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On the Semantics of Petri Nets

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Abstract

Petri Place/Transition (PT) nets are one of the most widely used models of concurrency. However, they still lack, in our view, a satisfactory semantics: on the one hand the "token game" is too intensional, even in its more abstract interpretations in term of nonsequential processes and monoidal categories; on the other hand, Winskel's basic unfolding construction, which provides a coreflection between nets and finitary prime algebraic domains, works only for safe nets.

In this paper we extend Winskel's result to PT nets. We start with a rather general category **PTNets** *of PT nets, we introduce a category* **DecOcc** *of decorated (nondeterministic) occurrence nets and we define adjunctions between* **PTNets** *and* **DecOcc** *and between* **DecOcc** *and* **Occ***, the category of occurrence nets. The role of* **DecOcc** *is to provide natural unfoldings for PT nets, i.e. acyclic safe nets where a notion of family is used for relating multiple instances of the same place.*

The unfolding functor from **PTNets** *to* **Occ** *reduces to Winskel's when restricted to safe nets, while the standard coreflection between* **Occ** *and* **Dom***, the category of finitary prime algebraic domains, when composed with the unfolding functor above, determines a chain of adjunctions between* **PTNets** *and* **Dom***.*

Introduction

Petri nets, introduced by C.A. Petri in [Pet62] (see also [Pet73, Rei85]), are a widely used model of concurrency. This model is attractive from a theoretical point of view because of its simplicity and because of its intrinsically concurrent nature, and has often been used as a semantic basis on which to interpret concurrent languages (see for example [Win82, Old87, vGV87, DDM88]).

For *Place/Transition (PT) nets*, having a satisfactory semantics—one that does justice to their truly concurrent nature, yet is abstract enough—remains in our view an unresolved problem. Certainly, many different semantics have been proposed in the literature; we briefly discuss some of them below.

At the most basic operational level we have of course the "*token game*". To account for computations involving many different transitions and for the *causal connections* between transition events, various notions of *process* have been proposed [Pet77, GR83, BD87], but process models do not provide a satisfactory semantic denotation for a net as a whole. In fact, they specify only the meaning of single, deterministic computations, while the accurate description of the fine interplay between concurrency and nondeterminism is one of the most valuable features of nets.

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Other semantic investigations have capitalized on the *algebraic structure* of PT nets, first noticed by Reisig [Rei85] and later exploited by Winskel to identify a sensible notion of *morphism* between nets [Win84, Win87]. More recently, a different interpretation of the algebraic structure of PT nets in terms of monoidal categories has been proposed in a paper by two of the authors [MM90].

One particular advantage of the algebraic approaches based on category theory is that they provide useful net combinators, associated to standard categorical constructions such as product and coproduct, which can be used to give a simple account of corresponding compositional operations at the level of a concurrent programming language, such as various forms of *parallel* and *non-deterministic* composition [Win87, MM90].

A unification of the process-oriented and algebraic views has recently been proposed by two of the present authors in joint work with P. Degano [DMM89], by showing that the *commutative processes* [BD87] of a net N are isomorphic to the arrows of a symmetric monoidal category $\mathcal{T}[N]$. Moreover, they introduced the *concatenable processes* of N—a slight variation of Goltz-Reisig processes [GR83] on which sequential composition is defined—and structured them as the arrows of the symmetric monoidal category $\mathcal{P}[N]$. That would individuate in the category of the symmetric monoidal categories a semantic domain for PT nets. However, in spite of accounting for algebraic and process aspects in a simple unified way, this semantics is still too concrete, and a more abstract semantics—one allowing greater semantic identifications between nets—would be clearly preferable.

A very attractive formulation for the semantics that we seek would be an *adjoint functor* assigning an abstract denotation to each PT net and preserving certain compositional properties in the assignment. This is exactly what Winskel has done for the subcategory of safe nets [Win86]. In that work—which builds on the previous work [NPW81]—the denotation of a safe net is a *Scott domain* [Sco70], and Winskel shows that there exists a coreflection—a particularly nice form of adjunction—between the category **Dom** of (coherent) *finitary prime algebraic domains* and the category **Safe** of *safe Petri nets*. This construction is completely satisfactory: from the intuitive point of view it gives the "truly concurrent" semantics of safe nets in the most universally accepted type of model, while from the formal point of view the existence of an adjunction guarantees its "naturality". Winskel's coreflection factorizes through the chain of coreflections

$$\underline{\textbf{Safe}} \underset{\hookleftarrow}{\xrightarrow{\mathcal{U}_w[\_]}} \underline{\textbf{Occ}} \underset{\mathcal{N}[\_]}{\overset{\mathcal{E}[\_]}{\rightleftarrows}} \underline{\textbf{PES}} \underset{\mathcal{P}r[\_]}{\overset{\mathcal{L}[\_]}{\rightleftarrows}} \underline{\textbf{Dom}}$$

where **PES** is the category of *prime event structures* (with binary conflict relation), which is equivalent to **Dom**, **Occ** is the category of *occurrence nets* [Win86], and $\hookrightarrow$ is the inclusion functor.

Recently, various attempts have been made to extend this chain or, more generally, to identify a suitable semantic domain for PT nets. Among them, we recall [Pra91], where, in order to obtain a model "mathematically more attractive than Petri nets", a *geometric* model of concurrency based on n-categories as models of higher dimensional automata is introduced, but the modelling power obtained is not greater than that of ordinary PT nets; [HKT92], in which the authors give semantics to PT nets in terms of generalized *trace languages* and discuss how using their work it could perhaps be possible to obtain a concept of unfolding for PT nets; and [Eng91], where the unfolding of Petri nets is given in term of a *branching process*. However, the nets considered in [Eng91] are not really PT nets because their transitions are restricted to have pre- and post-sets where all places have no multiplicities.

The present work extends Winskel's approach from safe nets to the category of PT nets. We define the *unfoldings* of PT nets and relate them by an *adjunction* to occurrence nets and therefore—exploiting the already existing adjunctions—to prime event structures and finitary prime algebraic domains. The adjunctions so obtained are extensions of the correspondent Winskel's coreflections.

The category **PTNets** that we consider is quite general. Objects are PT nets in which markings may be infinite and transitions are allowed to have infinite pre- and post-sets, but, as usual, with finite multiplicities. The only technical restriction we impose, with respect to the natural extension to nets with infinite markings of the general formulation in [MM90], is the usual condition that transitions must have non-empty pre-sets. Actually, the objects of **PTNets** strictly include those of the categories considered in [Win86, Win87]. Although a technical restriction applies to the morphisms—they are required to map places belonging to the initial marking or to the post-set of the same transition to disjoint multisets—they are still quite general. In particular, the category **PTNets** has *initial* and *terminal* objects, and has *products* and *coproducts* which model, respectively, the operations of parallel and non-deterministic composition of nets as in [Win87] and [MM90]. It is worth remarking that, while coproducts do *not* exist in the categories of generally marked, non-safe PT nets considered in the above cited works, they do in **PTNets**. However, due to the lack of space, such a result cannot be given here. It will be presented in a forthcoming full version of this work.

Concerning the organization of the paper, in Section 1 we introduce a new kind of nets, the *decorated occurrence nets*, which naturally represent the unfoldings of PT nets and can account for the multiplicities of places in transitions. They are occurrence nets in which places belonging to the post-set of the same transition are partitioned into *families*. Families are used to relate places corresponding in the unfolding to multiple instances of the same place in the original net. When all the families of a decorated occurrence net have cardinality one, we have (a net isomorphic to) an ordinary occurrence net. Therefore, **Occ** is (isomorphic to) a full subcategory of **DecOcc**, the category of decorated occurrence nets.

