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ON THE SEMIGROUP OF BINARY RELATIONS ON A FINITE SET

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Let $\Omega = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $n > 1$ different elements.

By a binary relation ϱ on the set Ω we mean a subset of $\Omega \times \Omega$. The diagonal $\Delta_\Omega = \Delta$ is the set $\{(a_1, a_1), \ldots, (a_n, a_n)\}$. The universal relation is the set $\omega = \Omega \times \Omega$. The empty relation is denoted by z.

Let B_Ω be the set of all binary relations on Ω. If $\varrho \in B_\Omega$, we denote

$$a_i\varrho = \{x \in \Omega \mid (a_i, x) \in \varrho\}\,, \quad \varrho a_i = \{y \in \Omega \mid (y, a_i) \in \varrho\}\,.$$

If M is a subset of Ω, then $M\varrho$ is defined as the set $\bigcup_{a_i \in M} a_i\varrho$.

Further we denote

$$\mathrm{pr}_1(\varrho) = \bigcup_{i=1}^{n} \varrho a_i\,, \quad \mathrm{pr}_2(\varrho) = \bigcup_{j=1}^{n} a_j\varrho\,.$$

In B_Ω a multiplication can be introduced. Let be $\varrho, \sigma \in B_\Omega$. Then $(a, b) \in \varrho\sigma$ if there is an $x \in \Omega$ such that $(a, x) \in \varrho$ and $(x, b) \in \sigma$. If $\mathrm{pr}_2(\varrho) \cap \mathrm{pr}_1(\sigma)$ is empty, we define $\varrho\sigma = z$. The multiplication just introduced is associative, so that B_Ω becomes a semigroup with Δ as the unit element and z the zero element.

For further notations we note the following. $\varrho \subset \sigma$ means that ϱ is a subset of σ. We shall often say that ϱ is a *subrelation* of σ. In particular $z \subset \sigma$ for any $\sigma \in B_\Omega$. In B_Ω we can perform the operations of union and intersection. If $\varrho_1 \cap \varrho_2 = z$, we shall say that the relations ϱ_1, ϱ_2 are *disjoint*.

The semigroup B_Ω contains many interesting subsemigroups.

There exist many papers concerning the abstract characterization of B_Ω and its subsemigroups. Also there exist investigations under what conditions a given semigroup can be represented as a semigroup of binary relations of a given type (e.g. reflexive relations) on a suitably chosen set Ω.

Various questions concerning B_Ω have been treated by J. Riguet ([21]), P. Dubreil ([4]), V. V. Wagner ([36]), B. M. Šajn ([33], [34], [35]), K. A. Zareckij

([41]–[43]) and many others. An extensive list of publications concerning B_Ω (even in the case that n is infinite) can be found in V. V. WAGNER [36] and B. M. ŠAJN [35].

Very interesting results in the case n = finite have been recently obtained by S. I. MONTAGUE - R. I. PLEMMONS ([44]) and R. I. PLEMMONS - M. T. WEST ([45]).

Relations on compact spaces have been studied by A. D. WALLACE, R. D. BEDNAREK and others. (See the bibliography in [38] and [39].)

In this paper we shall deal in essential with one relation $\varrho \in B_\Omega$, the cyclic semigroup generated by ϱ, and with some further relations which naturally appear in this study. These are primarily unions of some powers of ϱ.

Since a relation can be considered as an oriented graph there is on some places a close connection with some recent results obtained by graph-theoretical methods in the study of some questions concerning non-negative matrices. [See mainly the work of A. L. DULMAGE - N. S. MENDELSHON ([5]–[8]) and B. R. HEAP - M. S. LYNN ([11]–[13]).]

Our method can be considered as an algebraization of some graph-theoretical methods. It also leads to new problems which hardly occur using purely graph-theoretical methods.

Some of our results (mainly those concerning irreducible relations in § 7) are implicitly (i.e. in other forms) known. Some of them are going back to G. FROBENIUS ([9]). Our proofs are new and they arise naturally from general considerations using in essential simple semigroup methods.

Since n is finite a number of interesting arithmetical questions will arise.

Some problems concerning the structure of B_Ω as a whole will be treated elsewhere.

1. THE POWERS OF A RELATION

In all of the paper $\Omega = \{a_1, \ldots, a_n\}$ is a set with card $\Omega = n > 1$.

Lemma 1,1. *For any $\varrho \in B_\Omega$ and any $s \geqq 1$ we always have*

$$\varrho^s \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n .$$

Proof. If $\varrho = z$, there is nothing to prove. Suppose first $s = n + 1$. The elements of ϱ^{n+1} are products of $n + 1$ couples (a_i, a_j). Such a product is z except the case when the subscripts follow in the following order:

$$(1{,}1) \qquad (i_1, i_2), (i_2, i_3), \ldots, (i_{n+1}, i_{n+2}) .$$

Since the numbers $i_1, i_2, \ldots, i_{n+1}$ cannot be all different, there exist two integers, say $m < l$, such that there is a segment in (1,1) of the form

$$\ldots (i_{m-1}, i_m), (i_m, i_{m+1}), \ldots, (i_{l-1}, i_m), (i_m, i_{l+1}), \ldots$$

We now can delete $(i_m, i_{m+1}), \ldots, (i_{l-1}, i_m)$ without changing the value of the product corresponding to $(1,1)$. The product contains then at most n factors, i.e. it is contained in $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$.

Now $\varrho^{n+1} \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$ implies $\varrho^{n+2} \subset \varrho^2 \cup \varrho^3 \cup \ldots \cup \varrho^{n+1} \subset \varrho \cup \varrho^2 \cup \ldots \ldots \cup \varrho^n$ and by repeating the same argument we have $\varrho^s \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$ for any $s \geqq 1$.

A binary relation σ is called *transitive* if $\sigma^2 \subset \sigma$.

For any binary relation the set $\bar{\varrho} = \varrho \cup \varrho^2 \cup \varrho^3 \cup \ldots$ is transitive and it is called the *transitive closure* of ϱ.

Our result may be stated as follows:

Lemma 1,2. *For any binary relation ϱ on a set Ω with* card $\Omega = n$ *the transitive closure of ϱ is the relation $\bar{\varrho} = \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n$.*

Let $\varrho \in B_\Omega$. Consider the sequence

$$(1,2) \qquad \varrho, \varrho^2, \varrho^3, \ldots$$

This sequence contains only a finite number of different elements (relations).

Let $k = k(\varrho)$ be the least integer such that $\varrho^k = \varrho^l$ for some $l > k$. Let further $l = k + d$ $(d \geqq 1)$ be the least integer satisfying this relation. Then the sequence (1, 2) is of the form

$$\varrho, \ldots, \varrho^{k-1} \mid \varrho^k, \ldots, \varrho^{k+d-1} \mid \varrho^k, \ldots, \varrho^{k+d-1} \mid \ldots$$

It is well-known from the elements of the theory of semigroups that the set $G(\varrho) = \{\varrho^k, \varrho^{k+1}, \ldots, \varrho^{k+d-1}\}$ is a cyclic group (with respect to the multiplication of relations). The unit element of $G(\varrho)$ is ϱ^r, where $k \leqq r \leqq k + d - 1$. More precisely: Let $\beta \geqq 1$ be the uniquely determined integer such that $k \leqq \beta d \leqq k + d - 1$. Then $r = \beta d$.

