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ON THE SECTIONAL CURVATURES OF R-SPACES

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Introduction

Let $\mathfrak{g}$ be a real semi-simple Lie algebra without compact factors, $\mathfrak{k}$ a maximal compactly imbedded subalgebra of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{k}$. We denote by B the Killing form of $\mathfrak{g}$. We regard the subspace $\mathfrak{p}$ as a Euclidean space with the inner product $\langle\ ,\ \rangle$ induced by the restriction of B to $\mathfrak{p}$. Let Int $(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$ and, the Lie algebra of Int $(\mathfrak{g})$ being identified with $\mathfrak{g}$, K the connected Lie subgroup of Int $(\mathfrak{g})$ corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Then K leaves the subspace $\mathfrak{p}$ invariant and acts on the Euclidean space $\mathfrak{p}$ as an isometry group. Let S be the unit sphere of $\mathfrak{p}$ and N an orbit of an element H_0 in S. Denoting by K^* the stabilizer of H_0 in K, the space N may be identified with the quotient space K/K^* and is called an *R-space*. We always assume that dim $N\geqq 2$, and the space N is *substantial*, i.e. there exist no proper subspaces of $\mathfrak{p}$ containing N. The aim of this paper is to study the sectional curvatures of N with respect to the K-invariant Riemannian metric $\langle\ ,\ \rangle$ induced by the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$.

It is known (Takeuchi-Kobayashi [8]) that if the pair (K, K^*) is a symmetric pair, the metric $\langle\ ,\ \rangle$ on $N=K/K^*$ coincides with the K-invariant Riemannian metric defined by a K-invariant inner product on $\mathfrak{k}$, and so the sectional curvatures of N are always non-negative, and N has a positive sectional curvature along each plane section if and only if the pair (K, K^*) is of rank 1.

In this paper we shall show that in general cases the space N may have both positive and negative sectional curvatures. Indeed, the curvatures are related with the restricted root system $\mathfrak{r}$ of $\mathfrak{g}$. Let Δ be a fundamental root system of $\mathfrak{r}$. Then a subsystem Δ_1 of Δ corresponds to the space N (See section 3), and we have:

(I) *If the restricted root system $\mathfrak{r}$ is irreducible and the cardinality $|\Delta-\Delta_1|$ of $\Delta-\Delta_1$ is not less than 2, the space N has both positive and negative sectional curvatures.*

Furthermore we shall characterize the R-spaces with strictly positive sectional curvatures:

(II) *The space N has a strictly positive sectional curvature along each plane section if and only if N is the unit sphere S or the pair (K, K^*) is a symmetric pair of rank* 1.

The class of R-spaces includes the spaces $K/K^*=SO(3)/Q$, $SU(3)/T$, $Sp(3)/Sp(1)\times Sp(1)\times Sp(1)$ and $F_4/\mathrm{Spin}\ (8)$, where Q (resp. T) denotes the subgroup of all diagonal matrics in $SO\ (3)$ (resp. in $SU\ (3)$). Although none of these pairs is a symmetric pair of rank 1, Wallach [9] proved that each of these spaces has a K-invariant Riemannian metric with strictly positive sectional curvatures. The charcterization (II) shows that the Riemannian metric of Wallach is not the same as ours.

I wish to express my sincere gratitude to Professor M. Takeuchi for his kind guidance and encouragements.

1. Preliminaries

1.1. The assumptions and the notation are the same as those in Introduction. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$. We shall identify $\mathfrak{a}$ with the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ by means of the duality defined by the Killing form B of $\mathfrak{g}$.

For an element $\lambda\in\mathfrak{a}$, we define subspaces $\mathfrak{k}_\lambda$ and $\mathfrak{p}_\lambda$ of $\mathfrak{g}$ as follows:

$$(1.1)\qquad \begin{cases} \mathfrak{k}_\lambda=\{X\in\mathfrak{k};\ ad(H)^2X=B(\lambda, H)^2X, \text{ for all } H\in\mathfrak{a}\}\ , \\ \mathfrak{p}_\lambda=\{X\in\mathfrak{p};\ ad(H)^2X=B(\lambda, H)^2X, \text{ for all } H\in\mathfrak{a}\} \end{cases}$$