Then, we show an adjunction $\langle (\_)^+, \mathcal{U}[\_] \rangle : \underline{\mathbf{DecOcc}} \rightharpoonup \underline{\mathbf{PTNets}}$ whose right adjoint $\mathcal{U}[\_]$ gives the unfoldings of PT nets. This adjunction restricts to Winskel's coreflection from **Occ** to **Safe** as illustrated by the following commutative diagrams:

$$\begin{array}{ccc} \underline{\mathbf{PTNets}} & \xrightarrow{\mathcal{U}[\_]} & \underline{\mathbf{DecOcc}} \\ \uparrow & & \uparrow \\ \underline{\mathbf{Safe}} & \xrightarrow{\mathcal{U}_w[\_]} & \underline{\mathbf{Occ}} \end{array} \qquad\qquad \begin{array}{ccc} \underline{\mathbf{PTNets}} & \xleftarrow{(\_)^+} & \underline{\mathbf{DecOcc}} \\ \uparrow & & \uparrow \\ \underline{\mathbf{Safe}} & \hookleftarrow & \underline{\mathbf{Occ}} \end{array}$$

i.e. the left and the right adjoint, when restricted respectively to **Safe** and **Occ**, coincide with the correspondent adjoints of Winskel's coreflection.

In Section 2, we relate decorated occurrence nets to occurrence nets by showing an adjunction $\langle \mathcal{D}[\_], \mathcal{F}[\_] \rangle : \underline{\mathbf{Occ}} \rightharpoonup \underline{\mathbf{DecOcc}}$, where $\mathcal{F}[\_]$ is the *forgetful* functor which forgets about families. Moreover, the diagram

$$\begin{array}{ccc} \underline{\mathbf{PTNets}} & \xrightarrow{\mathcal{U}[\_]} & \underline{\mathbf{DecOcc}} \\ \uparrow & & \downarrow \mathcal{F}[\_] \\ \underline{\mathbf{Safe}} & \xrightarrow{\mathcal{U}_w[\_]} & \underline{\mathbf{Occ}} \end{array} \tag{1}$$

commutes.

Therefore, we get the desired adjunction between **Dom** and **PTNets** as the composition of the chain of adjunctions

$$\begin{array}{ccccccc}
\underline{\mathbf{PTNets}} & \underset{(_)^+}{\overset{\mathcal{U}[_]}{\rightleftarrows}} & \underline{\mathbf{DecOcc}} & & & & \\
 & & \mathcal{D}[_] \uparrow\ \downarrow \mathcal{F}[_] & & & & \\
 & & \underline{\mathbf{Occ}} & \underset{\mathcal{N}[_]}{\overset{\mathcal{E}[_]}{\rightleftarrows}} & \underline{\mathbf{PES}} & \underset{\mathcal{P}r[_]}{\overset{\mathcal{L}[_]}{\rightleftarrows}} & \underline{\mathbf{Dom}}
\end{array}$$

It follows from the commutative diagram (1) that, when **PTNets** is restricted to **Safe**, all the right adjoints in the above chain coincide with the corresponding functors defined by Winskel. In this sense, this work generalizes the work of Winskel and gives an abstract, truly concurrent semantics for PT nets. Moreover, the existence of left adjoints guarantees the "naturality" of this generalization.

1 PT Net Unfoldings

In this section we define the categories **PTNets** of *Place/Transition (PT) nets* and **DecOcc** of *decorated occurrence nets*. We define the *unfolding* of a PT net as a decorated occurrence net and show that it is a functor from **PTNets** to **DecOcc** which has a left adjoint.

A *pointed set* is a pair (S, s) where S is a set and $s \in S$ is a chosen element of S: the pointed element. Morphisms of pointed sets are functions that preserve the pointed elements. Therefore, pointed set morphisms provide a convenient way to treat partial functions between sets as total functions.

Given a set S, we denote by $S^{\mathcal{M}}$ the set of *multisets* of S, i.e. the set of all functions from S to the set of natural numbers ω, and by $S^{\mathcal{M}_\infty}$ the set of *multisets* with (possibly) *infinite multiplicities*, i.e. the functions from S to $\omega_\infty = \omega \cup \{\infty\}$. For $\mu \in S^{\mathcal{M}_\infty}$, we write $[\![\mu]\!]$ to denote the subset of S consisting of those elements s such that $\mu(s) > 0$.

A multiset $\mu \in S^{\mathcal{M}_\infty}$ can be represented as a formal sum $\bigoplus_{s \in S} \mu(s) \cdot s$. Given an arbitrary index set I and $\{\eta_i \in \omega_\infty \mid i \in I\}$, we define $\Sigma_{i \in I} \eta_i$ to be the usual sum in ω if only finitely many η_i are nonzero and ∞ otherwise. Then, we can give meaning to linear combinations of multisets, i.e. multisets of multisets, by defining

$$\bigoplus_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \cdot \mu = \bigoplus_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \cdot \left(\bigoplus_{s \in S} \mu(s) \cdot s \right) = \bigoplus_{s \in S} \Big(\Sigma_{\mu \in S^{\mathcal{M}_\infty}} \eta_\mu \mu(s) \Big) \cdot s.$$

A $(\_)^{\mathcal{M}_\infty}$–homomorphism from $S_0^{\mathcal{M}_\infty}$ to $S_1^{\mathcal{M}_\infty}$ is a function $g : S_0^{\mathcal{M}_\infty} \to S_1^{\mathcal{M}_\infty}$ such that

$$g(\mu) = \bigoplus_{s \in S_0} \mu(s) \cdot g(1 \cdot s),$$

where $1 \cdot s$ is the formal sum corresponding to the function which yields 1 on s and zero otherwise. Actually, it is worth noticing, that $(\_)^{\mathcal{M}_\infty}$ can be seen as an endofunctor on **Set**, the category of sets. As such, it defines a *commutative monad* [MM90] which sends S to $S^{\mathcal{M}_\infty}$, whose multiplication is the operation of linear combination of multisets and whose unit maps $s \in S$ to $1 \cdot s$. In these terms, $S^{\mathcal{M}_\infty}$ is a $(\_)^{\mathcal{M}_\infty}$–algebra and a $(\_)^{\mathcal{M}_\infty}$–homomorphism is a homomorphism between $(\_)^{\mathcal{M}_\infty}$-algebras.

We will regard $S^{\mathcal{M}}$ also as a pointed set whose pointed element is the empty multiset, i.e. the function which always yields zero, that, in the following, we denote by 0. In the paper, we will often denote a multiset $\mu \in S^{\mathcal{M}_\infty}$ by $\bigoplus_{i \in I} \eta_i s_i$ where $\{s_i \mid i \in I\} = [\![\mu]\!]$ and $\eta_i = \mu(s_i)$, i.e. as a sum whose summands are all nonzero. In case of multisets in $S^{\mathcal{M}}$, instead of η_i, we will use $n_i, m_i, \ldots$, the standard variables for natural numbers. Moreover, given $S' \subseteq S$, we will write $\bigoplus S'$ for $\bigoplus_{s \in S'} 1 \cdot s$.

Definition 1.1 *(PT Nets)*
A PT net *is a structure* $N = \left(\partial^0_N, \partial^1_N : (T_N, 0) \to S^{\mathcal{M}}_N, u^I_N\right)$ *where* S_N *is a set of* places*;* T_N *is a pointed set of* transitions*;* $\partial^0_N, \partial^1_N$ *are pointed set morphisms; and* $u^I_N \in S^{\mathcal{M}}_N$ *is the* initial marking*. Moreover, we assume the standard constraint that* $\partial^0_N(t) = 0$ *if and only if* $t = 0$*.*
A morphism *of PT nets from* N_0 *to* N_1 *consists of a pair of functions* $\langle f, g\rangle$ *such that:*

i. $f : T_{N_0} \to T_{N_1}$ *is a pointed set morphism;*

ii. $g : S^{\mathcal{M}_\infty}_{N_0} \to S^{\mathcal{M}_\infty}_{N_1}$ *is a* $(\_)^{\mathcal{M}_\infty}$*–homomorphism;*

iii. $\partial^0_{N_1} \circ f = g \circ \partial^0_{N_0}$ *and* $\partial^1_{N_1} \circ f = g \circ \partial^1_{N_0}$*, i.e.* $\langle f, g\rangle$ *respects source and target;*

iv. $g(u^I_{N_0}) = u^I_{N_1}$*, i.e.* $\langle f, g\rangle$ *respects the initial marking;*

v. $\forall b \in [\![u^I_{N_1}]\!]$, $\exists! a \in [\![u^I_{N_0}]\!]$ *such that* $b \in [\![g(a)]\!]$
$\forall b \in [\![\partial^1_{N_1}(f(t))]\!]$, $\exists! a \in [\![\partial^1_{N_0}(t)]\!]$ *such that* $b \in [\![g(a)]\!]$.