We introduce a further constant which is associated with any relation on a finite set Ω. We have just seen that there is an integer $r \geqq 1$ such that $\varrho^{2r} = \varrho^r$. Denote by $t \geqq 1$ the least integer $s \geqq 1$ such that ϱ^s is transitive, i.e. $\varrho^{2s} \subset \varrho^s$. Such a number exists and we clearly have $t \leqq r$. We state it explicitly:

Lemma 1,3. *To any binary relation ϱ on Ω there exists a least integer $t = t(\varrho) \geqq 1$ such that ϱ^t is transitive.*

Remark. The integer t may be much larger than n. Later we shall see that for some classes of relations (called irreducible relations) it is aproximatively of order at most n^2. Note also that ϱ^t may be equal to z.

To any binary relation ϱ we have associated four integers $k = k(\varrho)$, $d = d(\varrho)$, $r = r(\varrho)$, $t = t(\varrho)$. We shall try to clarify the relationship between them.

Lemma 1,4. *If ϱ^s, $s \geqq 1$, is transitive, then $\varrho^r \subset \varrho^s$. More generally: $\varrho^r \subset \varrho^{s+ld}$ for any integer $l \geqq 0$.*

Proof. The sequence $\varrho^s, \varrho^{2s}, \varrho^{3s} \ldots$ contains a unique idempotent (idempotent relation) namely ϱ^r. Hence there is an integer $v \geqq 1$ such that $\varrho^{vs} = \varrho^r$. The transitivity implies

$$\varrho^s \supset \varrho^{2s} \supset \ldots \supset \varrho^{vs} = \varrho^r .$$

Further $\varrho^r \subset \varrho^s$ implies $\varrho^{r+ld} \subset \varrho^{s+ld}$. Since $\varrho^{r+ld} = \varrho^r$, our Lemma is proved.

Remark. Clearly, if ϱ^s is transitive also $\varrho^{2s}, \varrho^{3s}, \ldots$ are transitive.

Lemma 1,5. *If ϱ^s is transitive and $\bar{\varrho} \cap \Delta \neq z$, then $\bar{\varrho} \cap \Delta \subset \varrho^s$.*

Proof. Let $(a_i, a_i) \in \Delta \cap \bar{\varrho}$. Then there is a h_i, $1 \leqq h_i \leqq n$, such that $(a_i, a_i) \in \varrho^{h_i}$. For any integer $l > 0$ we clearly have $(a_i, a_i) \in \varrho^{h_i l}$. Since some power of ϱ^{h_i} is ϱ^r, we have $(a_i, a_i) \in \varrho^r$ and by Lemma 1,4 $(a_i, a_i) \in \varrho^r \subset \varrho^s$.

Lemma 1,6. *If $\Delta \cap \bar{\varrho} = \Delta$, then the sequence (1, 2) contains a unique transitive relation, namely ϱ^r.*

Proof. Let ϱ^s be transitive. By Lemma 1,5 $\Delta \subset \varrho^s$. Hence $\varrho^s = \varrho^s\Delta \subset \varrho^s\varrho^s = \varrho^{2s}$. On the other side by transitivity $\varrho^{2s} \subset \varrho^s$. Hence $\varrho^s = \varrho^{2s}$, and since there is a unique idempotent in the sequence (1, 2), we have $\varrho^s = \varrho^r$, q.e.d.

Lemma 1,7. *The group $G(\varrho) = \{\varrho^k, \ldots, \varrho^{k+d-1}\}$ contains exactly one transitive relation (namely ϱ^r).*

Proof. Suppose that ϱ^s, $k \leqq s \leqq k + d - 1$, is transitive. By Lemma 1,4 $\varrho^r \subset \varrho^s$; hence $\varrho^{r+s} \subset \varrho^{2s} \subset \varrho^s$. On the other side ϱ^r is the unit element of $G(\varrho)$. Hence $\varrho^{r+s} = \varrho^r\varrho^s = \varrho^s$. The inclusion $\varrho^s \subset \varrho^{2s} \subset \varrho^s$ implies $\varrho^s = \varrho^{2s}$ and ϱ^s (being an idempotent) is equal to ϱ^r. This proves our Lemma.

Lemma 1,8. *If ϱ^s is transitive, then $d \mid s$. In particular, $d \mid t$.*

Proof. We first show that $\varrho^r = \varrho^{r+s}$. Since $\varrho^{2(r+s)} = \varrho^{2r}\varrho^{2s} \subset \varrho^r\varrho^s = \varrho^{r+s}$, we conclude that ϱ^{r+s} is transitive. Further $\varrho^{r+s} \in G(\varrho)$, hence (by Lemma 1,7) $\varrho^{r+s} = \varrho^r$.

Suppose now that $d \nmid s$ and write $s = l_1 d + l_2$, where $l_1 \geqq 0$ is an integer and $0 < l_2 < d$. We then have

$$\varrho^r = \varrho^{r+l_1 d+l_2} = \varrho^{r+l_1 d}\,\varrho^{l_2} = \varrho^r\varrho^{l_2} = \varrho^{r+l_2} .$$

This contradicts to the fact that $G(\varrho)$ is of order d.

Lemma 1,9. *If ϱ^s is transitive and $d > 1$, then none of the relations*

$$\varrho^{s+1}, \varrho^{s+2}, \ldots, \varrho^{s+d-1}$$

is transitive. In particular: None of the relations

$$\varrho^{t+1}, \varrho^{t+2}, \ldots, \varrho^{t+d-1}$$

is transitive.

Proof. If $\varrho^{s+\lambda}$, $1 \leqq \lambda \leqq d - 1$ were transitive, Lemma 1,8 would imply $d \mid s$ and $d \mid s + \lambda$, which is impossible.

We can visualise the situation by arranging the powers of ϱ in the following way:

$$\begin{array}{llll} \varrho, \varrho^2, \ldots, \varrho^{t-1} \mid \varrho^t, & \varrho^{t+1}, & \ldots, \varrho^{t+d-1}, \\ \varrho^{t+d}, & \varrho^{t+d+1}, & \ldots, \varrho^{t+2d-1}, \\ \varrho^{t+2d}, & \varrho^{t+2d+1}, & \ldots, \varrho^{t+3d-1}, \\ \vdots & & \\ \varrho^{r}, & \varrho^{r+1}, & \ldots, \varrho^{r+d-1}. \end{array}$$

Since $d \mid t$ and $d \mid r$ there is necessarily an integer $l \geqq 0$ such that $r = t + ld$. The $l + 1$ rows contain certainly all different powers of ϱ. The last row contains at least one element $\in G(\varrho)$ which does not occur in the foregoing row. This means: It may happen that to obtain all different powers of ϱ it is not necessary to consider the whole last row, but in any case the first element ϱ^r contained in it.

All transitive relations are contained in the column $\varrho^t, \varrho^{t+d}, \ldots, \varrho^r$, but, in general, we cannot state that all these powers are transitive.

Lemma 1,4 implies the following

Corollary. *If $t < r$, we always have*

$$\varrho^r \subset \varrho^t \cap \varrho^{t+d} \cap \ldots \cap \varrho^{r-d}. \tag{1,3}$$

If $t = r$, then by Lemma 1,7 there is a unique transitive relation and the assertion of our Corollary is trivial.

Remark. If $t < r$, then $r \geqq 2d$ and we have $2(r - d) \geqq 2(r - \frac{1}{2}r) = r$. Since $d \mid r - 2d$, we have $\varrho^{2(r-d)} = \varrho^{r+(r-2d)} = \varrho^r \subset \varrho^{r-d}$, hence ϱ^{r-d} is transitive. This says that on the right hand side of (1, 3) the first and last member is always transitive.

