

On the Semiboundedness of the $(\phi^4)_2$ Hamiltonian \star

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Received February 1, 1973

Abstract. An elementary alternate proof of the semiboundedness of the locally correct Hamiltonian $H_0 + \int :\phi^4(x): g(x) dx$ of the $(\phi^4)_2$ quantum field theory model. The interaction operator is expressed as the sum of a positive operator and operators which are “tiny” relative to N^ε for any $\varepsilon > 0$, where N is the number operator.

The semiboundedness of the space cut-off $(\phi^4)_2$ Hamiltonian was first proved by Nelson [1]. Alternative proofs and generalizations of this result have been given by various authors (see [2] and the references therein). In this note we give an elementary alternate proof in which the interaction operator $V = \int :\phi^4(x): g(x) dx$ ($g \geq 0$, $g \in L^1 \cap L^2$) is expressed as the sum of a positive operator and operators which are “tiny” relative to N^ε for any $\varepsilon > 0$ (here N is the number operator). The proof is based on the formal identity $:\phi^4: = (:\phi^2: - 2c)^2 - 6c^2$ where c is the infinite constant $\int w(k)^{-1} dk$.

In our notation

$$a(k) a^+(p) - a^+(p) a(k) = \delta(k - p)$$

$$N = \int a^+(k) a(k) dk$$

$$H_0 = \int a^+(k) a(k) w(k) dk$$

$$\phi(x) = \int [a(k) + a^+(-k)] \exp(ikx) w(k)^{-1/2} dk$$

where $w(k) = (k^2 + m^2)^{1/2}$ and m is the mass of the free field ϕ .

Let $b > 0$ and define

$$f_n(k) = \begin{cases} w(k)^{-1/2} & |k| \leq n^b \\ 0 & |k| > n^b \end{cases}$$

Let $a_n = a_n(x) = \int a(k) \exp(ikx) f_n(k) dk$ and let a_n^+ be the adjoint of a_n and let

$$c_n = a_n a_n^+ - a_n^+ a_n = \|f_n\|^2 \quad \text{for } n = 0, 1, \dots$$

\star Supported by the National Research Council of Canada.

Let $Y = Y(x)$ and C be the symmetric operators defined on “ n -particle states” ψ_n by

$$Y \psi_n = [a_{n+2}^+{}^2 + 2a_n^+ a_n + a_n^2] \psi_n$$

$$C \psi_n = c_n \psi_n.$$

We now apply the positive operator $\int dx g(x)(Y - 2C)^2$ to the n -particle state ψ_n and Wick-order the terms in the resulting expression, using the commutation relations $[a_n, a_n^+] = c_n$:

$$\begin{aligned} & \left[\int g(x)(Y - 2C)^2 dx \right] \psi_n \\ &= \left[\int g(x) \{ a_{n+4}^+{}^2 a_{n+2}^+{}^2 + 2a_{n+2}^+{}^2 a_n^+ a_n \right. \\ & \quad + 2a_{n+2}^+{}^3 a_{n+2} + a_{n+2}^+{}^2 a_{n+2}^2 + 5a_n^+{}^2 a_n^2 \\ & \quad + 2a_n^+ a_n^3 + 2a_{n-2}^+ a_{n-2} a_n^2 + a_{n-2}^2 a_n^2 \\ & \quad + 2(c_{n+2} - c_n) a_{n+2}^+{}^2 + 4(c_{n+2} - c_n) a_{n+2}^+ a_{n+2} \\ & \quad + 2(c_n - c_{n-2}) a_n^2 + 2c_{n+2} + 4c_n^2 \\ & \quad \left. + 4c_n(a_{n+2}^+ a_{n+2} - a_n^+ a_n) \} dx \right] \psi_n. \end{aligned}$$