This, with the obvious componentwise composition of morphisms, gives the category **PTNets**.□

A PT net is thus a graph whose arcs are the transitions and whose nodes are the multisets on the set of places, i.e. *markings* of the net. As usual, transitions are restricted to have pre- and post-sets, i.e. sources and targets, in which each place has only finitely many tokens, i.e. finite multiplicity. The same is required for the initial marking. To be consistent with the use of zero transitions as a way to treat partial mappings, they are required to have empty pre- and post-sets. Moreover, they are the only transitions which can have empty pre-sets. To simplify notation, we assume the standard constraint that $T_N \cap S_N = \emptyset$—which of course can always be achieved by an appropriate renaming.

Morphisms of PT nets are graph morphisms in the precise sense of respecting source and target of transitions, i.e. they make the diagram

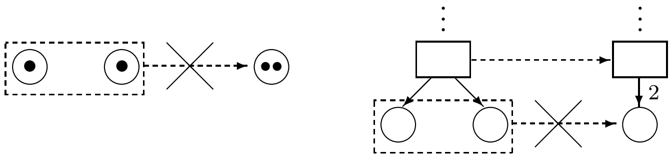
$$
\begin{array}{ccccc}
T_{N_0} & \underset{\partial^1_{N_0}}{\overset{\partial^0_{N_0}}{\rightrightarrows}} & S^{\mathcal{M}}_{N_0} & \hookrightarrow & S^{\mathcal{M}_\infty}_{N_0} \\
{\scriptstyle f}\big\downarrow & & & & \big\downarrow{\scriptstyle g} \\
T_{N_1} & \underset{\partial^1_{N_1}}{\overset{\partial^0_{N_1}}{\rightrightarrows}} & S^{\mathcal{M}}_{N_1} & \hookrightarrow & S^{\mathcal{M}_\infty}_{N_1}
\end{array}
$$

commute. Moreover they map initial markings to initial markings. To simplify notation, we will sometimes use a single letter to denote a morphism $\langle f, g\rangle$. In these cases, the type of the argument will identify which component we are referring to.

A $(\_)^{\mathcal{M}_\infty}$–homomorphism $g : S^{\mathcal{M}_\infty}_{N_0} \to S^{\mathcal{M}_\infty}_{N_1}$, which constitutes the place component of a morphism $\langle f, g\rangle$, is completely defined by its behaviour on S_{N_0}, the generators of $S^{\mathcal{M}_\infty}_{N_0}$.

Therefore, we will often define morphisms between nets giving their transition components and a map $g : S_{N_0} \to S_{N_1}^{\mathcal{M}_\infty}$ for their place components: it is implicit that the latter have to be thought of as lifted to the correspondent $(\_)^{\mathcal{M}_\infty}$–homomorphisms.

The last condition in the definition means that morphisms are not allowed to map two different places in the initial marking or in the post-set of some transition to two multisets having a place in common. This is pictorially described in the figure below, where dashed arrows represent the forbidden morphisms. We use the standard graphical representation of nets in which circles are places, boxes are transitions, the initial marking is given by the number of "tokens" in the places, and sources and targets are directed arcs whose weights represent multiplicities. Unitary weights are omitted.



Such a condition will play an important role while establishing the adjunction between **PTNets** and **DecOcc**. In fact, it is crucial for showing the *universality* of the *counit* of the adjunction.

Now, we recall the definition of a well-known class of nets: *safe nets*.

Definition 1.2 *(Safe Nets)*
A PT net N is safe *if and only if*

$$\forall t \in T_N,\ \bigoplus [\![\partial_N^i(t)]\!] = \partial_N^i(t),\ \textit{for } i = 0, 1 \textit{ and } \forall v \in \mathcal{R}[N],\ \bigoplus [\![v]\!] = v,$$

where $\mathcal{R}[N]$ is the set of reachable markings of N (see, for instance, [Rei85]).
This defines the category **Safe** *as a full subcategory of* **PTNets***.* □

Observe that $\bigoplus [\![v]\!] = v$ is a compact way of saying that each $s \in S$ has multiplicity at most one in v. Therefore this definition is exactly the classical definition of safe nets.

Another important class of nets is that of *occurrence nets*. In the following, we will use ${}^\bullet a$ to mean the *pre-set* of a, that is ${}^\bullet a = \{t \in T_N \mid a \in [\![\partial_N^1(t)]\!]\}$. Symmetrically, the *post-set* of a is indicated as $a^\bullet = \{t \in T_N \mid a \in [\![\partial_N^0(t)]\!]\}$. These notations are extended in the obvious way to the case of sets of places.

Definition 1.3 *(Occurrence Nets)*
A (non-deterministic) occurrence net *is a safe net Θ such that*

i. $a \in [\![u_\Theta^I]\!]$ *if and only if* ${}^\bullet a = \emptyset$*;*

ii. $\forall a \in S_\Theta,\ |{}^\bullet a| \leq 1$*, where* $|\_|$ *gives the cardinality of sets;*

iii. $\prec$ *is irreflexive, where* $\prec$ *is the transitive closure of the relation*

$$\prec^1 = \{(a,t) \mid a \in S_\Theta,\, t \in T_\Theta,\, t \in a^\bullet\} \cup \{(t,a) \mid a \in S_\Theta,\, t \in T_\Theta,\, t \in {}^\bullet a\};$$

moreover, $\forall t \in T_\Theta,\ \{t' \in T_\Theta \mid t' \prec t\}$ *is finite;*

iv. the binary "conflict" relation $\#$ *on* $T_\Theta \cup S_\Theta$ *is irreflexive, where*

$$\forall t_1, t_2 \in T_\Theta,\ t_1 \#_m t_2 \Leftrightarrow [\![\partial_\Theta^0(t_1)]\!] \cap [\![\partial_\Theta^0(t_2)]\!] \neq \emptyset \textit{ and } t_1 \neq t_2,$$

$$\forall x, y \in T_\Theta \cup S_\Theta,\ x \# y \Leftrightarrow \exists t_1, t_2 \in T_\Theta :\ t_1 \#_m t_2 \textit{ and } t_1 \preceq x \textit{ and } t_2 \preceq y,$$

where $\preceq$ *is the reflexive closure of* $\prec$*.*

This defines the category **Occ** *as a full subcategory of* **Safe** □

It is easy to see that Winskel's categories of safe nets, called **<u>Net</u>**, and of occurrence nets, here called **<u>Occ</u>**$_{\mathbf{W}}$, are *full* subcategories of **<u>Safe</u>** and **<u>Occ</u>**. In fact, the objects of **<u>Net</u>** (**<u>Occ</u>**$_{\mathbf{W}}$) are the objects of **<u>Safe</u>** (**<u>Occ</u>**) with sets of places, initial markings and post-sets which are non-empty, and without *isolated places*—places belonging neither to the initial marking nor to the pre- or post-set of any transition—while the morphisms between any pair of nets in **<u>Net</u>** (**<u>Occ</u>**$_{\mathbf{W}}$) coincide with the morphisms between the same pair of nets in **<u>Safe</u>** (**<u>Occ</u>**). However, since all the results in [Win86] easily extend to **<u>Safe</u>** and **<u>Occ</u>**, in the following we will ignore any difference between **<u>Safe</u>** and **<u>Net</u>** and between **<u>Occ</u>** and **<u>Occ</u>**$_{\mathbf{W}}$.