We are also able to give an estimation for the number $t = t(\varrho)$.

Lemma 1,10. *For any binary relation ϱ we have $r/n \leqq t \leqq r$.*

Proof. It is sufficient to prove $t \geqq r/n$. By definition of $t = t(\varrho)$ we have $\varrho^t \supset \supset \varrho^{2t} \supset \varrho^{3t} \supset \dots$ Let a_i be any element $\in \Omega$. We then have

$$(1,4) \qquad a_i\varrho^t \supset a_i\varrho^{2t} \supset a_i\varrho^{3t} \supset \dots$$

a) If $a_i \in a_i\varrho^t$, we get (multiplying by ϱ^t) $a_i\varrho^t \subset a_i\varrho^{2t}$ and by (1,4) $a_i\varrho^t \subset a_i\varrho^{2t} \subset \subset a_i\varrho^t$. Hence $a_i\varrho^t = a_i\varrho^{2t}$.

b) If $a_i \notin a_i\varrho^t$, then $a_i\varrho^t$ contains at most $n - 1$ different elements $\in \Omega$. The set $a_i\varrho^{2t}$ is either equal to $a_i\varrho^t$ or contains at most $n - 2$ elements $\in \Omega$. Repeating this argument we conclude that there is either an integer $1 \leqq l_i \leqq n - 1$ such that $a_i\varrho^{l_it} = a_i\varrho^{(l_i+1)t}$ or $a_i\varrho^{nt} = \emptyset$. In the first case we have certainly $a_i\varrho^{(n-1)t} = a_i\varrho^{nt}$.

In both cases a) and b) we have $a_i\varrho^{nt} = a_i\varrho^{2nt}$ for every $a_i \in \Omega$. Hence $\varrho^{nt} = \varrho^{2nt}$, so that $\varrho^{nt} = \varrho^r$. This implies $nt \geqq r$, q.e.d.

Remark. This estimation is sharp in the sense that there are relations for which $t = r/n$. Let be, e.g., $\Omega = \{a_1, a_2, \dots, a_n\}$ and $\varrho = \bigcup_{i=2}^{n} \{(a_i, a_1), (a_i, a_2), \dots, (a_i, a_{i-1})\}$. It is easy to see that $r = n$ (as a matter of fact we have $\varrho^n = z$ but $\varrho^{n-1} \neq z$). On the other side $t = 1$ (since ϱ itself is transitive), so that we have $t = r/n = 1$.

We summarise:

Theorem 1,1. *To any binary relation ϱ there is a least integer $r = r(\varrho)$ such that ϱ^r is an idempotent $\in B_\Omega$. Further there exists a least integer $t = t(\varrho) \leqq r$ such that ϱ^t is transitive.*

a) *If ϱ^s is transitive, then $d \mid s$. In particular, $d \mid t$ and $d \mid r$.*

b) *The group $G(\varrho)$ contains a unique transitive relation (namely ϱ^r).*

c) *We always have $t(\varrho) \geqq r/n$.*

d) *If $t < r$, then (1,3) holds.*

Consider now the (formally infinite) sequence

$$\varrho^t, \varrho^{t+d}, \varrho^{t+2d}, \dots, \varrho^r = \varrho^{t+(l-1)d}, \varrho^{t+ld}, \varrho^{t+(l+1)d}, \dots,$$

where all powers beginning with ϱ^r are equal. Since the greatest common divisor (g.c.d.) of the sequence of integers $t, t + d, t + 2d, \dots,$ is exactly the number d, we have proved:

Theorem 1,2. *The integer $d(\varrho) =$ card $G(\varrho)$ is the greatest commom divisor of all integers $s > 0$ such that ϱ^s is transitive.*

The group $G(\varrho) = \{\varrho^k, \varrho^{k+1}, \dots, \varrho^{k+d-1}\}$ is cyclic. There exists therefore an integer u, $k \leqq u \leqq k + d - 1$, such that

$$G(\varrho) = \{\varrho^u, \varrho^{2u}, \dots, \varrho^{du}\}.$$

The number u is, in general, not uniquely determined. It is clear that we may choose any integer u with $(u, d) = 1$.

In what follows we shall choose $u = r + 1$. This is possible since $d \mid r$ implies $(r + 1, d) = 1$.

Denote $\delta = \varrho^{r+1}$. We then have $\delta^2 = \varrho^{2(r+1)} = \varrho^r \varrho^{r+2} = \varrho^{r+2}$, $\delta^3 = \varrho^{r+3}, \ldots$ $\ldots, \delta^d = \varrho^r$. This choice of δ will be consequently used over all the paper (and will be very convenient in particular in § 6).

The group $G(\varrho)$ can be then written in the form

$$G(\varrho) = \{\delta, \delta^2, \ldots, \delta^d\}$$

and δ^d is the unit element $\in G(\varrho)$.

In conclusion to the foregoing we mention two transitive relations which are intimately connected with any relation ϱ.

Lemma 1,11. *For any binary relation* ϱ *the following relations are transitive*:

a) $\sigma = \delta \cup \delta^2 \cup \ldots \cup \delta^d$;
b) $\tau = \delta \cap \delta^2 \cap \ldots \cap \delta^d$.

Proof. a)

$$\sigma^2 = \bigcup_{l=1}^{d} \delta^l \bigcup_{l_1=1}^{d} \delta^{l_1} = \bigcup_{h=2}^{2d} \delta^h .$$

Since each member δ^{l+l_1} is contained in $G(\varrho)$, we clearly have $\sigma^2 \subset \sigma$. We have moreover $\sigma^2 = \sigma$, since also $\delta = \delta^{d+1}$ is contained in σ^2.

b) We have

$$\tau^2 = \tau(\delta \cap \delta^2 \cap \ldots \cap \delta^d) \subset \tau\delta \cap \tau\delta^2 \cap \ldots \cap \tau\delta^d .$$

Now for any l, $1 \leqq l \leqq d$, we have

$$(1,5) \qquad \tau\delta^l = (\delta \cap \delta^2 \cap \ldots \cap \delta^d)\, \delta^l \subset \delta^{l+1} \cap \delta^{l+2} \cap \ldots \cap \delta^{l+d} = \\ = \delta \cap \delta^2 \cap \ldots \cap \delta^d .$$

Since the right hand side of (1,5) is exactly τ, we have $\tau^2 \subset \tau$. (Note that τ may also be the empty relation.)

Remark. An analogous argument shows that $\tau_1 = \varrho^t \cap \varrho^{t+d} \cap \ldots \cap \varrho^{r-d}$ is also a transitive relation.

The following natural question arises. What is the relation between

$$\sigma = \delta \cup \delta^2 \cup \ldots \cup \delta^d$$

and

$$\sigma_0 = \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n .$$

(We write here – for a while – σ_0 instead of $\bar{\varrho}$.)

First $\sigma\varrho = \{\delta \cup \delta^2 \cup \ldots \cup \delta^d\}\, \varrho = \{\delta^2 \cup \delta^3 \cup \ldots \cup \delta^d \cup \delta\} = \sigma$. Therefore $\sigma\varrho^l = \sigma$ for every integer $l \geqq 1$. In particular, $\sigma\sigma_0 = \sigma\{\varrho \cup \ldots \cup \varrho^n\} = \sigma$.

