ \mathsf{e}^{\pm}(z)\mathsf{k}^{\pm}_1(z) & \mathsf{k}^{\pm}_2(z) + \mathsf{e}^{\pm}(z)\mathsf{k}^{\pm}_1(z)\mathsf{f}^{\pm}(z) \end{pmatrix} \tag{3.3}$$

in terms of the generating functions in the indeterminate z:

$$\mathsf{e}^{+}(z) = (q - q^{-1})\sum_{k=0}^{\infty} q^{k(c/2-1)}\mathsf{x}^{-}_{-k}z^{k} \ , \quad \mathsf{e}^{-}(z) = -(q - q^{-1})\sum_{k=1}^{\infty} q^{k(c/2+1)}\mathsf{x}^{-}_{k}z^{-k} \ , \tag{3.4}$$

$$\mathsf{f}^{+}(z) = (q - q^{-1})\sum_{k=1}^{\infty} q^{-k(c/2+1)}\mathsf{x}^{+}_{-k}z^{k} \ , \quad \mathsf{f}^{-}(z) = -(q - q^{-1})\sum_{k=0}^{\infty} q^{-k(c/2-1)}\mathsf{x}^{+}_{k}z^{-k} \ , \tag{3.5}$$

$$\mathsf{k}^{+}_j(z) = \sum_{k=0}^{\infty} \mathsf{k}^{+}_{j,-k}z^{k} \ , \qquad \mathsf{k}^{-}_j(z) = \sum_{k=0}^{\infty} \mathsf{k}^{-}_{j,k}z^{-k} \ , \quad j = 1, 2 \ . \tag{3.6}$$

The defining relations are the following:

$$\mathsf{k}^{+}_{i,0}\mathsf{k}^{-}_{i,0} = \mathsf{k}^{-}_{i,0}\mathsf{k}^{+}_{i,0} = 1 \ , \tag{3.7}$$

$$\tilde{R}(z/w)\ (L^{\pm}(z) \otimes \mathbb{I})\ (\mathbb{I} \otimes L^{\pm}(w)) = (\mathbb{I} \otimes L^{\pm}(w))\ (L^{\pm}(z) \otimes \mathbb{I})\ \tilde{R}(z/w) \ , \tag{3.8}$$

$$\tilde{R}(q^{c}z/w)\ (L^{+}(z) \otimes \mathbb{I})\ (\mathbb{I} \otimes L^{-}(w)) = (\mathbb{I} \otimes L^{-}(w))\ (L^{+}(z) \otimes \mathbb{I})\ \tilde{R}(q^{-c}z/w) \ . \tag{3.9}$$

For (3.8), the expansion direction of $\tilde{R}(z/w)$ can be chosen in z/w or w/z, but for (3.8) the expansion direction is only in z/w. The Hopf algebra structure is characterized as follows. The coproduct[2] Δ, antipode $\mathcal{S}$ and counit $\mathcal{E}$ are such that:

$$\Delta(L^{\pm}(z)) = (L^{\pm}(zq^{\pm(1\otimes c/2)}))_{[1]}(L^{\pm}(zq^{\mp(c/2\otimes 1)}))_{[2]} \ , \tag{3.11}$$

$$\mathcal{S}(L^{\pm}(z)) = L^{\pm}(z)^{-1} \ , \quad \mathcal{E}(L^{\pm}(z)) = \mathbb{I} \ . \tag{3.12}$$

The complete isomorphism between the FRT presentation of Theorem 3.1 and Drinfeld second presentation of $U_q(\widehat{gl_2})$ is given in [GJ02, Section 4] (see also [FMu02]). Following [GJ02, Section 4], introduce the generating

[2]The index $[j]$ characterizes the 'quantum space' $V_{[j]}$ on which the entries of $L^{\pm}(z)$ act. With respect to the ordering $V_{[1]} \otimes V_{[2]}$, one has:

$$((T)_{[1]}(T')_{[2]})_{ij} = \sum_{k=1}^{2}(T)_{ik} \otimes (T')_{kj} \ . \tag{3.10}$$

functions

$$\mathsf{k}_i^{\pm}(z)=\mathsf{k}_{i,0}^{\pm}\exp\left(\pm(q-q^{-1})\sum_{n=1}^{\infty}a_{i,\mp n}z^{\pm n}\right) \tag{3.13}$$

in terms of the new generators $a_{i,\mp n}$. In terms of Drinfeld generators h_m, the new generators $a_{1,m}, a_{2,m}$ decompose as:

$$a_{1,m}=\frac{1}{q^m+q^{-m}}(\mathsf{h}_m+\gamma_m)\ ,\qquad a_{2,m}=-\frac{1}{q^m+q^{-m}}(q^{2m}\mathsf{h}_m-\gamma_m)\ . \tag{3.14}$$

where γ_m are central elements of $U_q(\widehat{gl_2})$. For our purpose, introduce the surjective map $\gamma'_D : U_q(\widehat{gl_2})\rightarrow U_q^{Dr}$ that is defined as follows. Let γ'_m be Laurent polynomials in $C^{1/2}$, that will be specified later on. We define:

$$\gamma'_D(q^{c/2})\mapsto C^{1/2}\ , \tag{3.15}$$

$$\gamma'_D(\mathsf{x}_k^{\pm})\mapsto \mathsf{x}_k^{\pm}\ , \tag{3.16}$$

$$\gamma'_D(a_{1,m})\mapsto \frac{1}{q^m+q^{-m}}(\mathsf{h}_m+\gamma'_m)\ ,\qquad \gamma_D(a_{2,m})\mapsto -\frac{1}{q^m+q^{-m}}(q^{2m}\mathsf{h}_m-\gamma'_m)\ , \tag{3.17}$$

$$\gamma'_D(\mathsf{k}_{2,0}^{\mp}(\mathsf{k}_{1,0}^{\mp})^{-1})\mapsto \mathsf{K}^{\pm 1}\ ,\qquad \gamma_D(\mathsf{k}_{1,0}^{\pm}\mathsf{k}_{2,0}^{\pm})\mapsto 1\ . \tag{3.18}$$

Note that the map γ'_D slightly differs from the map chosen in [B20, eq. (5.62)-(5.65)].

3.2. **Alternating subalgebras of U_q^{Dr}.** Certain 'alternating' subalgebras of $U_q(\widehat{sl_2})$ have been introduced in [B20], that are now reviewed for further analysis.

Definition 3.2.

$$\begin{aligned} U_q^{Dr,\triangleright,\pm} &= \{C^{\mp k/2}\mathsf{K}^{-1}\mathsf{x}_k^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{k+1}^{\mp}, \mathsf{h}_{k+1}\,|\,k\in\mathbb{N}\}\ ,\\ U_q^{Dr,\triangleleft,\pm} &= \{C^{\mp k/2}\mathsf{x}_{-k}^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{-k-1}^{\mp}\mathsf{K}, \mathsf{h}_{-k-1}\,|\,k\in\mathbb{N}\}\ . \end{aligned}$$

We call $U_q^{Dr,\triangleright,\pm}$ and $U_q^{Dr,\triangleleft,\pm}$ the right and left alternating subalgebras of U_q^{Dr}. The subalgebra generated by $\{\mathsf{K}^{\pm 1}, C^{\pm 1/2}\}$ is denoted $U_q^{Dr,\diamond}$.

The defining relations of the alternating subalgebras are identified using (A.4)-(A.9). Consider for instance $U_q^{Dr,\triangleright,+}$. If we denote $A_k^+ = C^{-k/2}\mathsf{K}^{-1}\mathsf{x}_k^+$, $A_\ell^- = C^{\ell/2}\mathsf{x}_\ell^-$ and $B_k=\mathsf{K}^{-1}\psi_k$, using the relations in Appendix A one gets the defining relations:

$$\big[\mathsf{h}_k,\mathsf{h}_\ell\big]=0\ ,\quad \big[\mathsf{h}_k,B_\ell\big]=0\ , \tag{3.19}$$

$$\big[\mathsf{h}_k,A_\ell^{\pm}\big]=\pm\frac{[2k]_q}{k}A_{k+\ell}^{\pm}\ , \tag{3.20}$$

$$A_{k+1}^{\pm}A_\ell^{\pm}-q^{\pm 2}A_\ell^{\pm}A_{k+1}^{\pm}=q^{\pm 2}A_k^{\pm}A_{\ell+1}^{\pm}-A_{\ell+1}^{\pm}A_k^{\pm}\ , \tag{3.21}$$

$$\big[A_k^+,A_\ell^-\big]_{q^{-1}}=\frac{q^{-1}B_{k+\ell}}{q-q^{-1}}\ . \tag{3.22}$$

The defining relations of the other alternating subalgebras can be similarly written.