We now introduce the category of *decorated occurrence nets*, a type of occurrence nets in which places are grouped into families. They allow a convenient treatment of multiplicity issues in the unfolding of PT nets. We will use the following notations:

$[n, m]$	for the segment $\{n, \ldots, m\}$ of ω;
$[n]$	for $[1, n]$;
$[k]_i$	for the i-th block of length k of $\omega - \{0\}$, i.e. $[ik] - [(i-1)k]$.

Definition 1.4 *(Block Functions)*
We call a function $f : [n] \to [m]$ *a* block function *if and only if* $n = km$ *and* $f([k]_i) = \{i\}$, *for* $i = 1, \ldots, m$. □

The place component g of a PT net morphism $\langle f, g\rangle : N_0 \to N_1$ can be thought of as a *multirelation* (with possibly infinite multiplicities) between S_{N_0} and S_{N_1}, namely the multirelation g such that $a\, g\, \eta b$ if and only if $g(a)(b) = \eta$. Indeed, this is a (generalization of a) widely used formalization of net morphisms due to Winskel [Win84, Win87]. In the case of morphisms between occurrence nets, we have that g is a *relation* and that the inverse relation g^{op}, defined by $bg^{op}a$ if and only if agb, restricts to (total) functions $g^{op}_{\emptyset} : [\![u^I_{N_1}]\!] \to [\![u^I_{N_0}]\!]$ and $g^{op}_{\{t\}} : [\![\partial^1_{N_1}(f(t))]\!] \to [\![\partial^1_{N_0}(t)]\!]$ for each $t \in T_{N_0}$. We will use these functions in the next definition.

Definition 1.5 *(Decorated Occurrence Nets)*
A decorated occurrence net *is an occurrence net* Θ *such that:*

i. S_Θ *is of the form* $\bigcup_{a\in A_\Theta}\{a\} \times [n_a]$, *where the set* $\{a\} \times [n_a]$ *is called the family of* a. *We will use* a^F *to denote the family of* a *regarded as a multiset;*

ii. $\forall a \in A_\Theta,\ \forall x, y \in \{a\} \times [n_a],\ {}^\bullet x = {}^\bullet y$.

A morphism *of decorated occurrence nets* $\langle f, g\rangle : \Theta_0 \to \Theta_1$ *is a morphism of occurrence nets which respects families, i.e. for each* $[\![a^F]\!] \subseteq S_{\Theta_0}$, *given* $x = {}^\bullet[\![a^F]\!]$*—which is a singleton set or the empty set by ii above and the definition of occurrence nets—we have:*

i. $g(a^F) = \bigoplus_{i\in I_a} b^F_i$, *for some index set* I_a*;*

ii. $\pi_a \circ g^{op}_i \circ in_{b_i}$ *is a block function, where*
π_a *is the projection of* $\{a\} \times [n_a]$ *to* $[n_a]$,
in_a *is the bijection from* $[n_a]$ *to* $\{a\} \times [n_a]$, *and*
$g^{op}_i : \{b_i\} \times [n_{b_i}] \to \{a\} \times [n_a]$ *is* g^{op}_x *restricted to* $\{b_i\} \times [n_{b_i}]$.

This defines the category **<u>DecOcc</u>**. □

A family is thus a collection of finitely many places with the same pre-set, and a decorated occurrence net is an occurrence net where each place belongs to exactly one family. Families, and therefore decorated occurrence nets, are capable of describing *relationships* between places by grouping them together. We will use families to relate places which are *instances* of the same place obtained in a process of unfolding. Therefore, morphisms treat families in a special way: they map families to families (condition i) and they do that in a unique pre-determined way (condition ii). This is because what we want to describe is that a^F is mapped to b^F. Hence, since the way to map a family to another family is fixed by definition, in the following we will often define morphisms by just saying what families are sent to what families.

Observe that the full subcategory of **DecOcc** consisting of all nets Θ such that $S_\Theta = \bigcup_{a\in A_\Theta}\{a\}\times[1]$ is (isomorphic to) **Occ**. Observe also that, since the initial marking consists exactly of the elements with empty pre-set and, by point ii in Definition 1.5, elements of a family have the same pre-set, for a decorated occurrence net u^I_Θ is of the form $\bigoplus_{i\in I} a_i^F$.

We have seen that for occurrence nets and decorated occurrence nets simple concepts of causal dependence ($\prec$) and conflict ($\#$) can be defined. The orthogonal concept is that of concurrency.

Definition 1.6 *(Concurrent Elements)*
Given a (decorated) occurrence net Θ (which defines $\prec$, $\preceq$ and $\#$), we can define

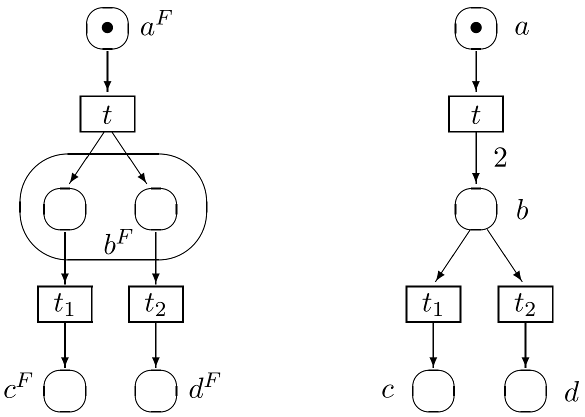
- *For $x, y \in T_\Theta \cup S_\Theta$, x co y iff $\neg(x \prec y$ or $y \prec x$ or $x\#y)$;*
- *For $X \subseteq T_\Theta \cup S_\Theta$, $\mathrm{Co}(X)$ iff $(\forall x, y \in X,\ x \text{ co } y)$ and $|\{t \in T_\Theta \mid \exists x \in X,\ t \preceq x\}| \in \omega$.* □

As a first step in relating the categories **DecOcc** and **PTNets**, we define a functor from decorated occurrence nets to PT nets.

Proposition 1.7 *($(\_)^+$: from **DecOcc** to **PTNets**)*
Given the decorated occurrence net $\Theta = \left(\partial^0_\Theta, \partial^1_\Theta : (T_\Theta, 0) \to (\bigcup_{a\in A_\Theta}\{a\}\times[n_a])^{\mathcal{M}}, u^I_\Theta\right)$ let $(\_)^+$ denote the $(\_)^{\mathcal{M}_\infty}$–homomorphism from $S_\Theta^{\mathcal{M}_\infty}$ to $A_\Theta^{\mathcal{M}_\infty}$ such that $(a,j)^+ = a$.
Then, we define Θ^+ to be the net $\left((\_)^+ \circ \partial^0_\Theta,\ (\_)^+ \circ \partial^1_\Theta : (T_\Theta, 0) \to A_\Theta^{\mathcal{M}}, (u^I_\Theta)^+\right)$.
Given a morphism $\langle f, g\rangle : \Theta_0 \to \Theta_1$, let $\langle f, g\rangle^+ : \Theta_0^+ \to \Theta_1^+$ be $\langle f, (\_)^+ \circ g \circ in\rangle$ where $in : A_{\Theta_0}^{\mathcal{M}_\infty} \to S_{\Theta_0}^{\mathcal{M}_\infty}$ is the $(\_)^{\mathcal{M}_\infty}$–homomorphism such that $in(a) = (a, 1)$.
*Then, $(\_)^+ :$ **DecOcc** $\to$ **Occ** is a functor.* □

The following example shows the result of applying $(\_)^+$ to a decorated occurrence net. In all the pictures to follow, a family is represented by drawing its elements from left to right in accordance with its ordering, and enclosing them into an oval. Families of cardinality one are not explicitly indicated.