By Lemma 1,1 we have $\varrho\sigma_0 \subset \sigma_0$. This implies

$$\sigma_0 \supset \varrho\sigma_0 \supset \varrho^2\sigma_0 \supset \ldots$$

We easily obtain an upper bound for the lenght of this chain. Let a_i be any element $\in \Omega$. We then have

$$(1,6) \qquad a_i\sigma_0 \supset a_i\varrho\sigma_0 \supset a_i\varrho^2\sigma_0 \supset \ldots$$

a) If $a_i \in a_i\sigma_0 = a_i\{\varrho \cup \ldots \cup \varrho^n\}$, there is an l, $1 \leqq l \leqq n$, such that $a_i \in a_i\varrho^l$, hence $a_i\sigma_0 \subset a_i\varrho^l\sigma_0$. By (1,6)

$$a_i\sigma_0 \subset a_i\varrho^l\sigma_0 \subset a_i\varrho^{l-1}\sigma_0 \subset \ldots \subset a_i\varrho\sigma_0 \subset a_i\sigma_0 .$$

Hence $a_i\sigma_0 = a_i\varrho\sigma_0$. Using the fact that $\varrho\sigma_0 = \sigma_0\varrho$ and multiplying successively by $\varrho, \varrho^2, \ldots$ we have $a_i\sigma_0 = a_i\varrho\sigma_0 = \ldots = a_i\varrho^{n-1}\sigma_0$.

b) If $a_i \notin a_i\sigma_0$, then $a_i\sigma_0$ contains at most $n - 1$ elements $\in \Omega$. $a_i\varrho\sigma_0$ is either equal to $a_i\sigma_0$ or it contains at most $n - 2$ elements $\in \Omega$. Repeating this argument we see that there is either an integer l_i, $0 \leqq l_i \leqq n - 2$, such that $a_i\varrho^{l_i}\sigma_0 = a_i\varrho^{l_i+1}\sigma_0$ or $a_i\varrho^{n-1}\sigma_0 = \emptyset$. In the first case we have the more $a_i\varrho^{n-2}\sigma_0 = a_i\varrho^{n-1}\sigma_0$.

Both cases a) and b) imply $a_i\varrho^l\sigma_0 = a_i\varrho^{n-1}\sigma_0$ for every $l \geqq n - 1$ and every $a_i \in \Omega$. Hence $\varrho^l\sigma_0 = \varrho^{n-1}\sigma_0$ for $l \geqq n - 1$.

Now

$$\varrho^k \cup \ldots \cup \varrho^{k+d-1} = \varrho^{k+\alpha d} \cup \ldots \cup \varrho^{k+\alpha d+d-1}$$

for any integer $\alpha \geqq 0$. Choose α such that $l = k + \alpha d \geqq n - 1$. We then have

$$\sigma = \sigma\sigma_0 = \{\delta \cup \ldots \cup \delta^d\}\, \sigma_0 = \{\varrho^l \cup \varrho^{l+1} \cup \ldots \cup \varrho^{l+d-1}\}\, \sigma_0 =$$
$$= \varrho^l\sigma_0 \cup \ldots \cup \varrho^{l+d-1}\sigma_0 = \varrho^{n-1}\sigma_0 \cup \ldots \cup \varrho^{n-1}\sigma_0 =$$
$$= \varrho^{n-1}\sigma_0 = \varrho^{n-1}\{\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n\} .$$

We have proved:

Theorem 1,3. *For any binary relation ϱ we have*

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d = \varrho^n \cup \varrho^{n+1} \cup \ldots \cup \varrho^{2n-1} .$$

Remark. The result of Theorem 1,3 is sharp in the sense that there are relations for which $\delta \cup \ldots \cup \delta^d = \varrho^{n-1} \cup \varrho^n \cup \ldots \cup \varrho^{2n-2}$ does not hold. Let, e.g., $\Omega = \{a_1, a_2\}$ and $\varrho = \{(a_2, a_1)\}$. Then $\delta = z$ while $\varrho^{n-1} = \varrho$. Note explicitly that though it may happen that $\sigma \subsetneqq \sigma_0$ we always have $\sigma = \varrho^{n-1}\sigma_0$. It is worth to mention (and we shall show it later on examples) that d may be much larger then n. [In contradistiction to this there is an important class of relations, called irreducible relations, for which we always have $d \leqq n$. See § 6.]

By revising the proof we easily find that we have proved a somewhat sharper result, namely: To any binary relation ϱ there is an integer l, $0 \leqq l \leqq n - 1$ such that $\delta \cup \ldots \cup \delta^d = \varrho^{l+1} \cup \ldots \cup \varrho^{l+n}$.

In particular we prove:

Corollary 1. *If $\Delta \subset \sigma_0$, then*

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d = \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n .$$

Proof. In this case $a_i \in a_i\sigma_0$ for every $a_i \in \Omega$. Hence, by the proof of Theorem 1,3, $a_i\sigma_0 = a_i\varrho\sigma_0$ for every $a_i \in \Omega$. Therefore $\sigma_0 = \varrho\sigma_0$, and $\sigma_0 = \varrho^l\sigma_0$ for $l \geqq 1$. This implies

$$\sigma = \sigma\sigma_0 = \{\delta \cup \ldots \cup \delta^d\}\, \sigma_0 = \delta\sigma_0 \cup \ldots \cup \delta^d\sigma_0 = \sigma_0 .$$

We also mention

Corollary 2. *For a binary relation ϱ we have*

$$\delta \cup \ldots \cup \delta^d = \varrho \cup \ldots \cup \varrho^n$$

if and only if

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = \varrho^n \cup \varrho^{n+1} \cup \ldots \cup \varrho^{2n-1} .$$

We conclude with a further (in no way trivial) result:

Theorem 1,4. *For any binary relation ϱ we have*

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d = (\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n)^n .$$

Proof. We have

$$\sigma_0^2 = (\varrho \cup \ldots \cup \varrho^n)\, \sigma_0 = \varrho\sigma_0 \cup \varrho^2\sigma_0 \cup \ldots \cup \varrho^n\sigma_0 .$$

Now, since

$$\varrho\sigma_0 \supset \varrho^2\sigma_0 \supset \varrho^3\sigma_0 \supset \dots,$$

we obtain $\sigma_0^2 = \varrho\sigma_0$. Further $\sigma_0^3 = \varrho\sigma_0^2 = \varrho^2\sigma_0$, $\sigma_0^4 = \varrho^3\sigma_0, \dots$ and finally $\sigma_0^n = \varrho^{n-1}\sigma_0 = \varrho^{n-1}(\varrho \cup \varrho^2 \cup \dots \cup \varrho^n) = \sigma$.

Remark. Here again the result is sharp in the sense that, in general, the exponent n cannot be replaced by a smaller one.

2. SOME REMARKS ON THE MATRIX REPRESENTATION OF BINARY RELATIONS

To any $\varrho \in B_\Omega$ we can associate an $n \times n$ "matrix" $M(\varrho) = (e_{ij})$ with elements 0 and 1 by writting $e_{ij} = 1$ on the place (i, j) if $(a_i, a_j) \in \varrho$ and $e_{ij} = 0$ otherwise.

We define the product $M(\varrho)\, M(\sigma)$ by the usual multiplication of matrices, where for the elements 0 and 1 the addition and multiplication is defined by the following rules:

$$\begin{aligned} 0 + 0 &= 0\,, & 1\,.\,0 &= 0\,,\\ 0 + 1 = 1 + 0 &= 1\,, & 0\,.\,1 = 1\,.\,0 &= 0\,,\\ 1 + 1 &= 1\,, & 1\,.\,1 &= 1\,. \end{aligned}$$

Further we define the sum of two "matrices" $M(\varrho) \cup M(\sigma)$ as the ordinary sum of two matrices, where the addition of the elements satisfies the above rules.