Then $\mathfrak{k}_{-\lambda}=\mathfrak{k}_\lambda$, $\mathfrak{p}_{-\lambda}=\mathfrak{p}_\lambda$ and $\mathfrak{p}_0=\mathfrak{a}$. It is known (Satake [4]) that if we put

$$\mathfrak{r}=\{\lambda\in\mathfrak{a};\ \lambda\neq 0,\ \mathfrak{p}_\lambda\neq\{0\}\}\ ,$$

$\mathfrak{r}$ is a root system in $\mathfrak{a}$. The root system $\mathfrak{r}$ is called the *restricted root system* of $\mathfrak{g}$. We denote by $\mathfrak{r}^+$ the set of positive roots of $\mathfrak{r}$ with respect to a linear order in the subspace $\mathfrak{a}$. Then we have the following orthogonal decompositions of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to the Killing form B (cf. Helgason [2]).

$$(1.2)\qquad \mathfrak{k}=\mathfrak{k}_0+\sum_{\lambda\in\mathfrak{r}^+}\mathfrak{k}_\lambda,\quad \mathfrak{p}=\mathfrak{a}+\sum_{\lambda\in\mathfrak{r}^+}\mathfrak{p}_\lambda$$

2. Second fundamental forms of R-spaces

2.1. In Introduction we assume that the point $H_0\in N$ is contained in the unit sphere S. Moreover we may assume $H_0\in S\cap\mathfrak{a}$ by virtue of the following lemma (cf. Helgason [2]).

Lemma 1. *For each element $X\in\mathfrak{p}$, there exists an element $k\in K$ such that kX is contained in the subspace $\mathfrak{a}$.*

Let $T_{H_0}(N)$ be the tangent space of N at H_0. We can identify the tangent space $T_{H_0}(N)$ with a subspace of $\mathfrak{p}$ in a canonical manner, and we have

$$T_{H_0}(N) = [\mathfrak{k}, H_0] .$$

Choose a linear order in the subspace $\mathfrak{a}$ such that $\langle\lambda, H_0\rangle \geqq 0$ for each positive root $\lambda \in \mathfrak{r}$ with respect to this order, and fix this order once for all. Put

$$\mathfrak{r}_1^+ = \{\lambda \in \mathfrak{r}^+ ; \langle\lambda, H_0\rangle = 0\} ,$$
$$\mathfrak{r}_2^+ = \{\lambda \in \mathfrak{r}^+ ; \langle\lambda, H_0\rangle > 0\} .$$

Then the tangent space $T_{H_0}(N)$ and the orthogonal complement $T_{H_0}^{\perp}(N)$ in $\mathfrak{p}$ are given by

$$\text{(2.1)} \qquad \begin{cases} T_{H_0}(N) = \sum_{\lambda \in \mathfrak{r}_2^+} \mathfrak{p}_\lambda , \\ T_{H_0}^{\perp}(N) = \mathfrak{a} + \sum_{\lambda \in \mathfrak{r}_1^+} \mathfrak{p}_\lambda . \end{cases}$$

Let

$$\alpha \colon T_{H_0}(N) \times T_{H_0}(N) \to T_{H_0}^{\perp}(N)$$

be the second fundamental form at H_0 of the submanifold N of $\mathfrak{p}$. In the same way as in Takagi-Takahashi [6] we get the following:

Proposition 2. *For $X_\lambda \in \mathfrak{p}_\lambda$, $Y_{\lambda'} \in \mathfrak{p}_{\lambda'}$ $(\lambda, \lambda' \in \mathfrak{r}_2^+, \lambda' \geqq \lambda)$, the second fundamental form α is given by*

$$\text{(2.2)} \qquad \alpha(X_\lambda, Y_{\lambda'}) = \begin{cases} \text{(i)} \quad -\dfrac{\langle X_\lambda, Y_\lambda\rangle}{\langle\lambda, H_0\rangle}\lambda & \textit{if } \lambda' = \lambda , \\ \text{(ii)} \quad \mathfrak{p}_\mu\textit{-component of } -\dfrac{1}{\langle\lambda, H_0\rangle^2}[[H_0, X_\lambda], Y_{\lambda'}] & \\ \qquad \textit{if } \lambda' > \lambda \textit{ and } \mu = \lambda' - \lambda \in \mathfrak{r}_1^+ , & \\ \text{(iii)} \quad 0 & \textit{otherwise} . \end{cases}$$