Example 1.8



A decorated occurrence net Θ and the net Θ^+

Nets obtained via $(\_)^+$ from decorated occurrence nets have a structure very similar to that of occurrence nets. In particular they have places whose pre-sets contain at most one transition and the places in the initial marking are exactly those with empty pre-set. Moreover, the causal dependence relation $\prec$, defined as in the case of occurrence nets, is irreflexive. Observe that, if Θ is (isomorphic to) an occurrence net, then Θ^+ is an occurrence net isomorphic to Θ. We will denote by $\underline{\mathbf{DecOcc}^+}$ the full subcategory of $\underline{\mathbf{PTNets}}$ consisting of (nets isomorphic to) nets of the form Θ^+.

Let $\underline{\mathcal{B}}$ range over $\underline{\mathbf{Occ}}$, $\underline{\mathbf{DecOcc}}$ and $\underline{\mathbf{DecOcc}^+}$. For any net in $\underline{\mathcal{B}}$, we can define the concept of *depth* of an element of the net, thanks to their nice tree-like structure.

Definition 1.9 *(Depth)*
Let Θ be a net in $\underline{\mathcal{B}}$. The depth *of an element in $T_\Theta \cup S_\Theta$ is inductively defined by:*

- $\mathrm{depth}(b) = 0$ *if* $b \in [\![u^I_\Theta]\!]$;
- $\mathrm{depth}(t) = \max\{\mathrm{depth}(b) \mid b \prec t\} + 1$;
- $\mathrm{depth}(b) = \mathrm{depth}(t)$ *if* $\{t\} = {}^\bullet b$. □

Given a net Θ in $\underline{\mathcal{B}}$ its *subnet* of depth n is the net $\Theta^{(n)}$ consisting of the elements of Θ whose depth is not greater than n. Clearly, for each $n \leq m$ there is a morphism $in_{n,m} : \Theta^{(n)} \to \Theta^{(m)}$ whose components are both set inclusions. In the following we will call such net morphisms simply *inclusions* and we will denote the inclusion of $\Theta^{(n)}$ in $\Theta^{(n+1)}$ by in_n.

Now, consider the category $\underline{\boldsymbol{\omega}} = \{0 \to 1 \to 2 \cdots\}$ and the class $\mathcal{D}$ of diagrams $D : \underline{\boldsymbol{\omega}} \to \underline{\mathcal{B}}$ such that $D(n \to n+1) : D(n) \to D(n+1)$ is an inclusion. For such a class we have the following results. The reader is referred to [ML71, III.3] for the definition of the categorical concepts involved.

Proposition 1.10 *(*$\mathrm{Colim}(D)$ *exists and Θ is the colimit of its subnets)*
i. For any $D \in \mathcal{D}$, the colimit of D in $\underline{\mathcal{B}}$ exists.
ii. Given a net Θ in $\underline{\mathcal{B}}$, let $D_\Theta : \underline{\boldsymbol{\omega}} \to \underline{\mathcal{B}}$ be the diagram such that $D_\Theta(n) = \Theta^{(n)}$ and $D_\Theta(n \to n+1) = in_n$. Then $\Theta = \mathrm{Colim}(D_\Theta)$. □

Next, we define a functor from $\underline{\mathbf{PTNets}}$ to $\underline{\mathbf{DecOcc}}$ which will be the right adjoint to $(\_)^+$. We start by giving the object component of such a functor. To this aim, given a net N, we define a family of decorated occurrence nets, one for each $n \in \omega$, where the n-th net approximates the unfolding of N up to depth n, i.e. it reflects the behaviour of the original net up to step sequences of length at most n. Clearly, the unfolding of N will be defined to be the colimit of an appropriate $\boldsymbol{\omega}$-shaped diagram built on the approximant nets. We will use the following notation: given $s \in X_1 \times \ldots \times X_n$, we denote by $s \downarrow X_i$ the projection of s on the X_i component. Moreover, given $S = \bigcup\{s_j \mid j \in J\}$, $S \downarrow X_i$ will be $\{s_j \downarrow X_i \mid j \in J\}$ and $S \overset{\oplus}{\downarrow} X_i$ will denote $\bigoplus_{j \in J} (s_j \downarrow X_i)$.

Definition 1.11 *(PT Nets Unfoldings: $\mathcal{U}[\_]^{(n)}$)*
Let $N = \left(\partial^0_N, \partial^1_N : (T_N, 0) \to S^{\mathcal{M}}_N, u^I_N\right)$ be a net in $\underline{\mathbf{PTNets}}$.
We define the nets $\mathcal{U}[N]^{(k)} = \left(\partial^0_k, \partial^1_k : (T_k, 0) \to S^{\mathcal{M}}_k, u^I_k\right)$, for $k \in \omega$, where:

- $S_0 = \bigcup\left\{ \{(\emptyset, b)\} \times [n] \mid u^I_N(b) = n \right\}$;
- $T_0 = \{0\}$, *and the ∂^i_0 with the obvious definitions;*
- $u^I_0 = \bigoplus S_0$;

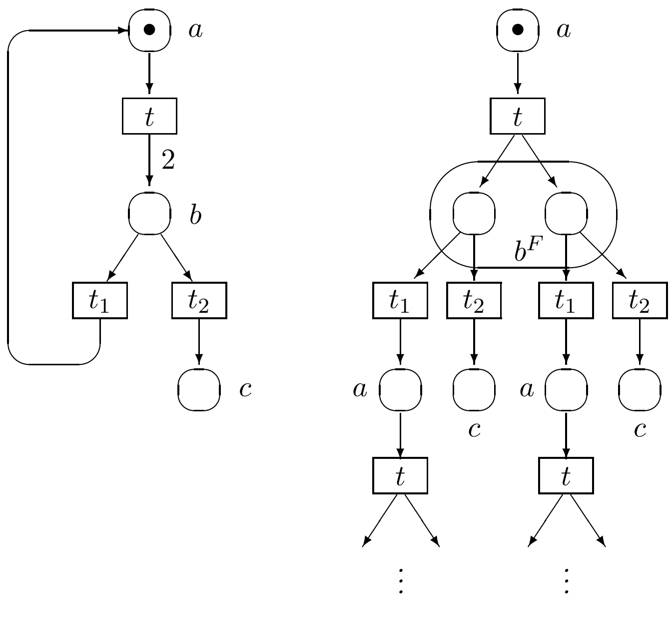
for $k > 0$,

- $T_k = T_{k-1} \cup \Big\{ (B,t) \;\Big|\; B \subseteq S_{k-1},\ \mathrm{Co}(B),\ t \in T_N,\ B \overset{\oplus}{\downarrow} S_N = \partial^0_N(t) \Big\}$;
- $S_k = S_{k-1} \cup \Big(\bigcup \Big\{ \Big\{ \big(\{t_0\}, b\big) \Big\} \times [n] \;\Big|\; t_0 \in T_k,\ b \in S_N,\ \partial^1_N(t_0 \downarrow T)(b) = n \Big\} \Big)$;
- $\partial^0_k(B,t) = \bigoplus B$, *and* $\partial^1_k(B,t) = \bigoplus \Big\{ \big((\{(B,t)\}, b), i\big) \in S_k \Big\}$;
- $u^I_k = \bigoplus \Big\{ \big((\emptyset, b), i\big) \in S_k \Big\} = \bigoplus S_0 = u^I_0$. □

Therefore, informally speaking, the net $\mathcal{U}[N]^{(0)}$ is obtained by exploding in families the initial marking of N, and $\mathcal{U}[N]^{(n+1)}$ is obtained, inductively, by generating a new transition for each possible subset of concurrent places of $\mathcal{U}[N]^{(n)}$ whose corresponding multiset of places of N constitutes the source of some transition t of N; the target of t is also exploded in families which are added to $\mathcal{U}[N]^{(n+1)}$. As a consequence, the transitions of the n-th approximant net are instances of transitions of N, in the precise sense that each of them corresponds to a unique occurrence of a transition of N in one of its step sequences of length at most n.