Finally $M(\varrho) \cap M(\sigma)$ is again an $n \times n$ "matrix", where the elements of the resulting matrix are constructed elementwise by means of the rules $0 \cap 0 = 0 \cap 1 = 1 \cap 0 = 0$, $1 \cap 1 = 1$.

The correspondence $\varrho \to M(\varrho)$ is an isomorphism of the semigroup B_Ω onto the semigroup of all such "matrices".

In particular

$$\varrho\sigma \to M(\varrho)\, M(\sigma) = M(\varrho\sigma)\,.$$

Also $\varrho \cup \sigma \to M(\varrho) \cup M(\sigma) = M(\varrho \cup \sigma)$ and $\varrho \cap \sigma \to M(\varrho) \cap M(\sigma) = M(\varrho \cap \sigma)$.

Note that $z \to M(z)$, where $M(z)$ is an $n \times n$ "zero-matrix" and $\omega \to M(\omega)$, where all entries in $M(\omega)$ are 1. Further $M(\Delta)$ is the "unit matrix" of order n.

The isomorphism $\varrho \to M(\varrho)$ seems to have been first considered in extenso by J. Riguet ([21]).

In the following we shall often use (mainly in examples) this matrix representation of binary relations.

In a series of papers ([23]–[29]) I have dealt with the properties of non-negative matrices. Since I have been mainly interested in properties that depend only on the distribution of zeros and non-zeros I used a method which may be considered in some sense as a converse approach. To any non-negative $n \times n$ matrix $M = (a_{ij})$ we may associate in an obvious way a binary relation ϱ, written in its matrix representation $M(\varrho)$, by writing 1 on the places where $a_{ij} > 0$ and 0 otherwise. The semigroup of all $n \times n$ non-negative matrices is in this way homomorphicaly mapped onto B_Ω. This has led me to the present investigations.

But the study of non-negative matrices even from the point of view of the distribution of zeros and non-zeros is not the same as the study of binary relations. The following simple example will provide some insight into the situation. Consider the binary relation ϱ on $\Omega = \{a_1, a_2\}$ with $M(\varrho) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Clearly, ϱ is an idempotent $\in B_\Omega$. But there does not exist a non-negative matrix $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $abc > 0$ which is idempotent. This makes it clear that the types of idempotents $\in B_\Omega$ are much richer that the types of idempotents in the semigroup of all non-negative matrices.

I recall – for further purposes – that a non-negative $n \times n$ matrix M is called reducible (more precisely: permutation-reducible) if there is a permutation matrix P such that PMP^{-1} is of the form $\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$, A_{11}, A_{22} being (non-negative) square matrices. Otherwise it is called irreducible. It can be easily shown that to any non-negative matrix M there is a permutation matrix P such that PMP^{-1} is of the form

$$\begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & \dots & 0 \\ \vdots & & & \\ A_{s1} & A_{s2} & \dots & A_{ss} \end{pmatrix}, \tag{2,1}$$

where A_{ii} are irreducible, including the case that some of the A_{ii} may be zero matrices of order 1.

Example 2,1. Let M be the non-negative matrix

$$\begin{pmatrix} a_1 & a_2 & 0 & a_4 \\ b_1 & b_2 & 0 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $a_i b_i c_i > 0$, $i = 1, 2, 4$. Choosing

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

we obtain

$$PMP^{-1} = \begin{pmatrix} \boxed{0} & 0 & 0 & 0 \\ b_4 & b_2 & b_1 & 0 \\ a_4 & a_2 & a_1 & 0 \\ c_4 & c_2 & c_1 & \boxed{0} \end{pmatrix}.$$

Here $A_{11} = 0$. It should be noted that there does not exist a permutation matrix Q such that in QMQ^{-1} the matrix A_{11} is a non-zero matrix. This remark should underline that the positions of the matrices A_{ii} in the main diagonal are not arbitrary.

We cann apply the preceding remarks to binary relations. If $\varrho \in B_\Omega$, we define ϱ^{-1} as follows: $(a_i, a_j) \in \bar{\varrho}^1 \Leftrightarrow (a_j, a_i) \in \varrho$.

A relation $\pi \in B_\Omega$ is called a *permutation relation* if and only if $\pi\pi^{-1} = \pi^{-1}\pi = \Delta$.

To any $\varrho \in B_\Omega$ there is a permutation relation π such that the matrix representation of $\sigma = \pi\varrho\pi^{-1}$ (i.e. $M(\sigma)$) is of the subdiagonal form (2,1), where A_{ii} is either a "zero matrix of order 1" or A_{ii} corresponds to an "irreducible subrelation "of σ in a sense, which will be later precisely defined.

It should be noted in advance that the notion of "reducible relation" will be defined in § 4 in a somewhat other form than the reducibility of non-negative matrices. (Roughly to say our definition will exclude superflous zero rows and zero columns.)

Note also that if $f(\varrho)$ is any of the functions $k(\varrho)$, $d(\varrho)$ $r(\varrho)$, $t(\varrho)$, we have $f(\varrho) = f(\pi\varrho\pi^{-1})$, so that it is sometimes useful to consider ϱ transformed to the "normal form" (2,1).

3. THE BEHAVIOUR OF THE "ROWS"

For a given binary relation ϱ and an element $a_i \in \Omega$ consider the sequence

$$(3,1) \qquad a_i\varrho,\ a_i\varrho^2,\ a_i\varrho^3,\ \ldots$$

The elements of this sequence are subsets of Ω (including sometimes the empty subset $\emptyset$).

Denote by $k_i = k_i(\varrho)$ the least integer such that $a_i\varrho^{k_i}$ occurs in (3,1) more than once. Let further $d_i = d_i(\varrho)$ be the least integer $\geqq 1$ such that $a_i\varrho^{k_i} = a_i\varrho^{k_i+d_i}$. Then (3,1) is of the form

$$a_i\varrho, \ldots, a_i\varrho^{k_i-1} \mid a_i\varrho^{k_i}, \ldots, a_i\varrho^{k_i+d_i-1} \mid a_i\varrho^{k_i}, \ldots$$

For any integers $u \geqq k_i$, $v \geqq k_i$ we have $a_i\varrho^u = a_i\varrho^v$ if and only if $u \equiv v \pmod{d_i}$.

Clearly $k_i \leqq k(\varrho)$ and $d_i \leqq d(\varrho)$. For further purposes denote $k^* = \max\limits_{i=1,\ldots,n} k_i$ and $d^* = [d_1, d_2, \ldots, d_n]$ (the least common multiple of the d_i's).

With k^* just defined we have $k^* \leqq k$, and $a_i\varrho^{k^*} = a_i\varrho^{k^*+\lambda_i d_i}$ for any integer $\lambda_i \geqq 1$. Choosing in particular $\lambda_i = d^*/d_i$ we have $a_i\varrho^{k^*} = a_i\varrho^{k^*+d^*}$ (for all $a_i \in \Omega$). Hence $\varrho^{k^*} = \varrho^{k^*+d^*}$. This implies $k^* \geqq k$ and $d \mid d^*$. Therefore $k = k^*$. On the other side $\varrho^k = \varrho^{k+d}$ implies $a_i\varrho^k = a_i\varrho^{k+d}$, hence $k \equiv k + d \pmod{d_i}$, i.e. $d_i \mid d$. Therefore $[d_1, d_2, \ldots, d_n] \mid d$, i.e. $d^* \mid d$. Finally $d = d^*$. We have proved:

Theorem 3,1. *For any binary relation ϱ we have*

a) $k(\varrho) = \max\limits_{i=1,2,\ldots,n} k_i(\varrho)$;

b) $d(\varrho) = [d_1, d_2, \ldots, d_n]$.