For $U_q(\widehat{sl_2})$, it is known that given a certain ordering the elements $\{\mathsf{x}_k^{\pm},\mathsf{h}_\ell,\mathsf{K}^{\pm},C^{\pm 1/2}\}$ generate a PBW basis. See [Be94, Proposition 6.1] with [BCP98, Lemma 1.5]. For the alternating subalgebras, PBW bases follow naturally. If one considers the subalgebra $U_q^{Dr,\triangleright,+}$, let us choose the ordering:

$$C^{1/2}\mathsf{x}_1^- < C\mathsf{x}_2^- < \cdots < \mathsf{h}_1 < \mathsf{h}_2 < \cdots < C^{-1/2}\mathsf{K}^{-1}\mathsf{x}_1^+ < \mathsf{K}^{-1}\mathsf{x}_0^+\ , \tag{3.23}$$

whereas for the subalgebra $U_q^{Dr,\triangleleft,-}$ we choose the ordering:

$$\mathsf{x}_0^- < C^{1/2}\mathsf{x}_{-1}^- < \cdots < \mathsf{h}_{-1} < \mathsf{h}_{-2} < \cdots < C^{-1}\mathsf{x}_{-2}^+\mathsf{K} < C^{-1/2}\mathsf{x}_{-1}^+\mathsf{K}\ . \tag{3.24}$$

It follows:

Proposition 3.3. *The vector space* $U_q^{Dr,\triangleright,+}$ *(resp.* $U_q^{Dr,\triangleleft,-}$*) has a linear basis consisting of the products* $x_1x_2\cdots x_n$ $(n\in\mathbb{N})$ *with* $x_i\in U_q^{Dr,\triangleright,+}$ *(resp.* $x_i\in U_q^{Dr,\triangleleft,-}$*) such that* $x_1\leq x_2\leq\cdots\leq x_n$.

Using the automorphism (A.12), PBW bases for $U_q^{Dr,\triangleright,-}$ and $U_q^{Dr,\triangleleft,+}$ are similarly obtained.

Extensions of the alternating subalgebras are now introduced, that will be useful in the analysis below.

Definition 3.4. ${U'}_q^{Dr,\triangleright,\pm}$ *(resp.* ${U'}_q^{Dr,\triangleleft,\pm}$*) denote the subalgebras of* U_q^{Dr} *generated by* $U_q^{Dr,\triangleright,\pm}$ *(resp.* $U_q^{Dr,\triangleleft,\pm}$*) and* $\{\mathsf{K}^{\pm1},C^{\pm1/2}\}$.

If one considers for instance ${U'}_q^{Dr,\triangleright,+}$, in addition to the relations (3.19)-(3.22) one has:

$$\big[\mathsf{h}_k,\mathsf{K}^{\pm1}\big]=0\ ,\quad \big[B_k,\mathsf{K}^{\pm1}\big]=0\ ,\quad C^{1/2}\ \text{ central}\ , \tag{3.25}$$

$$\mathsf{K}A_k^{\pm}\mathsf{K}^{-1}=q^{\pm2}A_k^{\pm}\ . \tag{3.26}$$

3.3. **The homomorphism** $\nu: U_q^{T,+}\to {U'}_q^{Dr,\triangleright,+}$**.** Consider the following Freidel-Maillet type equation (for a non-symmetric R-matrix)

$$\tilde{R}_{12}(z/w)\ (\tilde{K}(z)\otimes\mathbb{I})\ R^{(0)}\ (\mathbb{I}\otimes\tilde{K}(w))\ =\ (\mathbb{I}\otimes\tilde{K}(w))\ R^{(0)}\ (\tilde{K}(z)\otimes\mathbb{I})\ \tilde{R}_{21}(z/w)\ . \tag{3.27}$$

Assume there exists a matrix $\tilde{K}^0(z)$ with scalar entries and two quantum Lax operators $L(z),L^0$, such that the following relations hold (recall that $\tilde{R}_{21}(z)=P\tilde{R}_{12}(z)P$):

$$\tilde{R}_{12}(z/w)\ \tilde{K}_1^0(z)\ R^{(0)}\ \tilde{K}_2^0(w)\ =\ \tilde{K}_2^0(w)\ R^{(0)}\ \tilde{K}_1^0(z)\ \tilde{R}_{21}(z/w)\ , \tag{3.28}$$

$$\tilde{R}_{12}(z/w)L_1(z)L_2(w)\ =\ L_2(w)L_1(z)\tilde{R}_{12}(z/w)\ , \tag{3.29}$$

$$\tilde{R}_{21}(z/w)(L^0)_1(L^0)_2\ =\ (L^0)_2(L^0)_1\tilde{R}_{21}(z/w)\ , \tag{3.30}$$

$$(L^0)_1R^{(0)}L_2(w)\ =\ L_2(w)R^{(0)}(L^0)_1\ , \tag{3.31}$$

$$L_1(z)R^{(0)}(L^0)_2\ =\ (L^0)_2R^{(0)}L_1(z)\ . \tag{3.32}$$

Adapting [Sk88, Proposition 2], using the above relations one finds that :

$$\tilde{K}(z)\mapsto L(z\lambda)\tilde{K}^0(z)L^0 \tag{3.33}$$

satisfies (3.27) provided λ is invertible and $[\lambda,U_q(\widehat{gl_2})]=0$. For instance, define:

$$\tilde{K}^0(z)=\begin{pmatrix}\bar{\epsilon}_+ & \frac{\bar{k}_+(q+q^{-1})}{(q-q^{-1})}\\ \frac{\bar{k}_-(q+q^{-1})}{(q-q^{-1})} & \frac{\bar{\epsilon}_-}{z}\end{pmatrix}\ , \tag{3.34}$$

where $\bar{k}_\pm\in\mathbb{C}(q)$ and $\big[\bar{\epsilon}_\pm,U_q(\widehat{gl_2})\big]=0$. It satisfies (3.28). It follows:

Lemma 3.5. *The K-operator*

$$\tilde{K}(z)\mapsto\tilde{K}^-(z)=L^-(z\lambda)\tilde{K}^0(z)L^{-,0} \tag{3.35}$$

satisfies (3.27) for any invertible λ *such that* $[\lambda,U_q(\widehat{gl_2})]=0$.

Proof. By previous comment, it is sufficient to check that (3.29)-(3.32) hold. For the choices

$$L(z)\mapsto L^-(z)\quad\text{and}\quad L^0\mapsto L^{-,0}=diag((\mathsf{k}_{2,0}^-)^{-1},(\mathsf{k}_{1,0}^-)^{-1})\ , \tag{3.36}$$

eq. (3.29) holds by definition and it is checked that eqs. (3.30)-(3.32) hold. □

The R-matrices $R(u)$ (symmetric) and $\tilde{R}(z)$ (non-symmetric) given by (2.25) and (3.1), respectively, are related through the similarity transformations:

$$\begin{aligned}\Big(\frac{u}{v}q-\frac{v}{u}q^{-1}\Big)^{-1}R_{12}(u/v)\ &=\ \mathcal{M}(u)_1\mathcal{M}(v)_2\tilde{R}_{12}(u^2/v^2)\mathcal{M}(v)_2^{-1}\mathcal{M}(u)_1^{-1}\ ,\\ &=\ \mathcal{M}(u)_1^{-1}\mathcal{M}(v)_2^{-1}\tilde{R}_{21}(u^2/v^2)\mathcal{M}(v)_2\mathcal{M}(u)_1\quad\text{with}\quad\mathcal{M}(u)=\begin{pmatrix}u^{-1/2} & 0\\ 0 & u^{1/2}\end{pmatrix}\ .\end{aligned}$$

Using this transformation, one relates (3.27) to (2.33): there exists an injective homomorphism from the Freidel-Maillet algebra (2.33) to the Yang-Baxter algebra (3.7)-(3.9) given by:

$$K(u)\mapsto \mathcal{M}(u)\tilde{K}^-(qu^2)\mathcal{M}(u)\ . \tag{3.37}$$

The explicit expression for (3.37) is a generalization of the K-operator in [B20, Lemma 5.15]. Here the difference relies on the additional elements $\bar{\epsilon}_\pm\neq 0$ in (3.34).