3. Sectional curvatures of R-spaces

3.1. Let R be the curvature tensor at H_0 of the space N. By the equation of Gauss and the flatness of the space $\mathfrak{p}$ we have

$$\text{(3.1)} \qquad \langle R(X, Y)Y, X\rangle = \langle\alpha(X, X), \alpha(Y, Y)\rangle - \langle\alpha(X, Y), \alpha(X, Y)\rangle$$

for any $X, Y \in T_{H_0}(N)$ (cf. Kobayashi-Nomizu [3]). For $X, Y \in T_{H_0}(N)$ we denote by $K_{X,Y}$ the sectional curvature along the plane section spanned by X and Y, X and Y being assumed to be linearly independent.

Proposition 3. *If there exist roots $\lambda, \lambda' \in \mathfrak{r}_2^+$ such that $\langle\lambda, \lambda'\rangle < 0$, then the space N has both positive and negative sectional curvatures.*

Proof. Since $\lambda \neq \lambda'$, from (2.2) and (3.1) follows that

$$(3.2) \qquad K_{X_\lambda, Y_\lambda'} = \langle \alpha(X_\lambda, X_\lambda), \alpha(Y_{\lambda'}, Y_{\lambda'})\rangle - \langle \alpha(X_\lambda, Y_{\lambda'}), \alpha(X_\lambda, Y_{\lambda'})\rangle \leqq \frac{\langle \lambda, \lambda' \rangle}{\langle \lambda, H_0 \rangle \langle \lambda', H_0 \rangle}$$

for $X_\lambda \in \mathfrak{p}_\lambda$, $Y_{\lambda'} \in \mathfrak{p}_{\lambda'}$ with $\langle X_\lambda, X_\lambda \rangle = \langle Y_{\lambda'}, Y_{\lambda'} \rangle = 1$. Since $\langle \lambda, \lambda' \rangle < 0$, we have $K_{X_\lambda, Y_\lambda'} < 0$.

For each root $\mu \in \mathfrak{r}$ we denote by S_μ the reflection of $\mathfrak{a}$ with respect to μ. Then $\lambda'' = S_\lambda(\lambda')$ is a root in $\mathfrak{r}$. Since $\langle \lambda, \lambda' \rangle < 0$, we find that

$$\lambda'' = \lambda' - \frac{2\langle \lambda, \lambda' \rangle}{\langle \lambda, \lambda \rangle} \lambda \in \mathfrak{r}_2^+$$

and

$$\langle \lambda, \lambda'' \rangle = \langle \lambda, S_\lambda(\lambda') \rangle = \langle S_\lambda(\lambda), \lambda' \rangle = -\langle \lambda, \lambda' \rangle > 0\,.$$

For $Z_{\lambda''} \in \mathfrak{p}_{\lambda''}$ with $\langle Z_{\lambda''}, Z_{\lambda''} \rangle = 1$, since $\langle \lambda'' - \lambda, H_0 \rangle > 0$, we have $\alpha(X_\lambda, Z_{\lambda''}) = 0$ by (2.2) (iii). So in the same way as above we have $K_{X_\lambda, Z_\lambda''} > 0$.

3.2. In the following we assume that the root system $\mathfrak{r}$ is irreducible. Let Δ be the fundamental root system of $\mathfrak{r}$ consisting of simple roots with respect to the order chosen in section 2, and put