The computation of d_i and k_i can be sometimes simplified by means of the following

Lemma 3,1. *Let $a_i\varrho = \{a_\alpha, a_\beta, \ldots, a_\nu\}$. Then*

a) $d_i \mid [d_\alpha, d_\beta, \ldots, d_\nu]$;

b) $k_i \leqq \max\{k_\alpha, k_\beta, \ldots, k_\nu\} + 1$.

Proof. For any $\xi \in \{\alpha, \beta, \ldots, \nu\}$ we have

$$a_\xi\varrho^{k_\xi} = a_\xi\varrho^{k_\xi + \lambda_\xi d_\xi}.$$

Denote $k' = \max\{k_\alpha, \ldots, k_\nu\}$, $d' = [d_\alpha, d_\beta, \ldots, d_\nu]$. Then for any $\xi \in \{\alpha, \beta, \ldots, \nu\}$ and any integer $\lambda_\xi \geqq 1$ we have

$$a_\xi\varrho^{k'} = a_\xi\varrho^{k' + \lambda_\xi d_\xi}.$$

Put $\lambda_\xi = d'/d_\xi$. We then have

$$a_\xi\varrho^{k'} = a_\xi\varrho^{k'+d'}.$$

Summing through all ξ we obtain

$$(a_i\varrho)\,\varrho^{k'} = (a_i\varrho)\,\varrho^{k'+d'},$$

$$a_i\varrho^{k'+1} = a_i\varrho^{k'+d'+1}.$$

Hence $k_i \leqq k' + 1$ and $d_i \mid d'$. This proves our Lemma.

If $a_i \notin a_i\varrho$, then we certainly have $d_i \mid [d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n]$ so that we get the following Corollary:

Corollary. *If $a_i \notin a_i\varrho$, then*

$$d(\varrho) = [d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n].$$

We now turn our attention to the collection of sets

$$R_i = \{a_i\varrho^{k_i}, a_i\varrho^{k_i+1}, \ldots, a_i\varrho^{k_i+d_i-1}\}.$$

It follows from the definition of k_i and d_i that any segment of length d_i and beginning with $a_i\varrho^l$, where $l \geqq k_i$, gives (up to the order) the same sets. In particular,

$$R_i = \{a_i\varrho^{r+1}, a_i\varrho^{r+2}, \ldots, a_i\varrho^{r+d_i}\}.$$

Using our notation $\delta = \varrho^{r+1}$, $\delta^2 = \varrho^{r+2}, \ldots$ we get the following

Lemma 3,2. *With the notations introduced above we have*

$$\{a_i\varrho^{k_i}, \ldots, a_i\varrho^{k_i+d_i-1}\} = \{a_i\delta, a_i\delta^2, \ldots, a_i\delta^{d_i}\}.$$

Remark. Note that $a_i\delta^l = a_i\delta^{l_1}$ iff $l \equiv l_1 \pmod{d_i}$. In particular, $a_i\delta^{d_i} = a_i\delta^{\alpha d_i}$ for any integer $\alpha \geqq 1$.

Definition. We shall say that *the couple* (a_i, a_i) *is accessible by the relation* ϱ *if*

$$(a_i, a_i) \in \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = \bar{\varrho}.$$

A couple which is not accessible by ϱ is called *inaccessible by* ϱ.

If $(a_i, a_i) \in \varrho^l$ for some $l \geqq 1$, then $(a_i, a_i) \in \varrho^{lh}$ for any integer $h > 1$. In particular, we have

(3,2) $$(a_i, a_i) \subset \delta \cup \delta^2 \cup \ldots \cup \delta^d = \sigma.$$

Conversely, if $(a_i, a_i) \in \sigma$, then since $\sigma \subset \bar{\varrho}$, (a_i, a_i) is accessible by ϱ. Now (3,2) is equivalent ot the fact that

$$a_i \in a_i\delta \cup a_i\delta^2 \cup \ldots \cup a_i\delta^d.$$

With respect to Lemma 3,2 and the remark thereafter we have: The couple (a_i, a_i) is accessible by ϱ if and only if

(3,3) $$a_i \in a_i\delta \cup a_i\delta^2 \cup \ldots \cup a_i\delta^{d_i}.$$

Suppose now that (a_i, a_i) is accessible by ϱ. Then (3,3) implies that there is an integer l, $1 \leqq l \leqq d_i$, such that $a_i \in a_i\delta^l$. This implies

$$a_i\delta^l \subset a_i\delta^{2l} \subset \ldots \subset a_i\delta^{d_il} \subset a_i\delta^{(d_i+1)l} = a_i\delta^l.$$

Hence $a_i\delta^l = a_i\delta^{d_i}$. We have proved:

Lemma 3,3. *If* (a_i, a_i) *is accessible by* ϱ, *then among the sets*

$$a_i\delta, a_i\delta^2, \ldots, a_i\delta^{d_i}$$

there is a unique set containing a_i, *namely the set* $a_i\delta^{d_i}$.

Lemma 3,4. *Suppose that (a_i, a_i) is accessible by ϱ. If $a_i \in a_i\varrho^l$, then $d_i \mid l$.*

Proof. $a_i \in a_i\varrho^l$ implies

$$a_i \in a_i\varrho^l \subset a_i\varrho^{2l} \subset \ldots \subset a_i\varrho^{sl} \subset a_i\varrho^{(s+1)l} \subset \ldots$$

There exists therefore an integer s such that $a_i\varrho^{sl} = a_i\varrho^{(s+1)l}$. By definition of d_i we have $d_i \mid (s+1)\,l - sl$, i.e. $d_i \mid l$, q.e.d.

Corollary. *Suppose that (a_i, a_i) is accessible by ϱ. Denote by $h_i \geqq 1$ the least integer such that $a_i \in a_i\varrho^{h_i}$, then $d_i \mid h_i$.*

Notation. In the following h_i will always have the meaning introduced in the last Corollary.

Remark. Since $h_i \leqq n$ and $d_i \mid h_i$, we have $d_i \leqq n$. (This will be proved again and independently from the above considerations in Theorem 4,1.)

Example 3,1. In general it is not true that $d_i = h_i$. Let, e.g., ϱ be the relation with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

It is easy to compute that $a_1 \in a_1\varrho^2$, so that $h_1 = 2$, while $d_1 = 1$. [This follows also from the next Theorem 3,2, since we have $(a_1, a_1) = (a_1, a_3)(a_3, a_1) \in \varrho^2$, $(a_1, a_1) = (a_1, a_3)(a_3, a_2)(a_2, a_1) \in \varrho^3$, i.e. $a_1 \in a_1\varrho^2$ and $a_1 \in a_1\varrho^3$.]