The map (3.37) allows to establish the precise relation between the equitable generators $\{y^+_{-k},y^+_{k+1},z^+_{k+1},\tilde{z}^+_{k+1}\}$ and the generators of alternating subalgebras. In the expressions below, for normalization convenience we set:

$$\bar{k}_+=q^{-1}(q-q^{-1})\ ,\quad \bar{k}_-=q-q^{-1}\ ,\quad \bar{\epsilon}_+=q+q^{-1}\ ,\quad \bar{\epsilon}_-=q(q+q^{-1})C^{-1}\ ,\quad \lambda=C^{3/2}\ . \tag{3.38}$$

Proposition 3.6. *There exists an injective homomorphism* $\nu:\ U_q^{T,+}\to {U'_q}^{Dr,\triangleright,+}$ *such that:*

$$\mathcal{Y}_+(u)\mapsto g(u)\left(-\bar{k}_-(q^2+1)(qu^2)^{-1}\sum_{k=0}^{\infty}q^kC^{-k/2}\mathsf{K}^{-1}\mathsf{x}^+_k(qu^2\lambda)^{-k}+\bar{\epsilon}_+(qu^2)^{-1}\mathsf{K}^{-1}\right)\ , \tag{3.39}$$

$$\begin{aligned}\mathcal{Y}_-(u)\mapsto &\left(-\bar{k}_+(q^{-2}+1)\sum_{k=0}^{\infty}q^{k+1}C^{(k+1)/2}\mathsf{x}^-_{k+1}(qu^2\lambda)^{-k-1}\right.\\&\left.+\ \bar{\epsilon}_-q^{-1}(qu^2)^{-1}\left(\psi(u^2\lambda)+(q-q^{-1})^2\sum_{k,\ell=0}^{\infty}q^{k-\ell}C^{(k-\ell+1)/2}\mathsf{x}^-_{k+1}\mathsf{x}^+_\ell(qu^2\lambda)^{-k-\ell-1}\right)\right)g(u)\ ,\end{aligned} \tag{3.40}$$

$$\mathcal{Z}_+(u)\mapsto\left(\frac{\bar{\rho}}{q-q^{-1}}-\bar{\epsilon}_-\bar{k}_-q^{-1}(q^2-q^{-2})(qu^2)^{-1}\sum_{k=0}^{\infty}q^{-k}C^{-k/2}\mathsf{x}^+_k(qu^2\lambda)^{-k}\right)g(u)-\frac{\bar{\rho}}{q-q^{-1}}\ , \tag{3.41}$$

$$\begin{aligned}\mathcal{Z}_-(u)\mapsto g(u)&\left(\frac{\bar{\rho}}{q-q^{-1}}\mathsf{K}^{-1}\psi(u^2\lambda)-\ \bar{\epsilon}_+\bar{k}_+(q^2-q^{-2})\sum_{k=0}^{\infty}q^{-k+1}C^{(k+1)/2}\mathsf{x}^-_{k+1}\mathsf{K}^{-1}(qu^2\lambda)^{-k-1}\right.\\&\left.+\ \bar{\rho}(q-q^{-1})\sum_{k,\ell=0}^{\infty}q^{-k+\ell}C^{(k-\ell+1)/2}\mathsf{K}^{-1}\mathsf{x}^-_{k+1}\mathsf{x}^+_\ell(qu^2\lambda)^{-k-\ell-1}\right)-\frac{\bar{\rho}}{q-q^{-1}}\ ,\end{aligned} \tag{3.42}$$

where

$$g(u)=\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}\frac{(\mathsf{h}_n+\gamma'_n)}{q^n+q^{-n}}(qu^2\lambda)^{-n}\right)\qquad with\quad \gamma'_n=-\frac{(q-q^{-1})^{2n-1}}{n}\left(\frac{\bar{\epsilon}_+\bar{\epsilon}_-\lambda}{\bar{\rho}q}\right)^n\ . \tag{3.43}$$

Proof. The first part of the proof concerns the derivation of the expressions on the r.h.s of (3.39)-(3.42). Recall Lemma 3.5. Then, one expands explicitly (3.35) using (3.3). Consider for instance the entry $(\tilde{K}^-(z))_{11}$, where some commutation relations given in [B20, eqs. (5.55)-(5.56)] are used:

$$\begin{aligned}(\tilde{K}^-(z))_{11}&=\frac{\bar{k}_-(q+q^{-1})}{q-q^{-1}}\mathsf{k}^-_1(z\lambda)\underbrace{\mathsf{f}^-(z\lambda)(\mathsf{k}^-_{2,0})^{-1}}_{=q(\mathsf{k}^-_{2,0})^{-1}\mathsf{f}^-(z\lambda)}+\bar{\epsilon}_+\mathsf{k}^-_1(z\lambda)(\mathsf{k}^-_{2,0})^{-1}\\&=\frac{\bar{k}_-(q+q^{-1})}{q-q^{-1}}\underbrace{\mathsf{k}^-_1(z\lambda)(\mathsf{k}^-_{2,0})^{-1}}_{=\mathsf{K}^{-1}\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(z\lambda)^{-n}\right)}\mathsf{f}^-(z\lambda)+\bar{\epsilon}_+\underbrace{\mathsf{k}^-_1(z\lambda)(\mathsf{k}^-_{2,0})^{-1}}\qquad\text{by}\quad(3.13)\ .\end{aligned}$$

Inserting (3.5), one gets:

$$(\tilde{K}^-(z))_{11} = -\bar{k}_-(q+q^{-1})\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(z\lambda)^{-n}\right)\sum_{k=0}^{\infty}q^k q^{-ck/2}\mathsf{K}^{-1}\mathsf{x}_k^+(z\lambda)^{-k} + \bar{\epsilon}_+\mathsf{K}^{-1}\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(z\lambda)^{-n}\right) . \tag{3.44}$$

Applying γ'_D according to (3.15)-(3.18), one finds $\gamma'_D\left(\tilde{K}^-(z)_{11}\right)$ is a power series in the elements of ${U'_q}^{Dr,\triangleright,+}$. Proceeding similarly for the other entries, $\gamma'_D\left(K^-(z)_{ij}\right) \in {U'_q}^{Dr,\triangleright,+}\otimes\mathbb{C}[[z]]$. Also, the entries are reordered using the defining relations for Drinfeld's currents [DF93]. In particular, one introduces (A.10) and uses:

$$\mathsf{x}_{\ell+1}^- g(u) = q^{-2\ell}g(u)\mathsf{x}_{\ell+1}^- , \qquad \mathsf{x}_\ell^+ g(u) = q^{2\ell}g(u)\mathsf{x}_\ell^+ . \tag{3.45}$$

Then, using (3.37) one compares (2.32) to $\mathcal{M}(u)\gamma'_D\left(\tilde{K}^-(qu^2)\right)\mathcal{M}(u)$. This gives (3.39)-(3.42).

The second part of the proof concerns the identification of the elements γ'_n such that (2.34) holds, i.e. (2.39). In the r.h.s. of (2.37), insert the explicit expressions previously obtained to get the image of $\mathcal{C}(u)$ in ${U'_q}^{Dr,\triangleright,+}$. By [B20, Corollary 3.4, Remark 3.5] (recall the substitutions (2.35), (2.36)), $\nu(\mathcal{C}(u))$ is central: it can be reduced to a function of $C^{1/2}$. To determine this function, it is sufficient to extract all terms of $\nu(\mathcal{C}(u))$ that belong to the center. According to the ordering (3.23) and the reduction rules (A.4)-(A.9), one identifies the subset of terms in the images of $\{\mathcal{Y}_\pm(u),\mathcal{Z}_\pm(u)\}$ that are relevant. One finds:

$$\mathcal{Y}_+(u)\mapsto\bar{\epsilon}_+\mathsf{K}^{-1}(qu^2)^{-1}c(u)+\cdots , \qquad \mathcal{Y}_-(uq)\mapsto\bar{\epsilon}_-q^{-3}\mathsf{K}(qu^2)^{-1}c(uq)+\cdots ,$$
$$\mathcal{Z}_+(uq)\mapsto\frac{\bar{\rho}}{(q-q^{-1})}(c(uq)-1)+\cdots , \qquad \mathcal{Z}_-(u)\mapsto\frac{\bar{\rho}}{(q-q^{-1})}(c(u)-1)+\cdots ,$$

where

$$c(u)=\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}\frac{\gamma'_n}{q^n+q^{-n}}(qu^2\lambda)^{-n}\right) , \tag{3.46}$$

and the 'dots' correspond to terms that will not contribute. After simplifications, one gets the factorized expression:

$$\nu(\mathcal{C}(u))-\frac{\bar{\rho}}{(q-q^{-1})}=\left((q-q^{-1})q^{-2}\bar{\epsilon}_+\bar{\epsilon}_-(qu^2)^{-1}-\frac{\bar{\rho}}{(q-q^{-1})}\right)c(u)c(uq) . \tag{3.47}$$