Let us designate by V' the operator defined on n -particle states by the first eight terms of the above integral. If we examine the “four-creation” part of $V - V'$ we find it is of the form

$$\iiint W''(k, p, q, r) a^+(k) a^+(p) a^+(q) a^+(r) dk dp dq dr$$

where the kernel W'' is equal to zero when $|k|, |p| < (n+2)^b$ and $|q|, |r| < (n+4)^b$ and equal to the kernel W of V otherwise. By a modification of a proof given by Simon and Höegh-Krohn [3, p. 155], W'' is square integrable and $\|W''\|_2 \leq \text{const} [(n+2)^b]^{-\alpha}$ for a certain α , $0 < \alpha < 1$. The other terms in $(V - V')$ may be treated similarly with the result that $\|(V - V')\psi_n\| \leq \text{const} n^2 n^{-b\alpha} \|\psi_n\|$ for large n . If we now choose $b = 2/\alpha$ then $\|(V - V')\psi_n\| \leq \text{const} \|\psi_n\|$. We see that $(c_{n+2} - c_n) \leq \text{const} n^{-1}$ and that c_n grows like $\text{const} \log n$ for large n . Since $\|\int a_{n+2}^+{}^2 g(x) dx \psi_n\| \leq (n+2) \left[\iint |\tilde{g}(k+p)|^2 w(k)^{-1} w(p)^{-1} dk dp \right]^{1/2} \|\psi_n\| \leq \text{const}(n+2) \|\psi_n\|$ the term involving $a_{n+2}^+{}^2$ may be bounded by a constant for large n . The next two terms may be treated similarly. The terms $4c_n^2$ and $2c_{n+2}^2$ are of order $(\log n)^2$. The last term may be written $4c_n \iint [X_{n+2}(k, p) - X_n(k, p)] \cdot a(k)^+ a(p) dk dp \psi_n$ where $X_n(k, p)$ equals $\tilde{g}(k-p) w(k)^{-1/2} w(p)^{-1/2}$ for $|k|, |p| \leq n^{2/\alpha}$ and equals zero elsewhere. $\iint |X_{n+2} - X_n|^2 dk dp$ may be bounded by a sum of four integrals of the type

$$\sup_p |\tilde{g}(p)|^2 n^{-2/\alpha} [(n+2)^{2/\alpha} - n^{2/\alpha}] \int_{-(n+2)^{2/\alpha}}^{(n+2)^{2/\alpha}} w(k)^{-1} dk$$

which are bounded by $\text{const} n^{-1} \log n$ for large n . Hence the last term is bounded by $\text{const} n^{1/2} (\log n)^{3/2} \|\psi_n\|$ for large n .

We conclude that V differs from the positive operator $\int (Y - 2C)^2 \cdot g(x) dx$ by an operator A , all of whose nonzero matrix elements $\langle \psi_{n+m} | A \psi_n \rangle$, $m = 0, \pm 2, \pm 4$ are bounded in magnitude by $n^{\varepsilon+1/2} \|\psi_{n+m}\| \|\psi_n\|$ for some $\varepsilon > 0$ and large n . It follows that the operator $N^{\varepsilon+1/2} + V$ is bounded below for any $\varepsilon > 0$, which of course implies that the locally correct Hamiltonian $H_0 + V$ is bounded below.

We can improve our estimate on the last term discussed above without essentially changing our estimates on the other terms by using a less sharp momentum cut-off in the definition of a_n . To do this we redefine $f_n(k)$ as the continuous function

$$f_n(k) = \begin{cases} w(k)^{-1/2} & 0 \leq |k| \leq n^{2/\alpha} \\ w(n^{2/\alpha})^{-1/2} [1 - (k - n^{2/\alpha}) n^{\beta-2/\alpha}] & n^{2/\alpha} < |k| < n^{2/\alpha} + n^{2/\alpha-\beta} \\ 0 & |k| \geq n^{2/\alpha} + n^{2/\alpha-\beta}, \end{cases}$$

where $0 < \beta < 1$.

With this choice of f_n one can show that the integral $\iint |X_{n+2} - X_n|^2 dk dp$ may be bounded by $\text{const} n^{\beta-2} \log n$ for large n so that our last term is bounded by $\text{const} n^{\beta/2} (\log n)^{3/2}$ for large n . By taking β sufficiently small we see that $N^\varepsilon + V$ is bounded below for any $\varepsilon > 0$.

The author would like to thank Professor D. Masson for reading the manuscript.

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