Definition 1.12 *(PT Net Unfoldings: $\mathcal{U}[\_]$)*
*We define $\mathcal{U}[N]$ to be the colimit of the diagram $D : \underline{\boldsymbol{\omega}} \to$ **DecOcc** such that $D(n) = \mathcal{U}[N]^{(n)}$ and $D(n \to n+1) = in_n$. Since for all $n \in \omega$, $\mathcal{U}[N]^{(n)}$ is a decorated occurrence net of depth n and moreover for any $n \in \omega$ there is an inclusion $in_n : \mathcal{U}[N]^{(n)} \to \mathcal{U}[N]^{(n+1)}$, then D belongs to $\mathcal{D}$ and so, by Proposition 1.10 (i), the colimit exists and is a decorated occurrence net.* □

Example 1.13



A PT Net N and (part of) its unfolding $\mathcal{U}[N]$

The correspondence between elements of the unfolding and elements of the original net is formalized by the folding morphism, which will also be the counit of the adjunction.

Proposition 1.14 *(Folding Morphism)*
Consider the map $\epsilon_N = \langle f_\epsilon, g_\epsilon \rangle : \mathcal{U}[N]^+ \to N$ *defined by*

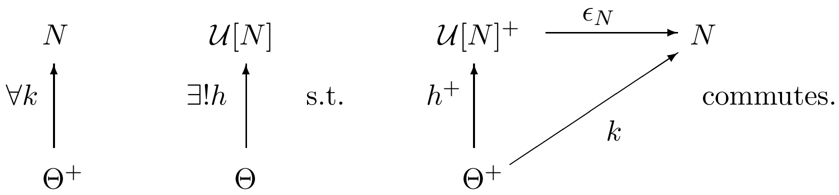
- $f_\epsilon(B, t) = t$ *and* $f_\epsilon(0) = 0$;
- $g_\epsilon(\bigoplus_i (x_i, y_i)) = \bigoplus_i y_i$.

Then, ϵ_N *is a morphism in* **PTNets**, *called the* folding *of* $\mathcal{U}[N]$ *into* N. □

Finally, we are ready to prove that $\mathcal{U}[\_]$ is right adjoint to $(\_)^+$.

Theorem 1.15 *(*$(\_)^+ \dashv \mathcal{U}[\_]$*)*
The pair $\langle (\_)^+, \mathcal{U}[\_] \rangle$: **DecOcc** $\rightharpoonup$ **PTNets** *constitutes an adjunction.*

Proof. Let N be a PT Net and $\mathcal{U}[N]$ its unfolding. By [ML71, Theorem 2, pg. 81], it is enough to show that the folding $\epsilon_N : \mathcal{U}[N]^+ \to N$ is universal from $(\_)^+$ to N, i.e. for any decorated occurrence net Θ and any morphism $k : \Theta^+ \to N$ in **PTNets**, there exists a unique $h : \Theta \to \mathcal{U}[N]$ in **DecOcc** such that $k = \epsilon_N \circ h^+$.



Consider the diagram in **DecOcc** given by $D_\Theta(n) = \Theta^{(n)}$, the subnet of Θ of depth n and $D_\Theta(n \to n+1) = in_n : \Theta^{(n)} \to \Theta^{(n+1)}$. We define a sequence of morphisms of nets $h_n : \Theta^{(n)} \to \mathcal{U}[N]$, such that for each n, $h_n = h_{n+1} \circ in_n$.
Since by Proposition 1.10 *(ii)* $\Theta = \mathrm{Colim}(D_\Theta)$, there is a unique $h : \Theta \to \mathcal{U}[N]$ such that $h \circ \mu_n = h_n$ for each n. At the same time, we show that

$$\forall n \in \omega,\ k \circ \mu_n^+ = \epsilon_N \circ h_n^+ \qquad (1)$$

and that the h_n are the unique sequence of morphisms $h_n : \Theta^{(n)} \to \mathcal{U}[N]$ such that (1) holds. Now, by functoriality of $(\_)^+$, we have that

$$\forall n \in \omega,\ k \circ \mu_n^+ = \epsilon_N \circ h^+ \circ \mu_n^+.$$

Therefore, since $(\_)^+ \circ D_\Theta = D_{\Theta^+}$ and, by Proposition 1.10 *(ii)*, $\Theta^+ = \mathrm{Colim}(D_\Theta^+) = \mathrm{Colim}((\_)^+ \circ D_\Theta)$, by the universal property of the colimit we must have $k = \epsilon_N \circ h^+$.

To show the uniqueness of h, let h' be such that $k = \epsilon_N \circ h'^+$. Then we have $k \circ \mu_n^+ = \epsilon_N \circ h'^+ \circ \mu_n^+$. But h_n is the unique morphism for which this happens. Therefore, for each n, $h_n = h' \circ \mu_n$ and so, by the universal property of the colimit, $h = h'$.

The definition of the h_n and the proof of their uniqueness proceed by induction on n, exploiting k, condition v in Definition 1.1 and the correspondence between the structure of the families of $\mathcal{U}[N]$ and the multiplicities originally present in N. □

Theorem 1.16 *(Correspondence with Winskel's Safe Net Unfoldings [Win86])*
Let N *be a safe net.*
Then, its unfolding $\mathcal{U}[N]$ *is (isomorphic to) an occurrence net and, therefore,* $\mathcal{U}[N]^+ \cong \mathcal{U}[N]$. *Moreover,* $\mathcal{U}[N]$ *is (isomorphic to) Winskel's unfolding of* N. *Finally, whenever* N *is (isomorphic to) an occurrence net, the unit of the adjunction* $(\_)^+ \dashv \mathcal{U}[\_]$, $\eta_N : N \to \mathcal{U}[N^+] \cong \mathcal{U}[N]$, *is an isomorphism.*
Therefore, the adjunction $\langle (\_)^+, \mathcal{U}[\_] \rangle$: **DecOcc** $\rightharpoonup$ **PTNets** *restricts to Winskel's coreflection* $\langle (\_)^+_{\mathbf{Occ}}, \mathcal{U}[\_]_{\mathbf{Safe}} \rangle$: **Occ** $\rightharpoonup$ **Safe**. □

2 PT Nets, Event Structures and Domains

In this section, we show an adjunction between occurrence nets and decorated occurrence nets. Composing this adjunction with that given in Section 1, we obtain an adjunction between **Occ** and **PTNets**. Moreover, exploiting Winskel's coreflections in [Win86], we obtain adjunctions between **PES** and **PTNets** and between **Dom** and **PTNets**, as explained in the Introduction.

We first define a functor from decorated occurrence nets to occurrence nets. It is simply the *forgetful* functor which forgets about the structure of families.

Definition 2.1 *($\mathcal{F}[\_]$: from* **DecOcc** *to* **Occ***)*
Given a decorated occurrence net Θ define $\mathcal{F}[\Theta]$ to be the occurrence net Θ. Furthermore, given $\langle f, g\rangle : \Theta_0 \to \Theta_1$, define $\mathcal{F}[\langle f, g\rangle]$ to be $\langle f, g\rangle$. □

In order to define a left adjoint for $\mathcal{F}[\_]$, we need to identify, for any occurrence net Θ, a decorated occurrence net which is, informally speaking, a "saturated" version of Θ, in the sense that it can match in a *unique* way the structure of the families of any decorated occurrence net built on (subnets of) Θ. Because of the uniqueness requirement, saturating occurrence nets is a delicate matter: we need to identify a suitable set of families which can "factorize" uniquely all the others. To this aim are devoted the following definition and lemma, where the relation $\mapsto$ is introduced to capture the behaviour of decorated occurrence net morphisms on families and *prime strings* are meant to represent—in a sense that will be clear later—exactly the families which we must add to Θ in order to saturate it.

In the following, given a string s on an alphabet Σ, as usual we denote the i-th element of s by s_i and its length by $|s|$. Moreover, σ^n, for $\sigma \in \Sigma$ and $n \in \omega$, will denote the string consisting of the symbol σ repeated n times.