The condition $\nu(\mathcal{C}(u))=0$ leads to

$$\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}(q\lambda)^{-n}\gamma'_n(qu^2)^{-n}\right)=1-\frac{(q-q^{-1})^2}{\bar{\rho}q^2}\bar{\epsilon}_+\bar{\epsilon}_-(qu^2)^{-1} . \tag{3.48}$$

Taking the logarithm on both sides, the corresponding formal power series are identified. It yields to (3.43). □

Recall (2.29), (2.30). Identifying the leading terms of the power series, from the proposition above with (3.38) one gets for instance:

Example 3.7. *The image in* ${U'_q}^{Dr,\triangleright,+}$ *of* $U_q^{T,+}$ *is such that:*

$$y_0^+ \mapsto C^{-1}\mathsf{K}-q^{-1}(q-q^{-1})C^{-1}\mathsf{x}_1^- ,$$
$$y_1^+ \mapsto \mathsf{K}^{-1}-q(q-q^{-1})\mathsf{K}^{-1}\mathsf{x}_0^+ ,$$
$$\tilde{z}_1^+ \mapsto -(q-q^{-1})^2\left(q^{-1}C^{-3/2}\mathsf{h}_1+(q+q^{-1})C^{-1}\mathsf{x}_0^+\right)+(q-q^{-1})C^{-1} ,$$
$$z_1^+ \mapsto (q-q^{-1})^2\left(qC^{-3/2}\mathsf{h}_1-(q+q^{-1})C^{-1}\mathsf{x}_1^-\mathsf{K}^{-1}+q(q^2-q^{-2})C^{-1}\mathsf{x}_1^-\mathsf{K}^{-1}\mathsf{x}_0^+\right)+(q-q^{-1})C^{-1} .$$

Using (A.13), it is checked that the images of $\{y_i^+\}_{i=0,1}$ match with (2.6).

Remark 3.8. *Alternative expressions for (3.39)-(3.42) can be written using the commutations relations (3.45) and* $\big[\psi(z),\mathsf{K}^{\pm1}\big]=\big[\psi(z),g(u)\big]=0$.

Remark 3.9. *The image of the equitable generators in terms of Lusztig's root vectors and* $\{K_0,K_1\}$ *is obtained as follows. According to the definitions of the root vectors* $\{E_{n\delta+\alpha_i},E_{n\delta}|i=0,1\}\in U_q^{DJ,+}$ *(or* $\{F_{n\delta+\alpha_i},F_{n\delta}|i=0,1\}\in U_q^{DJ,-}$*) given in* [Be94, BCP98]*, one uses the correspondence:*

$$\mathsf{x}_k^{+}=E_{k\delta+\alpha_1}\ ,\qquad \mathsf{x}_{k+1}^{-}=-C^{-k-1}\mathsf{K}E_{k\delta+\alpha_0}\ ,\qquad \mathsf{h}_{k+1}=C^{-(k+1)/2}E_{(k+1)\delta}\ ,\tag{3.49}$$

$$\mathsf{x}_{-k}^{-}=F_{k\delta+\alpha_1}\ ,\qquad \mathsf{x}_{-k-1}^{+}=-F_{k\delta+\alpha_0}\mathsf{K}^{-1}C^{k+1}\ ,\qquad \mathsf{h}_{-k-1}=C^{(k+1)/2}F_{(k+1)\delta}\tag{3.50}$$

for $k\in\mathbb{N}$ *and* $\mathsf{K}=K_1$, $C\mathsf{K}^{-1}=K_0$ *in Proposition 3.6, where* $\bar{k}_\pm,\bar{\epsilon}_\pm,\lambda$*, are chosen such that (2.6) is recovered at the leading order of the power series.*

3.4. **The homomorphism** $\delta:U_q^{T,+}\to {U'_q}^{Dr,\triangleright,+}\otimes U_q^{T,+}$**.** For $U_q^{T,+}$, a comodule algebra structure can be exhibited as follows. Starting from any K-operator satisfying (3.27) and following standard arguments [Sk88], left or right coactions can be constructed using the FRT presentation. Consider the K-operator in the r.h.s. of (3.35). A new K-operator can be constructed using a dressing procedure [Sk88], which leads naturally to a left or right coaction map. For instance[3]

Proposition 3.10. $U_q^{T,+}$ *is a left comodule algebra over* ${U'_q}^{Dr,\triangleright,+}$ *with coaction map* $\delta:U_q^{T,+}\to {U'_q}^{Dr,\triangleright,+}\otimes U_q^{T,+}$ *such that*

$$\delta(\tilde{K}^-(z))=(\gamma'_D\otimes\gamma'_D)\Big((L^-(zq^{(1\otimes c/2)}))_{[1]}(\tilde{K}^-(zq^{(c/2\otimes1)}))_{[2]}(L^{-,0})_{[1]}\Big)\ .\tag{3.52}$$

Proof. By construction, the r.h.s. satisfies (3.27) for the non-symmetric R-matrix (3.1). For $U_q^{T,+}$ to be a comodule algebra, we need to check:

$$(\Delta\otimes id)\circ\delta\ =\ (id\otimes\delta)\circ\delta\ ,\tag{3.53}$$

$$(\mathcal{E}\otimes id)\circ\delta\ \cong\ id\ .\tag{3.54}$$

Firstly, consider (3.53). Apply the l.h.s. of (3.53) to $\tilde{K}^-(z)$ in (3.35) and use the Lax operator coproduct rule (3.11). Compare the result with the r.h.s. of (3.53) applied on $\tilde{K}^-(z)$. Both expressions coincide. Secondly, consider (3.54). Apply the l.h.s of (3.54) to $\tilde{K}^-(z)$ in (3.35) and use the counit rule (3.12). Thus, we conclude that $U_q^{T,+}$ is a left comodule algebra. □

If needed, the image of the equitable generators by δ can be extracted in a straightforward manner. As an example, for simplicity let us consider a specialization of (3.52), namely the left coaction map $\bar{\delta}:U_q^{T,+}\to {U'_q}^{Dr,\triangleright,+}/_{C=1}\otimes U_q^{T,+}$. For the symmetric R-matrix (2.25) using (3.37) and (3.52) at $c=0$ (i.e. $C=1$) it yields to:

$$\bar{\delta}(K(u))=(\gamma'_D\otimes1)\Big(\big(\mathcal{M}(u)L^-(qu^2)\mathcal{M}(u)^{-1}\big)_{[1]}(K(u))_{[2]}(L^{-,0})_{[1]}\Big)/_{C_{[1]}=1}\ .\tag{3.55}$$

Now, recall the generating functions (2.29), (2.30).

[3] With respect to the ordering $V_{[1]}\otimes V_{[2]}$:

$$((T)_{[1]}(T')_{[2]}(T'')_{[1]})_{ij}=\sum_{k,\ell=1}^{2}(T)_{ik}(T'')_{\ell j}\otimes(T')_{k\ell}\ .\tag{3.51}$$

Lemma 3.11. *There exists a coaction map* $\bar\delta: U_q^{T,+} \to U_q'^{Dr,\triangleright,+}/_{C=1} \otimes U_q^{T,+}$ *such that:*