$$\Delta_1 = \{\lambda \in \Delta; \langle \lambda, H_0 \rangle = 0\}\,.$$

Let $\mathfrak{h}$ be a Cartan subalgebra containing $\mathfrak{a}$. Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$, $\mathfrak{h}^C$ the subspace of $\mathfrak{g}^C$ spanned by $\mathfrak{h}$, and $\mathfrak{h}_0$ the real part of $\mathfrak{h}^C$. Let σ be the conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$, and choose a σ-order in the sense of Satake [4] on $\mathfrak{h}_0$, extending our order on $\mathfrak{a}$. In the root system of $\mathfrak{g}^C$ relative to $\mathfrak{h}^C$, let $\hat{\Delta}$ be the fundamental root system consisting of simple roots with respect to this order, and denote the Satake diagram of $\hat{\Delta}$ by the same symbol $\hat{\Delta}$. Then $\hat{\Delta}$ defines our fundamental root system Δ by the projection p of $\mathfrak{h}_0$ onto $\mathfrak{a}$: $\Delta = p(\hat{\Delta}) - \{0\}$. Let $\hat{\Delta}_1 = p^{-1}(\Delta_1)$. It is known (Takeuchi [7]) that isomorphic pairs $(\hat{\Delta}, \hat{\Delta}_1)$ of Satake diagrams give rise to isomorphic pairs (K, K^*): We say that the pair $(\hat{\Delta}, \hat{\Delta}_1)$ is isomorphic to the pair $(\hat{\Delta}', \hat{\Delta}_1')$ if there exists an isomorphism φ of $\hat{\Delta}$ onto $\hat{\Delta}'$ such that φ maps $\hat{\Delta}_1$ onto $\hat{\Delta}_1'$, and the pair (K, K^*) is said to be isomorphic to the pair $(K', K^{*\prime})$ if there exists an isomorphism f of K onto K' such that f maps K^* onto $K^{*\prime}$.

For a set A we will denote by $|A|$ the cardinality of A.

Proposition 4. *If the root system $\mathfrak{r}$ is irreducible and $|\Delta - \Delta_1| \geqq 2$, the space N has both positive and negative sectional curvatures.*

Proof. Assume that $\lambda, \mu \in \Delta - \Delta_1$ and $\lambda \neq \mu$. Then there exist simple roots $\lambda_0, \lambda_1, \cdots, \lambda_s \in \Delta$ such that the following conditions are satisfied:

$$(3.3) \qquad \begin{cases} \text{(i)} & \lambda_0 = \lambda, \quad \lambda_s = \mu \\ \text{(ii)} & \langle \lambda_i, \lambda_j \rangle = \begin{cases} < 0 & \text{if } j = i+1 \\ = 0 & \text{otherwise} \end{cases} \end{cases}$$
$$\text{where } 0 \leqq i < j \leqq s$$

Put

$$\lambda'' = S_{\lambda_{s-1}} S_{\lambda_{s-2}} \cdots S_{\lambda_1}(\lambda_0) \text{ and } \lambda'' = \lambda_0 + \sum_{i=1}^{s-1} m_i \lambda_i .$$

Then by (3.3) we have $m_i > 0$ and $\lambda'' \in \mathfrak{r}_2^+$. Since

$$\langle \lambda'', \mu \rangle = m_{s-1} \langle \lambda_{s-1}, \lambda_s \rangle < 0 ,$$

the proposition follows from Proposition 3.

An element H of $\mathfrak{a}$ is called regular if $\langle \lambda, H \rangle \neq 0$ for any $\lambda \in \mathfrak{r}$. Then the following Corollary is an immediate consequence of Proposition 4.

Corollary. *If the root system $\mathfrak{r}$ is irreducible and the rank of $\mathfrak{r}$ is not less than 2, then the K-orbit N through a regular element of $\mathfrak{a}$ has both positive and negative sectional curvatures.*

REMARK. If $\mathfrak{g}$ is the direct sum of r-copies of $\mathfrak{sl}(2, \boldsymbol{R})$ ($r \geqq 2$), the space N is the r-dimensional flat torus. According to the following result (A) and (B), it follows that, except for the above case, the space N has always a plane section along which the sectional curvature is strictly positive.

(A) *Let M be an n-dimensional manifold and G a compact connected transitive Lie transformation group of M. If the universal covering manifold $\tilde{M}$ of M satisfies*

$$H_i(\tilde{M}, \boldsymbol{Z}_2) = \{0\} \qquad \textit{for any } \ i > 0 ,$$

then G is the n-dimensional toral group, M is the n-dimensional torus and G acts on M as translations (M. Takeuchi).

(B) A simply connected complete Riemannian manifold with non-positive sectional curvatures is diffeomorphic with a Euclidean space (Theorem of Cartan-Hadamard, cf. [3]).