Definition 2.2 *(Prime Strings)*
Let Σ be an alphabet, i.e. a set of symbols. Define the binary relation $\mapsto$ on Σ^+, the language of non-empty strings on Σ, by

$$\sigma_1^{n_1} \dots \sigma_k^{n_k} \mapsto \sigma_1^{m_1} \dots \sigma_k^{m_k} \Leftrightarrow \sigma_i \neq \sigma_{i+1} \text{ and } \exists q \in \omega \text{ s.t. } qn_i = m_i,\ i = 1, \dots, k.$$

Define the language of prime strings *on Σ to be*

$$\Sigma^P = \Sigma^+ - \left\{\sigma_1^{n_1}\sigma_2^{n_2} \dots \sigma_k^{n_k} \mid \sigma_i \in \Sigma,\ \sigma_i \neq \sigma_{i+1},\ \gcd(n_1, \dots, n_k) > 1\right\},$$

where gcd *is the greatest common divisor.* □

Lemma 2.3 *(Prime Strings are primes)*
Given $s' \in \Sigma^+$ there exists a unique $s \in \Sigma^P$ s.t. $s \mapsto s'$. □

We start relating strings and nets by looking at sets of places as alphabets and by looking at families as strings on such alphabets.

Given a (decorated) occurrence net Θ and a transition $t \in T_\Theta$, we denote by $\Sigma_{\{t\}}$ the alphabet $[\![\partial^1_\Theta(t)]\!]$. By analogy, since the places in the initial marking are in the post-set of no transition, $\Sigma_\emptyset$ will consist of the places $[\![u^I_\Theta]\!]$; following the analogy, in the rest of the section u^I_Θ will also be denoted by $\partial^1_\Theta(\emptyset)$.

Since a family b^F of a decorated occurrence net Θ is nothing but an ordered subset of the initial marking or of the post-set of a transition, it corresponds naturally to a string in Σ^+_x where

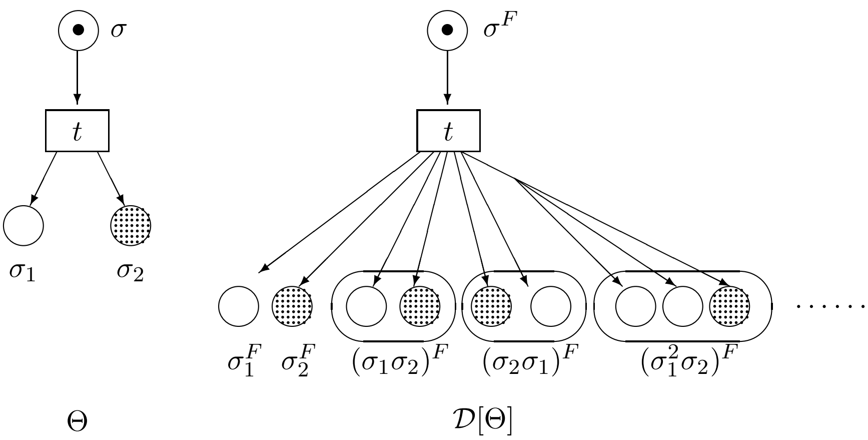
$x = {}^{\bullet}[\![b^F]\!]$, namely, the string of length $|[\![b^F]\!]|$ whose i-th element is (b, i). We will write $\hat{b}^F$ to indicate such a string.

Now, we can define the saturated net corresponding to an occurrence net Θ. It is the net $\mathcal{D}[\Theta]$ whose transitions are the transitions of Θ, and whose families are the prime strings on the alphabets defined by Θ. It is immediate to see that this construction is well-defined, i.e. that $\mathcal{D}[\Theta]$ is a decorated occurrence net.

Definition 2.4 *($\mathcal{D}[\_]$: from* **<u>Occ</u>** *to* **<u>DecOcc</u>***)*
Let Θ be a net in **<u>Occ</u>***.*
We define the decorated occurrence net $\mathcal{D}[\Theta] = \left(\partial^0_{\mathcal{D}[\Theta]}, \partial^1_{\mathcal{D}[\Theta]} : (T_\Theta, 0) \to S^{\mathcal{M}}_{\mathcal{D}[\Theta]}, u^I_{\mathcal{D}[\Theta]}\right)$, where

- $S_{\mathcal{D}[\Theta]} = \bigcup \Big\{ \{s\} \times \big[\,|\,s\,|\,\big] \;\mid\; s \in \Sigma^P_x \text{ and } (x = \{t\} \subseteq T_\Theta \text{ or } x = \emptyset) \Big\}$;
- $\partial^0_{\mathcal{D}[\Theta]}(t) = \bigoplus \Big\{ (s, i) \in S_{\mathcal{D}[\Theta]} \mid s_i \in [\![\partial^0_\Theta(t)]\!] \Big\}$;
- $\partial^1_{\mathcal{D}[\Theta]}(t) = \bigoplus \Big\{ (s, i) \in S_{\mathcal{D}[\Theta]} \mid s_i \in [\![\partial^1_\Theta(t)]\!] \Big\} = \bigoplus \Big\{ s^F \mid s \in \Sigma^P_{\{t\}} \Big\}$;
- $u^I_{\mathcal{D}[\Theta]} = \bigoplus \Big\{ s^F \mid s \in \Sigma^P_\emptyset \Big\}$. □

Example 2.5



An occurrence net Θ and (part of) the decorated occurrence net $\mathcal{D}[\Theta]$

We now select a candidate for the unit of the adjunction.

Proposition 2.6 *(Unit Morphism)*
Given an occurrence net Θ consider the map $\eta_\Theta : \Theta \to \mathcal{F}\mathcal{D}[\Theta]$ defined by:

$$\eta_\Theta(t) = t;$$
$$\eta_\Theta(a) = \bigoplus\{(s, i) \in S_{\mathcal{D}[\Theta]} \mid s_i = a\}.$$

Then η_Θ is a morphism in **<u>Occ</u>***.* □

In order to illustrate the above definition, considering again the net Θ of Example 2.5. For such a net we have that

$$\eta_\Theta(\sigma_1) = (\sigma_1, 1) \oplus (\sigma_1\sigma_2, 1) \oplus (\sigma_2\sigma_1, 2) \oplus (\sigma_1^2\sigma_2, 1) \oplus (\sigma_1^2\sigma_2, 2) \oplus \ldots;$$
$$\eta_\Theta(\sigma_2) = (\sigma_2, 1) \oplus (\sigma_1\sigma_2, 2) \oplus (\sigma_2\sigma_1, 1) \oplus (\sigma_1^2\sigma_2, 2) \oplus \ldots.$$

Before showing that η_Θ is universal, we need to develop further the relation between nets and strings. Since a morphism maps post-sets to post-sets, it naturally induces a (contravariant) mapping between the languages associated to transitions related by the morphism. To simplify notation, in the rest of this section, for k a morphism of nets, $k(\{t\})$ and $k(\emptyset)$, denote, respectively, $\{k(t)\}$ and $\emptyset$. Moreover, $\partial^1_\Theta(\{t\})$ denotes $\partial^1_\Theta(t)$.

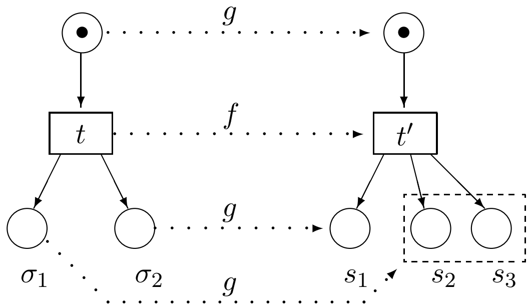
Definition 2.7 *($\mathcal{S}_k^x$: from $\Sigma_{k(x)}^+$ to Σ_x^+)*
Let Θ_0 and Θ_1 be (decorated) occurrence nets, let $k = \langle f, g\rangle : \Theta_0 \to \Theta_1$ be a morphism and let $x = \{t\} \subseteq T_{\Theta_0}$ or $x = \emptyset$ and y be such that $f(x) = y$. Then k induces a unique semigroup homomorphism $\mathcal{S}_k^x$ from Σ_y^+ to Σ_x^+ defined on the generators $b \in [\![\partial^1_{\Theta_1}(y)]\!]$ by

$$\mathcal{S}_k^x(b) = a \in [\![\partial^1_{\Theta_0}(x)]\!] \text{ s.t. } g(a) = b.$$

From the properties of safe net morphisms, it is easy to see that $\mathcal{S}_k^x$ is well-defined, i.e. there exists one and only one $a \in [\![\partial^1_{\Theta_0}(x)]\!]$ such that $g(a) = b$. □

To clarify the relation between $\mapsto$ and decorated occurrence net morphisms, observe that, in the condition of the previous definition, if Θ is a decorated occurrence net and k is a decorated occurrence net morphism, then $\hat{a}^F \mapsto \mathcal{S}_k^x(\hat{b}^F)$ if and only if $[\![b^F]\!] \subseteq [\![k(a^F)]\!]$.