$$
\begin{aligned}
\bar\delta(\mathcal{Y}_+(u)) &\mapsto (qu^2)^{-1}q\gamma'_D\left(\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\mathsf{f}^-(qu^2)\right)/_{C=1} \otimes \left(\frac{1}{\bar k_+(q+q^{-1})}\mathcal{Z}_-(u) + \frac{\bar k_-(q+q^{-1})}{(q-q^{-1})}\right)\\
&\quad + \gamma'_D\left(\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\right)/_{C=1}\otimes \mathcal{Y}_+(u)\ ,\\
\bar\delta(\mathcal{Y}_-(u)) &\mapsto q^{-1}\gamma'_D\left(\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\right)/_{C=1} \otimes \left(\frac{1}{\bar k_-(q+q^{-1})}\mathcal{Z}_+(u) + \frac{\bar k_+(q+q^{-1})}{(q-q^{-1})}\right)\\
&\quad + \gamma'_D\left(\mathsf{k}_2^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1} + q^{-1}\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\mathsf{f}^-(qu^2)\right)/_{C=1}\otimes \mathcal{Y}_-(u)\ ,\\
\bar\delta(\mathcal{Z}_+(u)) &\mapsto \gamma'_D\left(\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\right)/_{C=1}\otimes \mathcal{Z}_+(u) + \frac{\bar\rho}{q-q^{-1}}\left(\gamma'_D\left(\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\right)-1\right)/_{C=1}\otimes 1\\
&\quad + \bar k_-(q+q^{-1})\gamma'_D\left(\mathsf{k}_1^-(qu^2)(\mathsf{k}_{1,0}^-)^{-1}\mathsf{f}^-(qu^2)\right)/_{C=1}\otimes \mathcal{Y}_-(u)\ ,\\
\bar\delta(\mathcal{Z}_-(u)) &\mapsto \gamma'_D\left(\mathsf{k}_2^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1} + q\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\mathsf{f}^-(qu^2)\right)/_{C=1}\otimes \mathcal{Z}_-(u)\\
&\quad + \frac{\bar\rho}{q-q^{-1}}\gamma'_D\left(\mathsf{k}_2^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1} + q\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\mathsf{f}^-(qu^2)-1\right)/_{C=1}\otimes 1\\
&\quad + \bar k_+ qu^2(q+q^{-1})\gamma'_D\left(\mathsf{e}^-(qu^2)\mathsf{k}_1^-(qu^2)(\mathsf{k}_{2,0}^-)^{-1}\right)/_{C=1}\otimes \mathcal{Y}_+(u)\ .
\end{aligned}
$$

Proof. Compute (3.55) using (3.3), (3.37) and (2.32) . Compare the entries of the resulting matrix to $\bar\delta(K(u))$ with (2.32). Applying (3.15)-(3.18) to (3.4), (3.5) and (3.13), one finds $\gamma'_D\left(\bar\delta(K(u))_{ij}\right) \in U_q'^{Dr,\triangleright,+}/_{C=1}\otimes U_q^{T,+}\otimes \mathbb{C}[[u^2]]$. □

Other examples of left and right coaction maps can be derived along the same lines. Now, expanding the power series on both sides of the above equations using (2.29), (2.30), (3.4)-(3.6) with (3.13), one gets the image by $\bar\delta$ of the generators of $U_q^{T,+}$.

Example 3.12.

$$
\begin{aligned}
\bar\delta(y_1^+) &= -q(q-q^{-1})\mathsf{K}^{-1}\mathsf{x}_0^+\otimes 1 + \mathsf{K}^{-1}\otimes y_1^+\ ,\\
\bar\delta(y_0^+) &= -q^{-1}(q-q^{-1})\mathsf{x}_1^-\otimes 1 + \mathsf{K}\otimes y_0^+\ .
\end{aligned}
$$

Using (A.13) at $C=1$, for (2.6) one finds $\bar\delta$ coincides with Δ, see (A.3).

Note that tensor product representations for $U_q^{T,+}$ can be obtained from [B20, Section 4], adapting the definitions of the generators and conventions. See [B20, Proposition 4.5].

4. The central extension of $U_q^{T,+}$

In this section, following [T19b] the central extension of $U_q^{T,+}$ and its center are considered. Below, they are denoted respectively $\mathcal{U}_q^{T,+}$ and C^+. Specializing the results of [B20], a Freidel-Maillet type presentation for $\mathcal{U}_q^{T,+}$ is given. Then, following [B20] the alternating subalgebras $U_q(\widehat{gl_2})^{\triangleright,+}$, $U_q'(\widehat{gl_2})^{\triangleright,+}$ and center $\mathsf{C}^\triangleright$ are introduced. By analogy with the analysis in previous section, the Freidel-Maillet type presentation is used to compute the images of the generators of $\mathcal{U}_q^{T,+}$ and C^+ in $U_q'(\widehat{gl_2})^{\triangleright,+}$ and $\mathsf{C}^\triangleright$, respectively.

The following definitions are straightforward adaptations of [T19b].

Definition 4.1. *$\mathcal{U}_q^{T,+}$ is the unital associative $\mathbb{C}(q)$-algebra with equitable generators $\{Y_{-k}^+, Y_{k+1}^+, Z_{k+1}^+, \tilde Z_{k+1}^+ | k\in\mathbb{N}\}$ subject to the relations (2.8)-(2.18) with the substitutions:*

$$y_{-k}^+\to Y_{-k}^+\ ,\quad y_{k+1}^+\to Y_{k+1}^+\ ,\quad z_{k+1}^+\to Z_{k+1}^+\ ,\quad \tilde z_{k+1}^+\to \tilde Z_{k+1}^+\ . \tag{4.1}$$

Definition 4.2. *The center C^+ is the subalgebra of $\mathcal{U}_q^{T,+}$ generated by the elements:*

$$\mathsf{C}_{n+1} = (q^2-q^{-2})\sum_{k=0}^{n} q^{-2n+2k-1}Y_{k+1}^+Y_{-n+k}^+ - \frac{(q-q^{-1})}{\bar\rho}\sum_{k=0}^{n+1} q^{2k-2n-2}Z_k^+\tilde Z_{n+1-k}^+\ ,\quad n\in\mathbb{N} \tag{4.2}$$

with $Z_0^+ = \tilde{Z}_0^+ = \bar{\rho}/(q-q^{-1})$.

Note that the automorphisms σ, S, given by (2.21), (2.22), naturally extend from $U_q^{T,+}$ to $\mathcal{U}_q^{T,+}$; See [T19b, Section 8] for details. Importantly, using the defining relations of $\mathcal{U}_q^{T,+}$ one shows that the central elements C_{n+1} are fixed under the action of σ, S [T19b, Proposition 8.3].

A Freidel-Maillet type presentation for $\mathcal{U}_q^{T,+}$ follows from [B20, Theorem 3.1], adapting the notations. We refer the reader to this work for the proof of the Theorem below. Introduce the generating functions:

$$\mathsf{Y}_+(u) = \sum_{k\in\mathbb{N}} Y_{k+1}^+ U^{-k-1} \ , \quad \mathsf{Y}_-(u) = \sum_{k\in\mathbb{N}} Y_{-k}^+ U^{-k-1} \ , \tag{4.3}$$

$$\mathsf{Z}_+(u) = \sum_{k\in\mathbb{N}} \tilde{Z}_{k+1}^+ U^{-k-1} \ , \quad \mathsf{Z}_-(u) = \sum_{k\in\mathbb{N}} Z_{k+1}^+ U^{-k-1} \ . \tag{4.4}$$

Theorem 4.3. *$\mathcal{U}_q^{T,+}$ has a presentation of Freidel-Maillet type. Let $\mathsf{K}(u)$ be a square matrix such that*

$$\mathsf{K}(u) = \begin{pmatrix} uq\mathsf{Y}_+(u) & \frac{1}{\bar{k}_-(q+q^{-1})}\mathsf{Z}_+(u) + \frac{\bar{k}_+(q+q^{-1})}{(q-q^{-1})} \\ \frac{1}{\bar{k}_+(q+q^{-1})}\mathsf{Z}_-(u) + \frac{\bar{k}_-(q+q^{-1})}{(q-q^{-1})} & uq\mathsf{Y}_-(u) \end{pmatrix} \tag{4.5}$$

with (4.3)-(4.4). The defining relations are given by:

$$R(u/v)\ (\mathsf{K}(u)\otimes\mathbb{I})\ R^{(0)}\ (\mathbb{I}\otimes\mathsf{K}(v)) \ = \ (\mathbb{I}\otimes\mathsf{K}(v))\ R^{(0)}\ (\mathsf{K}(u)\otimes\mathbb{I})\ R(u/v)\ . \tag{4.6}$$

In the Freidel-Maillet framework, a generating function for central elements of C^+ is derived from the quantum determinant of the K-operator (4.5). Introduce the generating function with coefficients (4.2)

$$\mathsf{C}(u) = \sum_{k=0}^{\infty} \mathsf{C}_{k+1} U^{-k-1} \ . \tag{4.7}$$

By previous comments, note that $\sigma(\mathsf{C}(u)) = \mathsf{C}(u)$. The following proposition is an alternative to [T19b, Section 13], adapted to the equitable case.

Proposition 4.4. *The quantum determinant*

$$\Gamma(u) = \mathrm{tr}_{12}\big(P_{12}^-(\mathsf{K}(u)\otimes I\!\!I)\ R^{(0)}(I\!\!I\otimes\mathsf{K}(uq))\big) \tag{4.8}$$

generates C^+.