3.3. Next we shall see under which conditions N has a strictly positive sectional curvature along each plane section.

Assume that the space N has strictly positive sectional curvatures. Decompose the root system $\mathfrak{r}$ into the sum of irreducible components $\mathfrak{r}^{(i)}$:

$$\mathfrak{r} = \mathfrak{r}^{(1)} \cup \cdots \cup \mathfrak{r}^{(m)}$$

If we put $(\mathfrak{r}^{(i)})_2^+ = \mathfrak{r}^{(i)} \cap \mathfrak{r}_2^+$ for $1 \leqq i \leqq m$, then the space N is substantial if and only if $(\mathfrak{r}^{(i)})_2^+ \neq \phi$ for each i. Thus the root system $\mathfrak{r}$ should be irreducible by (3.2). If $|\Delta| = 1$, then $\Delta_1 = \phi$ and the space N coincides with the unit sphere S itself. So we assume that $|\Delta| \geqq 2$ in the following. Then we have by Proposition 4 and (3.2):

$$|\Delta - \Delta_1| = 1 \tag{3.4}$$

$$\langle \lambda, \lambda' \rangle > 0 \quad \text{for all} \quad \lambda, \lambda' \in \mathfrak{r}_2^+ \tag{3.5}$$

We remark that the properties (3.4) and (3.5) depend only on the root system $\mathfrak{r}$. So we shall consider the following situation:

Let V be a real vector space and $\mathfrak{S}$ a *reduced* irreducible root system in V. Choosing a linear order in the space V, let $\mathfrak{S}^+$ denote the set of positive roots and Δ the fundamental root system of $\mathfrak{S}$ consisting of simple roots. The Dynkin diagram of Δ will be also denoted by the same symbol Δ. Let Δ_1 be a subset Δ of such that $|\Delta - \Delta_1| = 1$,

$$\Delta = \{\lambda_1, \lambda_2, \cdots, \lambda_l\} \quad \text{and} \quad \Delta - \Delta_1 = \{\lambda_1\} \ .$$

We put

$$\mathfrak{S}_2^+ = \{\lambda \in \mathfrak{S}^+ ; \lambda = \sum_{i=1}^{l} m_i \lambda_i, \ m_1 > 0\} \ .$$

We denote by λ_0 the highest root in $\mathfrak{S}$ and S_λ the reflection of V with respect to the root $\lambda \in \mathfrak{S}$. We fix an inner product $(\ , \ )$ on V invariant under the Weyl group of $\mathfrak{S}$. Since the root system $\mathfrak{S}$ is irreducible, such an inner product is unique up to a multiple of positive constant. The diagram obtained from the Dynkin diagram Δ by adding $-\lambda_0$ is called the *extended Dynkin diagram* of $\mathfrak{S}$. The table of extended Dynkin diagrams is seen, for instance, in Borel-de Siebenthal [1]. We shall consider the condition

$$(\lambda, \lambda') > 0 \quad \text{for all} \quad \lambda, \lambda' \in \mathfrak{S}_2^+ \tag{3.6}$$

and prove the following:

Proposition 5. *The pair* (Δ, Δ_1) *satisfies the condition* (3.6) *if and only if the Dynkin diagram* Δ *is of type* A_l *and* λ_1 *is one of the terminals of* Δ, *i.e. the vertex* λ_1 *is connected with only one vertex in* Δ.

Proof. Let Δ be of type A_l and λ_1 one of the terminals of Δ. We may assume that there exist an $(l+1)$-dimensional vector space $W \supset V$ with an inner product $(\ , \ )$ and an orthonormal basis $e_1, \cdots, e_{l+1}$ of W such that the following conditions are satisfied:

(i) The restriction to V of the inner product $(\ , \ )$ on W is invariant under the Weyl group of $\mathfrak{S}$.

(ii) $\mathfrak{S}=\{e_i-e_j;\ 1\leqq i, j\leqq l+1,\ i\neq j\}$
(iii) $\Delta=\{e_i-e_{i+1};\ 1\leqq i\leqq l\}$, $\lambda_1=e_1-e_2$

Then we have

$$\mathfrak{S}_2^+ = \{e_1-e_i;\ 2\leqq i\leqq l+1\}\ .$$

Hence the pair (Δ, Δ_1) satisfies the condition (3.6).