Example 2.8
Consider the following figure, where the morphism $\langle f, g\rangle$ is such that $g(\sigma_1) = s_2 \oplus s_3$ and $g(\sigma_2) = s_1$.

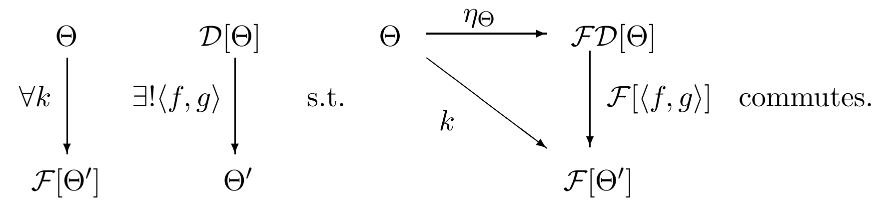


Then, for instance, we have that $\mathcal{S}^{\{t\}}_{\langle f,g\rangle}(s_1 s_2 s_3 s_2 s_1) = \sigma_2 \sigma_1^3 \sigma_2$. □

Finally, we show that $\mathcal{D}[\_]$ extends to a functor which is left adjoint to $\mathcal{F}[\_]$.

Theorem 2.9 *($\mathcal{D}[\_] \dashv \mathcal{F}[\_]$)*
*The pair $\langle \mathcal{D}, \mathcal{F}\rangle$: **<u>Occ</u>** $\rightharpoonup$ **<u>DecOcc</u>** constitutes an adjunction.*

Proof. Let Θ be an occurrence net. By [ML71, Theorem 2, pg. 81] it is enough to show that the morphism $\eta_\Theta : \Theta \to \mathcal{FD}[\Theta]$ is universal from Θ to $\mathcal{F}$, i.e. for any decorated occurrence net Θ' and any $k : \Theta \to \mathcal{F}[\Theta']$ in **<u>Occ</u>**, there exists a unique $\langle f, g\rangle : \mathcal{D}[\Theta] \to \Theta'$ in **<u>DecOcc</u>** such that $k = \mathcal{F}[\langle f, g\rangle] \circ \eta_\Theta$.



Given Θ' and k, we define $\langle f, g\rangle : \mathcal{D}[\Theta] \to \Theta'$ as follows:

$f(t) = k(t)$

$[\![b^F]\!] \subseteq [\![g(s^F)]\!] \Leftrightarrow s \mapsto \mathcal{S}_k^x(\hat{b}^F)$, where $x = {}^\bullet[\![s^F]\!]$ and $k(x) = {}^\bullet[\![b^F]\!]$

First remark that $\langle f, g\rangle$ is well-defined: if $s = \sigma_1^{n_1} \ldots \sigma_r^{n_r} \mapsto \mathcal{S}_k^x(\hat{b}^F)$ then there is one and only one way to have $[\![b^F]\!] \subseteq [\![g(s^F)]\!]$, namely

$$g(s, i) = \bigoplus \{b\} \times [q]_i,$$

where q is the unique integer such that $\sigma_1^{qn_1} \ldots \sigma_r^{qn_r} = \mathcal{S}_k^x(\hat{b}^F)$.

Let $x = \{t_0\}$ or $x = \emptyset$. Observe that $\forall a \in [\![\partial^1_{\Theta'}(x)]\!]$

$$\forall (b,j) \in [\![k(a)]\!]\ \exists!(s,i) \text{ such that } (s,i) \in [\![\partial^1_{\mathcal{D}[\Theta]}(x)]\!] \text{ and } (b,j) \in [\![g(s,i)]\!]. \qquad (1)$$

Moreover, (s,i) is the unique place in $\mathcal{D}[\Theta]$ such that $s_i = a$ and $(b,j) \in [\![g(s,i)]\!]$. (2)

Now, if $(b,j) \in [\![g(s,i)]\!]$ then $s \mapsto \mathcal{S}^x_k(\hat{b}^F)$ and therefore, by definition of $\mapsto$, we have $\mathcal{S}^x_k(\hat{b}^F)_{(i-1)q+1}, \ldots, \mathcal{S}^x_k(\hat{b}^F)_{iq} = s_i$. Thus, by definition of $\mathcal{S}^x_k$, $\{b\} \times [q]_i \subseteq [\![k(s_i)]\!]$. So we have $\bigcup\{[\![g(s,i)]\!] \mid s_i = a\} = [\![k(a)]\!]$. Obviously, all the $[\![g(s,i)]\!]$ are disjoint and $\bigoplus [\![g(s,i)]\!] = g(s,i)$, since the families are disjoint. Therefore,

$$\bigoplus \{g(s,i) \mid s_i = a\} = k(a). \qquad (3)$$

It is now easy to see that the diagram commutes. For transitions this is clear. Concerning places, we have:

$$\mathcal{F}[\langle f, g\rangle] \circ \eta_\Theta(a) = \bigoplus g\Big(\{s_i \mid s_i = a\}\Big) = k(a).$$

Now, consider any morphism $h : \mathcal{D}[\Theta] \to N$ which makes the diagram commute. Because of the definition of η_Θ on the transitions, h must be of the form $\langle f, g'\rangle$. It follows from the definitions of g and $\mathcal{S}^x_k$ and from Lemma 2.3 that $g = g'$. Therefore, $h = \langle f, g\rangle$.

From (1), (2) and (3), exploiting the properties of the occurrence net morphism h, it can be shown that $\langle f, g\rangle$ is a morphism. □

The next corollary summarizes the results we obtain by means of the adjunction $\langle \mathcal{D}[\_], \mathcal{F}[\_]\rangle$: **Occ** ⇀ **DecOcc** and by means of Winskel's coreflections $\langle \mathcal{N}[\_], \mathcal{E}[\_]\rangle$: **PES** ⇀ **Occ** and $\langle \mathcal{P}r[\_], \mathcal{L}[\_]\rangle$: **Dom** ⇀ **PES**.

Corollary 2.10 *(Extensions of Winskel's coreflections [Win86])*
The following are adjunctions whose right adjoints relate PT nets to, respectively, occurrence nets, prime event structures and prime algebraic domains.

- $\langle (\_)^+\,\mathcal{D}[\_], \mathcal{FU}[\_]\rangle$: **Occ** ⇀ **PTNets**;
- $\langle (\_)^+\,\mathcal{DN}[\_], \mathcal{EFU}[\_]\rangle$: **PES** ⇀ **PTNets**;
- $\langle (\_)^+\,\mathcal{DNP}r[\_], \mathcal{LEFU}[\_]\rangle$: **Dom** ⇀ **PTNets**.

Moreover, $\mathcal{FU}[\_]_{\mathbf{Safe}} = \mathcal{U}_w[\_]$ *and, therefore,* $\mathcal{EFU}[\_]_{\mathbf{Safe}} = \mathcal{EU}_w[\_]$ *and* $\mathcal{LEFU}[\_]_{\mathbf{Safe}} = \mathcal{LEU}_w[\_]$*, i.e. the semantics given to safe nets by the chain of adjunctions presented in this work coincides with the semantics given by Winskel's chain of coreflections.* □

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