Proof. Firstly, one shows that $\Gamma(u)$ is central i.e. $\big[\Gamma(u), (\mathsf{K}(u))_{ij}\big] = 0$. We refer the reader to [B20, Proposition 3.3] for details. Secondly, inserting (4.5) into the r.h.s. of (4.8), one gets

$$\Gamma(u) = \frac{1}{(q-q^{-1})}\left(\mathsf{C}(u) - \frac{\bar{\rho}}{(q-q^{-1})}\right) \tag{4.9}$$

where

$$\mathsf{C}(u) \ = \ (q-q^{-1})u^2q^2\mathsf{Y}_+(u)\mathsf{Y}_-(uq) - \frac{(q-q^{-1})}{\bar{\rho}}\mathsf{Z}_-(u)\mathsf{Z}_+(uq) - \mathsf{Z}_-(u) - \mathsf{Z}_+(uq)\ . \tag{4.10}$$

□

Also, the analogs of [T19b, Lemma 3.3, Lemma 2.8] take the following form. Recall Theorems 2.10, 4.3.

Lemma 4.5. *There exists a surjective homomorphism $\mathcal{U}_q^{T,+} \to U_q^{T,+}$ that sends*

$$\mathsf{K}(u) \mapsto K(u)\ , \qquad \Gamma(u) \mapsto -\frac{\bar{\rho}}{(q-q^{-1})^2}\ . \tag{4.11}$$

An embedding of $\mathcal{U}_q^{T,+}$ into a subalgebra of $U_q(\widehat{gl_2})$ can be obtained using the the FRT presentation of Theorem 3.1. To prepare the analysis below, some results from [B20] are needed. We refer to [B20, Section 5] for details and references. The definition below is a variation of [B20, Definition 5.12, eqs. (5.52)-(5.53)].

Definition 4.6.

$$U_q(\widehat{gl_2})^{\triangleright,\pm} = \{C^{\mp k/2}\mathsf{K}^{-1}\mathsf{x}_k^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{k+1}^{\mp}, a_{1,k+1}, a_{2,k+1} | k \in \mathbb{N}\} ,$$
$$U_q(\widehat{gl_2})^{\triangleleft,\pm} = \{C^{\mp k/2}\mathsf{x}_{-k}^{\pm}, C^{\pm(k+1)/2}\mathsf{x}_{-k-1}^{\mp}\mathsf{K}, a_{1,-k-1}, a_{2,-k-1} | k \in \mathbb{N}\} .$$

We call $U_q(\widehat{gl_2})^{\triangleright,\pm}$ and $U_q(\widehat{gl_2})^{\triangleleft,\pm}$ the right and left alternating subalgebras of $U_q(\widehat{gl_2})$. The subalgebra generated by $\{\mathsf{K}^{\pm 1}, C^{\pm 1/2}\}$ is denoted $U_q(\widehat{gl_2})^{\diamond}$.

Importantly, it is known that the elements

$$\gamma_m = q^m a_{1,m} + q^{-m} a_{2,m} \quad \text{for} \quad m \in \mathbb{Z}^* \tag{4.12}$$

generate the center $\mathcal{C}$ of $U_q(\widehat{gl_2})$ (see e.g. [FMu02]). It follows:

Definition 4.7. *The center $\mathcal{C}^{\triangleright}$ (resp. $\mathcal{C}^{\triangleleft}$) of $U_q(\widehat{gl_2})^{\triangleright,\pm}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,\pm}$) is generated by γ_m (resp. γ_{-m}) with $m \in \mathbb{N}^*$.*

Extensions of the alternating subalgebras of $U_q(\widehat{gl_2})$ are now introduced.

Definition 4.8. *$U'_q(\widehat{gl_2})^{\triangleright,\pm}$ (resp. $U'_q(\widehat{gl_2})^{\triangleleft,\pm}$) denote the subalgebras of $U_q(\widehat{gl_2})$ generated by $U_q(\widehat{gl_2})^{\triangleright,\pm}$ (resp. $U_q(\widehat{gl_2})^{\triangleleft,\pm}$) and $\{\mathsf{K}^{\pm 1}, C^{\pm 1/2}\}$.*

For the quantum algebra $U_q(\widehat{gl_2})$, it is known that $U_q(\widehat{gl_2}) \cong U_q^{Dr} \otimes \mathcal{C}$. Thus, for the alternating subalgebras analog properties hold. Recall Definition 3.4 and (3.14). One has:

$$U'_q(\widehat{gl_2})^{\triangleright,\pm} \cong U'^{Dr,\triangleright,\pm}_q \otimes \mathcal{C}^{\triangleright} , \qquad U'_q(\widehat{gl_2})^{\triangleleft,\pm} \cong U'^{Dr,\triangleleft,\pm}_q \otimes \mathcal{C}^{\triangleleft} .$$

The embedding of the Freidel-Maillet algebra (4.6) into the FRT presentation of $U_q(\widehat{gl_2})$ is now studied. Generalizing the results of previous section, the map (3.37) allows to establish the precise relation between the equitable generators $\{Y^+_{-k}, Y^+_{k+1}, Z^+_{k+1}, \tilde{Z}^+_{k+1}\}$ and the generators of alternating subalgebras of $U_q(\widehat{gl_2})$. Recall (3.46) and Proposition 3.6. Introduce the generating function in the central elements:

$$\mathsf{c}(u) = \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{\gamma_n}{q^n + q^{-n}}(qu^2\lambda)^{-n}\right) \in \mathcal{C}^{\triangleright} \otimes \mathbb{C}[[u^2]]. \tag{4.13}$$

Proposition 4.9. *There exists an injective homomorphism $\mu : \mathcal{U}_q^{T,+} \to U'_q(\widehat{gl_2})^{\triangleright,+}$ such that*

$$\mathsf{Y}_{\pm}(u) \mapsto \nu(\mathcal{Y}_{\pm}(u))c(u)^{-1}\mathsf{c}(u) , \tag{4.14}$$
$$\mathsf{Z}_{\pm}(u) \mapsto \left(\nu(\mathcal{Z}_{\pm}(u)) + \frac{\bar{\rho}}{(q-q^{-1})}\right) c(u)^{-1}\mathsf{c}(u) - \frac{\bar{\rho}}{(q-q^{-1})} . \tag{4.15}$$

Proof. One expands explicitly (3.35) using (3.3). For the entry $(\tilde{K}^-(z))_{11}$, we previously obtained (3.44). Inserting (3.14) and using (3.17), it factorizes as:

$$(\tilde{K}^-(z))_{11} = \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty} \frac{(\gamma_n - \gamma'_n)}{q^n + q^{-n}}(z\lambda)^{-n}\right) \gamma'_D((\tilde{K}^-(z))_{11}) . \tag{4.16}$$

Actually, other entries $(\tilde{K}^-(z))_{ij} \in U'_q(\widehat{gl_2})^{\triangleright,\pm} \otimes \mathbb{C}[[z]]$ and similarly factorize in terms of $\gamma'_D((\tilde{K}^-(z))_{ij})$. Thus, a solution of (4.6) is given by the K-operator:

$$\mathcal{M}(u)\left(\tilde{K}^-(qu^2)\right)\mathcal{M}(u) = \mathsf{c}(u)c(u)^{-1}\mathcal{M}(u)\gamma'_D\left(\tilde{K}^-(qu^2)\right)\mathcal{M}(u) \tag{4.17}$$

with (3.46), (4.13). Then, one compares (2.32) to (4.17) which gives (4.14)-(4.15). □

In terms of central elements of $U_q(\widehat{gl_2})$, the quantum determinant takes a rather simple form.

Corollary 4.10.

$$\Gamma(u) \overset{\mu}{\mapsto} \left(\frac{\bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1})^2 - \bar{k}_+\bar{k}_-(q+q^{-1})^2}{q^3u^2(q-q^{-1})^2}\right)\exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}\gamma_n(q^2u^2\lambda)^{-n}\right) \tag{4.18}$$

with (3.38).