It remains to prove that if the pair (Δ, Δ_1) satisfies (3.6), then the Dynkin diagram Δ is of type A_l and λ_1 is one of the terminals of Δ.

Lemma 6. *If the pair* (Δ, Δ_1) *satisfies* (3.6), *the inner product* $(\lambda_0, \lambda_1)\neq 0$.

Proof. Since the highest root λ_0 is contained in $\mathfrak{S}_2^+$, we have $(\lambda_0, \lambda_1)>0$.

Lemma 7. *If the pair* (Δ, Δ_1) *satisfies* (3.6), λ_1 *is one of the terminals of* Δ.

Proof. Suppose that λ_1 is not a terminal. Then there exist two simple roots $\lambda, \mu\in\Delta$ such that $(\lambda, \lambda_1)\neq 0$ and $(\mu, \lambda_1)\neq 0$. Since the Dynkin diagram Δ contains no cycles, we have $(\lambda, \mu)=0$. It is known that $(\lambda, \lambda_1)<0$ and $(\mu, \lambda_1)<0$. Then it follows that $\lambda+\lambda_1$ is a root (cf. Serre [5]). Since $(\lambda+\lambda_1, \mu)<0$, it follows also that $\lambda+\lambda_1+\mu$ is a root. On the other hand $\lambda_1+\lambda+\mu$ belongs to $\mathfrak{S}_2^+$ by definition, so that $(\lambda_1+\lambda+\mu, \lambda_1)>0$. Thus we see that $\lambda+\mu=(\lambda+\lambda_1+\mu)-\lambda_1$ is a root.

Now let

$$\lambda-p\mu, \cdots, \lambda-\mu, \lambda, \lambda+\mu, \cdots, \lambda+q\mu$$

be a μ-series of roots such that neither $\lambda-(p+1)\mu$ nor $\lambda+(q+1)\mu$ is a root. Then we have (cf. Serre [5]).

$$q-p = -\frac{2(\lambda, \mu)}{(\mu, \mu)}\ .$$

Since $(\lambda, \mu)=0$ and $p=0$, we get $q=0$, which contradicts to $\lambda+\mu\in\gamma$.

Lemma 8. *Suppose that the pair* (Δ, Δ_1) *satisfies the condition* (3.6). *If the Dynkin diagram* Δ *is one of the classical types, then* Δ *is of type* A_l *and* λ_1 *is one of the terminals of* Δ.

Proof. It suffices to show that the Dynkin diagram Δ is of type A_l. Suppose that Δ is of type $B_l(l\geqq 3)$ or of type $D_l(l\geq 4)$. Then, looking at their extended Dynkin diagrams, we see from Lemma 6 and Lemma 7 that there exist no pairs (Δ, Δ_1) satisfying the condition (3.6).

Suppose that the pair (Δ, Δ_1) satisfies (3.6) and the Dynkin diagram Δ is of type $C_l(l\geqq 2)$. Then we may assume that there exists an orthonormal basis $\{e_1, \cdots, e_l\}$ of V such that the following conditions are satisfied:

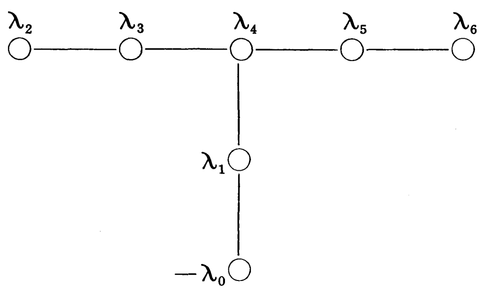
(i) $\mathfrak{S}=\{\pm e_i \pm e_j,\ \pm 2e_i;\ 1\leqq i, j\leqq l,\ i\neq j\}$
(ii) $\Delta=\{e_1-e_2,\ e_2-e_3,\ \cdots,\ e_{l-1}-e_l,\ 2e_l\}$
By Lemma 6 and Lemma 7 we should have $\lambda_1=e_1-e_2$. So we have

$$\mathfrak{S}_2^+ = \{e_1 \pm e_i,\ 2e_1;\ 2\leqq i\leqq l\}\ .$$

For $e_1-e_2,\ e_1+e_2\in\mathfrak{S}_2^+$ we have $(e_1-e_2,\ e_1+e_2)=0$, which is a contradiction.