Theorem 3,2. *Suppose that (a_i, a_i) is accessible by ϱ. Then d_i is the greatest common divisor of all integers λ for which $a_i \in a_i\varrho^\lambda$.*

Proof. Consider the sequence

$$a_i\varrho, \ldots, a_i\varrho^{k-1} \mid a_i\varrho^k, \ldots, a_i\varrho^r, \ldots, a_i\varrho^{k+d-1} \mid a_i\varrho^{k+d}, \ldots$$

Let $\lambda' < \lambda'' < \ldots < \lambda^{(u)}$ be the set of all $\lambda \leqq r$ for which $a_i \in a_i\varrho^\lambda$ holds. By Lemma 3,4 $d_i \mid \lambda'$, $d_i \mid \lambda''$, ..., $d_i \mid \lambda^{(u)}$. By Lemma 3,3 for $\lambda \geqq r + 1$ we have $a_i \in a_i\varrho^\lambda$ if and only if $\lambda = r + d_i, r + 2d_i, \ldots$ Each of these numbers is divisible by d_i and, moreover, d_i is clearly the greatest common divisor of all these numbers. This proves our statement.

We have seen that $a_\alpha\delta^d = a_\alpha\delta^{d_\alpha}$. This will be used in the proof of the next Theorem.

Theorem 3,3. *If $a_\alpha\delta^{d_\alpha} \neq \emptyset$, then there is at least one element $a_\xi \in a_\alpha\delta^{d_\alpha}$ such that (a_ξ, a_ξ) is accessible by ϱ.*

Proof. Suppose without loss of generality that $a_\alpha\delta^d = \{a_1, a_2, \ldots, a_s\}$, $1 \leqq s \leqq n$. Since $\delta^{2d} = \delta^d$, we have $\alpha_\alpha\delta^d = \{a_1\delta^d, a_2\delta^d, \ldots, a_s\delta^d\}$. Hence we can find integers $i_1, i_2, \ldots, i_{s-1}, i_s$ all $\in \{1, 2, \ldots, s\}$ such that

$$a_1 \in a_{i_1}\delta^d,\ a_{i_1} \in a_{i_2}\delta^d,\ a_{i_2} \in a_{i_3}\delta^d, \ldots, a_{i_{s-1}} \in a_{i_s}\delta^d .$$

The $s + 1$ integers $1 = i_0, i_1, \ldots, i_s$ cannot be all different. There is therefore a couple $l \neq l + h$ $(0 \leqq l \leqq s - 1,\ 1 \leqq l + h \leqq s)$ such that $i_l = i_{l+h}$. We than have

$$a_{i_l} \in a_{i_{l+1}}\delta^d,\ a_{i_{l+1}} \in a_{i_{l+2}}\delta^d, \ldots, a_{i_{l+h-1}} \in a_{i_{l+h}}\delta^d = a_{i_l}\delta^d .$$

This implies

$$a_{i_l} \in (a_{i_{l+2}}\delta^d)\,\delta^d = a_{i_{l+2}}\delta^d \subset a_{i_{l+3}}\delta^d \subset \ldots \subset a_{i_l}\delta^d .$$

The relation $a_{i_l} \in a_{i_l}\delta^d$ says that (a_{i_l}, a_{i_l}) is accessible by ϱ. This proves our Theorem.

Remark. Note that if $a_i\delta = \emptyset$, then $a_i\varrho$ need not be empty. Of course, in this case we have $d_i = 1$.

Example 3,2. In general it is not true that if $a_i\delta^d \neq \emptyset$, it contains only such elements a_ξ for which the couple (a_ξ, a_ξ) is accessible by ϱ. This shows the example of the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Since $\varrho = \varrho^2$, we have $\varrho = \delta^d$. Here $a_4\delta^d = a_4\varrho = \{a_1, a_2, a_3, a_4\}$, while (a_3, a_3) is inaccessible by ϱ.

Lemma 3,5. *If* $a_i\delta^{d_i} = \{a_\alpha, a_\beta, \ldots, a_\nu\}$, *then* $d_i \mid [d_\alpha, d_\beta, \ldots, d_\nu]$.

Remark. This is analogous to Lemma 3,1 and the result is non-trivial only if $a_i \notin a_i\delta^{d_i}$, i.e. (a_i, a_i) is inaccessible by ϱ.

Proof. By definition of $d_\alpha, d_\beta, \ldots, d_\nu$, we have for any integers $\lambda_\alpha \geqq 1$, $\lambda_\beta \geqq$ $\geqq 1, \ldots, \lambda_\nu \geqq 1$:

$$\begin{aligned} a_\alpha\delta &= a_\alpha\delta^{\lambda_\alpha d_\alpha + 1}, \\ a_\beta\delta &= a_\beta\delta^{\lambda_\beta d_\beta + 1}, \\ &\ldots\ldots\ldots\ldots\ldots \\ a_\nu\delta &= a_\nu\delta^{\lambda_\nu d_\nu + 1}. \end{aligned}$$

Denote $d^* = [d_\alpha, d_\beta, \ldots, d_\nu]$ and put $\lambda_\alpha = d^*/d_\alpha, \ldots, \lambda_\nu = d^*/d_\nu$. Summing all these sets we get

$$\{a_\alpha, a_\beta, \ldots, a_\nu\}\,\delta = \{a_\alpha, a_\beta, \ldots, a_\nu\}\,\delta^{d^*+1},$$

i.e.

$$a_i\delta^{d_i+1} = a_i\delta^{d_i+1+d^*}.$$

This implies $d_i \mid d^* = [d_\alpha, \ldots, d_\nu]$, q.e.d.

Remark. If (a_i, a_i) is inaccessible by ϱ, then for any $a_\alpha \in a_i\delta^d$ we have $a_i \notin a_\alpha\delta^d$. For, $a_i \in a_\alpha\delta^d$ would imply $a_i \in a_\alpha\delta^d \subset (a_i\delta^d)\,\delta^d = a_i\delta^d$, contrary to the assumption that (a_i, a_i) is inaccessible.

Lemma 3,6. *Let (a_i, a_i) be inaccessible by ϱ and $a_i\delta^d = \{a_\alpha, a_\beta, a_\gamma, \ldots, a_\nu\}$. If (a_α, a_α) is inaccessible, then $d_i \mid [d_\beta, d_\gamma, \ldots, d_\nu]$.*

Proof. $a_i\delta^d = \{a_\alpha, \ldots, a_\nu\}$ implies $a_i\delta^d = \{a_\alpha\delta^d, a_\beta\delta^d, \ldots, a_\nu\delta^d\} = \{a_\alpha, \ldots, a_\nu\}$. Therefore $a_\alpha\delta^d$ is a subset of $\{a_\alpha, a_\beta, \ldots, a_\nu\}$ which does not contain a_α. Hence (by Lemma 3,5) $d_\alpha \mid [d_\beta, d_\gamma, \ldots, d_\nu]$. Now $d_i \mid [d_\alpha, d_\beta, \ldots, d_\nu]$ implies $d_i \mid [d_\beta, d_\gamma, \ldots, d_\nu]$.

Remark. In Lemma 3,6 we have proved: If $a_i\delta^d = \{a_\alpha, \ldots, a_\nu\}$, then on the "right side" of $d_i \mid [d_\alpha, \ldots, d_\nu]$ it is allowed to delete any d_λ corresponding to an inaccessible couple (a_λ, a_λ). But we have not proved that it is allowed to delete simultaneously more than one of these numbers. This will be proved in Theorem 3,4.

Lemma 3,6 may be reformulated in the following way.