Proof. Applying μ to (4.10) and using (4.14), (4.15), one gets:

$$\mu(\mathsf{C}(u)) - \frac{\bar{\rho}}{(q-q^{-1})} = \mathsf{c}(u)\mathsf{c}(uq)c(u)^{-1}c(uq)^{-1}\left(\nu(\mathcal{C}(u)) - \frac{\bar{\rho}}{(q-q^{-1})}\right) . \tag{4.19}$$

Using (2.31), (3.47), (4.9), (4.13), the r.h.s. of (4.18) follows. □

Remark 4.11. *An alternative expression for $\mu(\Gamma(u))$ is derived as follows. Inserting (4.12) into (4.13) and using the second eq. of (3.14), (A.5) and (A.10), one gets:*

$$\Gamma(u) \overset{\mu}{\mapsto} \left(\frac{\bar{\epsilon}_+\bar{\epsilon}_-(q-q^{-1})^2 - \bar{k}_+\bar{k}_-(q+q^{-1})^2}{q^3u^2(q-q^{-1})^2}\right)\mathsf{g}(u)\psi(q^2u^2\lambda)\mathsf{g}(uq) \tag{4.20}$$

with

$$\mathsf{g}(u) = \exp\left(-(q-q^{-1})\sum_{n=1}^{\infty}a_{1,n}(qu^2\lambda)^{-n}\right) . \tag{4.21}$$

5. Concluding remarks

The results of this paper together with [B20] can be summarized as follows. In [B20], it was shown that the alternating presentation for an algebra U_q^+ and its central extension $\mathcal{U}_q^+$ introduced and studied in [T18, T19a, T19b] admits a presentation of Freidel-Maillet type. For $\mathcal{U}_q^+$, this presentation consists of a K-operator satisfying (4.6), which entries are generating functions in the alternating generators of $\mathcal{U}_q^+$ [T19a, T19b]; See [B20, Theorem 3.1]. To get the analog presentation for U_q^+, a condition for the quantum determinant of the K-operator is asserted. It reads (2.34). Now, by Lemmas 2.5, 2.6, two different embeddings of the alternating presentation of U_q^+ into $U_q(\widehat{sl_2})$ can be considered :

(i) The Drinfeld-Jimbo (or Chevalley) type: the alternating presentation for $U_q^{DJ,+}$ - the standard positive part of $U_q(\widehat{sl_2})$;

(ii) The equitable (or Ito-Terwilliger) type: the alternating presentation for $U_q^{IT,+}$ - the non-standard positive part of $U_q(\widehat{sl_2})$.

In this paper, the alternating presentation for the equitable type is denoted $U_q^{T,+}$, and its central extension $\mathcal{U}_q^{T,+}$. Its generators are called the equitable generators to avoid any confusion with the Drinfeld-Jimbo type.

The alternating presentations (i),(ii), are studied in details in [B20, Ter21] and the present paper. The approach followed in [B20] and here is based on Freidel-Maillet type presentations. In this framework, to the K-operator of U_q^+ (see [B20, eq. (3.8)] one associates a K-operator of Drinfeld-Jimbo type for (i), or of equitable type (2.32) for (ii). For each type, embeddings into Yang-Baxter subalgebras of U_q^{RS} [RS90, DF93] and Drinfeld second realization U_q^{Dr} [D88] are studied in details. They are characterized explicitly using L-operators and Drinfeld generators as follows:

(i') The K-operator of Drinfeld-Jimbo type is the image of (3.37) by γ'_D for $\bar{\epsilon}_\pm = 0, \bar{k}_+ = q^2, \bar{k}_- = -q^{-1}$ and $\gamma'_n = 0$, $\forall n$. As a corollary, the image of the alternating generators in U_q^{Dr} is obtained in terms of Drinfeld generators/root vectors from (3.39)-(3.42); See [B20, Propositions 5.27];

(ii') The K-operator of equitable type is the image of (3.37) by γ'_D for (3.38), (3.43). As a corollary, the image of the equitable generators in terms of Drinfeld generators is obtained. It follows from (3.39)-(3.42), thus generalizing (2.6) of [IT03]. See Remark 3.9 for the corresponding expressions in terms of root vectors.

Analogous results hold for their central extensions related with subalgebras of $U_q(\widehat{gl_2})$, see [B20, Section 5.2]) and Section 4 of this paper. Furthermore, for the central extension of any type (i) and (ii) the image of the quantum determinant (4.18) coincides, up to an overall factor, with a generating function for 'half' of central elements (4.12) of $U_q(\widehat{gl_2})$.

The results of [B20] and the present paper show that Freidel-Maillet type algebras provide a unified framework for $U_q^{DJ,+}$ and $U_q^{IT,+}$. Actually, this unified framework can be extended to $U_q(\widehat{sl_2})$ (see [B20, Section 6] for the Drinfeld-Jimbo type), thus providing an alternative to the FRT presentation for $U_q(\widehat{sl_2})$ [DF93]. For the Drinfeld-Jimbo type, the existence of a Freidel-Maillet type presentation can be understood from the FRT presentation using a Drinfeld twist. The most interesting case is the Freidel-Maillet presentation for $U_q(\widehat{sl_2})$ of equitable type. Details will be considered elsewhere. As a preliminary, for the case of $U_q(sl_2)$ the Freidel-Maillet type presentation unifying the Drinfeld-Jimbo and equitable presentations is introduced and studied in [B21].

Let us also mention that FRT presentations for higher rank affine Lie algebras have been recently achieved, see [JLM19, JLM20, LP21]. By analogy, Freidel-Maillet and alternating presentations of Drinfeld-Jimbo or equitable type for higher rank cases are expected.

From the perspective of physics, it seems natural to study further Freidel-Maillet type algebras of Drinfeld-Jimbo or equitable type. Indeed, it is known that FRT presentations of quantum algebras provide a powerful framework for the explicit construction and analysis of quantum integrable models such as spin chains, using the Bethe ansatz or q-vertex operators's techniques for instance. By analogy, it would be natural to investigate the class of quantum integrable models generated from Freidel-Maillet type presentations.

Acknowledgments: I thank Paul Terwilliger for many discussions, kind explanations of his work and important comments on the manuscript. Also, I thank him for sharing the unpublished results of [Ter21] which motivated the analysis of Section 4. P.B. is supported by C.N.R.S.

Appendix A. Drinfeld-Jimbo and Drinfeld (second realization) presentation of $U_q(\widehat{sl_2})$

For the quantum affine Kac-Moody algebra $U_q(\widehat{sl_2})$, two standard presentations are recalled. The *Drinfeld-Jimbo* presentation U_q^{DJ} and the *Drinfeld (second) presentation* U_q^{Dr}, see e.g. [CP94, p.392], [**?**].

A.1. **Drinfeld-Jimbo presentation U_q^{DJ}.** Define the extended Cartan matrix $\{a_{ij}\}$ ($a_{ii} = 2$, $a_{ij} = -2$ for $i \neq j$). The quantum affine algebra $U_q(\widehat{sl_2})$ over $\mathbb{C}(q)$ is generated by $\{E_j, F_j, K_j^{\pm 1}\}$, $j \in \{0, 1\}$ which satisfy the defining relations

$$K_iK_j = K_jK_i \ , \quad K_iK_i^{-1} = K_i^{-1}K_i = 1 \ , \quad K_iE_jK_i^{-1} = q^{a_{ij}}E_j \ , \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j \ , \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}}$$

together with the $q-$Serre relations ($i \neq j$)

$$\big[E_i, \big[E_i, \big[E_i, E_j\big]_q\big]_{q^{-1}}\big] = 0 \ , \tag{A.1}$$

$$\big[F_i, \big[F_i, \big[F_i, F_j\big]_q\big]_{q^{-1}}\big] = 0 \ . \tag{A.2}$$

The product $C = K_0K_1$ is the central element of the algebra.