Lemma 9. *If the Dynkin diagram Δ is one of the exceptional types, there exist no pairs (Δ, Δ_1) satisfying the condition* (3.6).

Proof. Suppose that the pair (Δ, Δ_1) satisfies the condition (3.6) and the Dynkin diagram Δ is of type E_6. By Lemma 6 and the extended Dynkin diagram we should have



We put

$$\mathfrak{S}' = \{\lambda\in\mathfrak{S};\ \lambda = \sum_{i=1}^{6} m_i\lambda_i,\ m_2 = m_6 = 0\}\ ,$$
$$\Delta' = \{\lambda_1,\ \lambda_3,\ \lambda_4,\ \lambda_5\}\ \text{ and }\ \Delta_1' = \{\lambda_3,\ \lambda_4,\ \lambda_5\}\ .$$

Then $\mathfrak{S}'$ is a root system of type D_4 with the fundamental root system Δ'. The subset $(\mathfrak{S}')_2^+$ of $\mathfrak{S}'$ corresponding to the pair (Δ', Δ_1') is given by $(\mathfrak{S}')_2^+=\mathfrak{S}_2^+\cap\mathfrak{S}'$. It follows from Lemma 8 that there exist two roots $\lambda,\ \mu\in(\mathfrak{S}')_2^+$ such that $(\lambda, \mu)\leqq 0$. This is a contradiction.

In the cases of type E_7 and E_8 we can prove the lemma in the same way, making use of Lemma 8 for Δ of type D_5 and D_7 respectively.

In the case of type F_4 we can prove in the same way, making use of Lemma 8 for Δ of type B_3.

In the case of type G_2 the lemma is easily proved.

By Lemma 8 and Lemma 9 we have the complete proof of Proposition 5.

Now we give the table of pairs $(\hat{\Delta}, \hat{\Delta}_1)$ of Satake diagrams (up to an isomorphism) with the following properties.

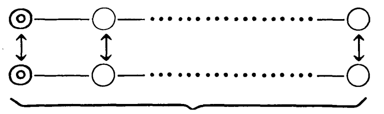
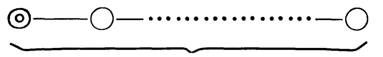
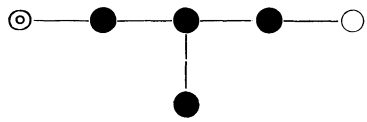
1) The Satake diagram $\hat{\Delta}$ has no compact factors.
2) The pair (Δ, Δ_1) of fundamental root systems obtained from $(\hat{\Delta}, \hat{\Delta}_1)$

by the projection has the properties:

(i) The Dynkin diagram Δ is of type A_l with $l\geqq 2$.

(ii) The set $\Delta-\Delta_1$ consists of one of the terminals of Δ.

Here the vertexes in $\hat{\Delta}-\hat{\Delta}_1$ are represented by ◎. $p^l(\boldsymbol{F})$ denotes the l-dimensional projective space over $\boldsymbol{F}$ where $\boldsymbol{F}$ denotes $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ (the algebra of real quaternions), or $\boldsymbol{K}$ (the algebra of real Cayley numbers).

	$(\hat{\Delta}, \hat{\Delta}_1)$	N
A	 l $(l\geqq 2)$	$SU(l+1)$ $/S(U(1)\times U(l))=P^l(\boldsymbol{C})$
AI	 l $(l\geqq 2)$	$SO(l+1)$ $/S(O(1)\times O(l))=P^l(\boldsymbol{R})$
AII	 $2l+1$ $(l\geqq 2)$	$Sp(l+1)$ $/Sp(1)\times Sp(l)=P^l(\boldsymbol{H})$
EIV		$F_4/\text{Spin}\,(9)=P^2(\boldsymbol{K})$

Each of the pairs (K, K^*) appeared in this table is a symmetric pair of rank 1 and the space N has strictly positive sectional curvatures as we have noted in Introduction. Thus we have proved the following:

Proposition 10. *The space N has strictly positive sectional curvatures if and only if N is the unit sphere S or the pair (K, K^*) is a symmetric pair of rank* 1.

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