The Hopf algebra structure is ensured by the existence of a comultiplication Δ , antipode $\mathcal{S}$ and a counit $\mathcal{E}$ with

$$\begin{aligned} \Delta(E_i) &= 1 \otimes E_i + E_i \otimes K_i \ , \\ \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i \ , \\ \Delta(K_i) &= K_i \otimes K_i \ , \end{aligned} \tag{A.3}$$

$$\mathcal{S}(E_i) = -E_i K_i^{-1} \ , \quad \mathcal{S}(F_i) = -K_i F_i \ , \quad \mathcal{S}(K_i) = K_i^{-1} \qquad \mathcal{S}(1) = 1$$

and

$$\mathcal{E}(E_i) = \mathcal{E}(F_i) = 0 \ , \quad \mathcal{E}(K_i) = 1 \ , \qquad \mathcal{E}(1) = 1 \ .$$

A.2. **Drinfeld's second realization U_q^{Dr}.** A second presentation for the quantum affine algebra $U_q(\widehat{sl_2})$, known as the *Drinfeld's second realization*, is now recalled. In [D88], it is shown that $U_q(\widehat{sl_2})$ is isomorphic to the associative algebra over $\mathbb{C}(q)$ with generators $\{\mathsf{x}_k^{\pm}, \mathsf{h}_\ell, \mathsf{K}^{\pm 1} | k \in \mathbb{Z}, \ell \in \mathbb{Z}\backslash\{0\}\}$, central elements $C^{\pm 1/2}$ and the following relations (see e.g. [CP94, Theorem 12.2.1]):

$$C^{1/2}C^{-1/2} = 1 \ , \quad \mathsf{K}\mathsf{K}^{-1} = \mathsf{K}^{-1}\mathsf{K} = 1 \ , \tag{A.4}$$

$$\big[\mathsf{h}_k, \mathsf{h}_\ell\big] = \delta_{k+\ell,0} \frac{1}{k} \big[2k\big]_q \frac{C^k - C^{-k}}{q - q^{-1}} \ , \tag{A.5}$$

$$\big[\mathsf{h}_k, \mathsf{x}_\ell^{\pm}\big] = \pm \frac{1}{k} \big[2k\big]_q C^{\mp |k|/2} \mathsf{x}_{k+\ell}^{\pm} \ , \tag{A.6}$$

$$\mathsf{K}\mathsf{x}_k^{\pm}\mathsf{K}^{-1} = q^{\pm 2}\mathsf{x}_k^{\pm} \ , \tag{A.7}$$

$$\mathsf{x}_{k+1}^{\pm}\mathsf{x}_\ell^{\pm} - q^{\pm 2}\mathsf{x}_\ell^{\pm}\mathsf{x}_{k+1}^{\pm} = q^{\pm 2}\mathsf{x}_k^{\pm}\mathsf{x}_{\ell+1}^{\pm} - \mathsf{x}_{\ell+1}^{\pm}\mathsf{x}_k^{\pm} \ , \tag{A.8}$$

$$\big[\mathsf{x}_k^+, \mathsf{x}_\ell^-\big] = \frac{(C^{(k-\ell)/2}\psi_{k+\ell} - C^{-(k-\ell)/2}\phi_{k+\ell})}{q - q^{-1}} \ , \tag{A.9}$$

where the ψ_k and ϕ_k are defined by the following equalities of formal power series in the indeterminate z:

$$\psi(z) \ = \ \sum_{k=0}^{\infty} \psi_k z^{-k} = \mathsf{K} \exp\left((q - q^{-1}) \sum_{k=1}^{\infty} \mathsf{h}_k z^{-k} \right) \ , \tag{A.10}$$

$$\phi(z) \ = \ \sum_{k=0}^{\infty} \phi_{-k} z = \mathsf{K}^{-1} \exp\left(-(q - q^{-1}) \sum_{k=1}^{\infty} \mathsf{h}_{-k} z \right) \ . \tag{A.11}$$

Note that there exists an automorphism such that:

$$\theta : \quad \mathsf{x}_k^{\pm} \mapsto \mathsf{x}_k^{\mp} \ , \quad \mathsf{h}_k \mapsto -\mathsf{h}_k \ , \quad \mathsf{K} \mapsto \mathsf{K} \ , \quad C \mapsto C^{-1}, \quad q \mapsto q^{-1} \ . \tag{A.12}$$

An isomorphism $U_q^{DJ} \to U_q^{Dr}$ is given by (see e.g [CP94, p. 393]:

$$K_0 \mapsto C\mathsf{K}^{-1} \ , \quad K_1 \mapsto \mathsf{K} \ , \quad E_1 \mapsto \mathsf{x}_0^+ \ , \quad E_0 \mapsto \mathsf{x}_1^- \mathsf{K}^{-1} \ , \quad F_1 \mapsto \mathsf{x}_0^- \ , \quad F_0 \mapsto \mathsf{K}\mathsf{x}_{-1}^+ \ . \tag{A.13}$$

Note that it is still an open problem to find the complete Hopf algebra isomorphism between U_q^{DJ} and U_q^{Dr}. Only partial information is known, see e.g. [CP91, Section 4.4].

References

[B20] P. Baseilhac, *The alternating presentation of $U_q(\widehat{gl_2})$ from Freidel-Maillet algebras*, Nucl. Phys. **B 967** (2021) 115400, `arXiv:2011.01572`.

[B21] P. Baseilhac, *Freidel-Maillet type presentations of $U_q(sl_2)$*, preprint 2021.

[Ba82] R. Baxter, *Exactly solvable models in statistical mechanics*, New York, Academic Press (1982).

[Be94] J. Beck, *Braid group action and quantum affine algebras*, Commun. Math. Phys. **165** 555-568.

[BCP98] J. Beck, V. Chari and A. Pressley, *An algebraic characterization of the affine canonical basis* Duke Math. J. **99** (1999) 3, 455-487, `arXiv:math/9808060`.

[CP91] V. Chari and A. Pressley, *Quantum affine algebras*, Commun. Math. Phys. **142** 261-283.

[CP94] V. Chari and A. Pressley, *A guide to quantum groups*, (1994) Cambridge University Press.

[C84] I.V. Cherednik, *Factorizing particles on the half-line and root systems*, Teor. Mat. Fiz. 61 (1984) 35-44.

[DF93] J. Ding and I. Frenkel, *Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{sl(n)})$*, Commun. Math. Phys. **156** 277-300.

[D86] V.G. Drinfeld, *Quantum groups*. Proc. ICM Berkeley **1** (1986) 789-820.

[D88] V.G. Drinfeld, *A new realization of yangians and quantum affine algebras*, Sov. Mat. Dokl. **36** (1988) 212-216.

[FRT89] L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Algebra and Analysis (Russian) **1.1** (1989) 118-206;
L.D. Faddeev, N.Y. Reshetikhin and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras, Yang-Baxter equation in Integrable systems*, Advanced Series in Mathematical Physics, Vol. **10**, Singapore: World Scientific 1989, 299-309.

[FM91] L. Freidel and J.M. Maillet, *Quadratic algebras and integrable systems*, Phys. Lett. **B 262** (1991) 278.

[FMu02] E. Frenkel and E. Mukhin, *The Hopf algebra* $RepU_q\widehat{gl}_\infty$, Selecta Mathematica **8** (2002) 537-635, arXiv:math/0103126v2.

[GJ02] Y. Gao and N. Jing, $U_q(\widehat{gl_N})$ *action on* $\widehat{gl_N}$*-modules and quantum toroidal algebras*, J. Algebra **273** (2004) 320-343, arXiv:math/0202292.

[IT03] T. Ito and P. Terwilliger, *Tridiagonal pairs and the quantum affine algebra* $U_q(\widehat{sl_2})$, The Ramanujan Journal **13** (2007) 39-62, arXiv:math/0310042.

[J85] M. Jimbo, *A q-difference analog of* $U(\widehat{g})$ *and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985) 63-69.

[JLM19] N. Jing, M. Liu and A. Molev, *Isomorphism between the R-matrix and Drinfeld presentations of quantum affine algebra: type C*. Jour. Math. Phys. **61** (2020) 031701, arXiv:1903.00204.

[JLM20] N. Jing, M. Liu and A. Molev, *Isomorphism between the R-Matrix and Drinfeld presentations of quantum affine algebra: Types B and D*, SIGMA **16** (2020) 043, arXiv:1911.03496.

[LP21] A. Liashyk and S.Z. Pakuliak, *On the R-matrix realization of quantum loop algebras*, arXiv:2106.10666.

[RS90] N. Yu. Reshetikhin and M. Semenov Tian-Shansky, *Central extensions of quantum current roups*, Lett. Math. Phys. **19** (1990) 133-142.

[Sk88] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A 21** (1988) 2375-2389.

[T18] P. Terwilliger, *Using Catalan words and a q-shuffle algebra to describe a PBW basis for the positive part of* $U_q(\widehat{sl_2})$, J. Algebra **525** (2019) 359-373, arXiv:1806.11228.

[T19a] P. Terwilliger, *The alternating PBW basis for the positive part of* $U_q(\widehat{sl_2})$, J. Math. Phys. **60** (2019) 071704, arXiv:1902.00721.

[T19b] P. Terwilliger, *The alternating central extension for the positive part of* $U_q(\widehat{sl_2})$, Nucl. Phys. **B 947** (2019) 114729, arXiv:1907.09872.

[Ter21] P. Terwilliger, *The algebra* U_q^+ *and its alternating central extension* $\mathcal{U}_q^+$, preprint 2021.

Institut Denis-Poisson CNRS/UMR 7013 - Université de Tours - Université d'Orléans Parc de Grammont, 37200 Tours, FRANCE

Email address: pascal.baseilhac@idpoisson.fr