Lemma 3,6a. *Let (a_i, a_i) be inaccessible by ϱ and $a_i\delta^d \subset \{a_\alpha, a_\beta, \ldots, a_\nu\}$. If (a_α, a_α) is inaccessible, then $d_i \mid [d_\beta, \ldots, d_\nu]$.*

Proof. If $a_\alpha \in a_i\delta^d$, we may delete d_α by Lemma 3,6. If a_α does not belong to $a_i\delta^d$, then (again by Lemma 3,6) we have $d_i \mid [d_\beta, \ldots, d_\nu]$, so that in both cases d_α may be deleted.

Theorem 3,4. *Let (a_i, a_i) be inaccessible, $a_i\delta^d \neq \emptyset$, and let $a_{\beta_1}, a_{\beta_2}, \ldots, a_{\beta_v}$ be those elements $\in a_i\delta^d$ for which $(a_{\beta_1}, a_{\beta_1}), \ldots, (a_{\beta_v}, a_{\beta_v})$ are accessible. Then $d_i \mid$ $\mid [d_{\beta_1}, d_{\beta_2}, \ldots, d_{\beta_v}]$.*

In other words: If $a_i\delta^d = \{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_u}\}$, then on the "right side" of $d_i \mid [d_{\alpha_1}, d_{\alpha_2}, \ldots, d_{\alpha_v}]$ we may delete all d_{α_j} corresponding to the inaccessible couples $(a_{\alpha_j}, a_{\alpha_j})$.

Proof. We prove it by induction. Suppose that if (a_j, a_j) is inaccessible and $a_j\delta^d \subset \{a_{\gamma_1}, \ldots, a_{\gamma_s}\}$, we have yet proved that on the "right side" of $d_j \mid [d_{\gamma_1}, \ldots, d_{\gamma_s}]$ it is allowed to delete simultaneously any m-tuple corresponding to inaccessible couples. We shall prove that it is allowed to delete any $(m + 1)$-tuple (if, of course, such an $(m + 1)$-tuple exists).

By Lemma 3,6a our statement is true for $m = 1$.

Suppose without loss of generality that $(a_{\alpha_1}, a_{\alpha_2}), \ldots, (a_{\alpha_m}, a_{\alpha_m}), (a_{\alpha_{m+1}}, a_{\alpha_{m+1}})$ are inaccessible. We have

$$a_i\delta^d = \{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_m}, a_{\alpha_{m+1}}, \ldots, a_{\beta_1}, \ldots, a_{\beta_v}\} = \{a_{\alpha_1}\delta^d, \ldots, a_{\alpha_{m+1}}\delta^d, \ldots, a_{\beta_v}\delta^d\} .$$

1. Suppose first that none of the elements $a_{\alpha_1}, \ldots, a_{\alpha_m}$ is contained in $a_{\alpha_{m+1}}\delta^d$. Then $a_{\alpha_{m+1}}\delta^d$ is a subset of $a_i\delta^d$ which does not contain $a_{\alpha_1}, \ldots, a_{\alpha_m}$ and, of course, it does not contain $a_{\alpha_{m+1}}$ also. Hence by Lemma 3,5

$$d_{\alpha_{m+1}} \mid [d_{\alpha_{m+2}}, d_{\alpha_{m+3}}, \ldots, d_{\beta_1}, \ldots, d_{\beta_v}] . \tag{3,4}$$

Now $d_i \mid [d_{\alpha_1}, \ldots, d_{\alpha_m}, d_{\alpha_{m+1}}, d_{\alpha_{m+2}}, \ldots]$ implies by the inductive supposition $d_i \mid [d_{\alpha_{m+1}}, d_{\alpha_{m+2}}, \ldots]$ and with respect to (3,4) we have $d_i \mid [d_{\alpha_{m+2}}, d_{\alpha_{m+3}}, \ldots, d_{\beta_1}, \ldots \ldots, d_{\beta_v}]$.

2. Suppose next that at least one of the elements $a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_m}$, say a_{α_1}, is contained in $a_{\alpha_{m+1}}\delta^d$. Then by the remark preceding Lemma 3,6 we have $a_{\alpha_{m+1}} \notin a_{\alpha_1}\delta^d$.

Now $a_{\alpha_1}\delta^d$ is a subset of $a_i\delta^d$ and

$$a_{\alpha_1}\delta^d \subset \{a_{\alpha_1}, \ldots, a_{\alpha_m}, a_{\alpha_{m+2}}, \ldots, a_{\beta_1}, \ldots, a_{\beta_v}\} .$$

Therefore

$$d_{\alpha_1} \mid [d_{\alpha_1}, \ldots, d_{\alpha_m}, d_{\alpha_{m+2}}, \ldots, d_{\beta_v}] .$$

By the inductive supposition it is allowed to delete the m numbers $d_{\alpha_1}, \ldots, d_{\alpha_m}$, i.e.

$$d_{\alpha_1} \mid [d_{\alpha_{m+2}}, \ldots, d_{\beta_v}] . \tag{3,5}$$

Now $d_i \mid [d_{\alpha_1}, \ldots, d_{\alpha_m}, d_{\alpha_{m+1}}, \ldots, d_{\beta_v}]$. By the inductive supposition we may delete $d_{\alpha_2}, \ldots, d_{\alpha_m}, d_{\alpha_{m+1}}$, so that

$$d_i \mid [d_{\alpha_1}, d_{\alpha_{m+2}}, \ldots, d_{\beta_v}] .$$

With respect to (3,5) we finally have $d_i \mid [d_{\alpha_{m+2}}, \ldots, d_{\beta_v}]$. This completes our induction.

3. We now easily conclude the proof of our Theorem.

With respect to the result just obtained we may delete in $d_i \mid [d_{\alpha_1}, d_{\alpha_2}, \ldots, d_{\beta_v}]$ simultaneously all d_{α_j} corresponding to inaccessible couples $(a_{\alpha_j}, a_{\alpha_j})$ so that we obtain $d_i \mid [d_{\beta_1}, \ldots, d_{\beta_v}]$, where all $(a_{\beta_l}, a_{\beta_l})$ are accessible. With respect to Theorem 3,3 (since $a_i\delta^d \neq \emptyset$) there remains always at least one d_{β_j}. This completes the proof of our Theorem.

We now sharpen Theorem 3,1.

Theorem 3,5. *Let ϱ be a binary relation for which $\bar{\varrho} \cap \Delta \neq z$. We then have $d(\varrho) = [d_{i_1}, d_{i_2}, \ldots, d_{i_l}]$, where $\{(a_{i_1}, a_{i_1}), \ldots, (a_{i_l}, a_{i_l})\} = \bar{\varrho} \cap \Delta$.*

Proof. By Theorem 3,1 we have $d(\varrho) = [d_1, d_2, \dots, d_n]$. If $a_i\delta^d = \emptyset$, then $d_i = 1$ and d_i can be deleted. If $a_i\delta^d \neq \emptyset$ and (a_i, a_i) is inaccessible, then by Theorem 3,4 d_i divides the least common multiple of some d_β which correspond to accessible couples (a_β, a_β). Hence again d_i may be deleted. Deleting the d_i's corresponding to these two types of "rows" we obtain our statement.

The next Theorem gives a further characterization of the number $d(\varrho)$.

Theorem 3,6. *Suppose that $\bar{\varrho} \cap \Delta \neq z$. Then $d(\varrho)$ is the least integer $s \geqq 1$ for which $\bar{\varrho} \cap \Delta \subset \delta^s$ holds.*

Proof. Suppose that $\bar{\varrho} \cap \Delta = \{(a_\alpha, a_\alpha), \dots, (a_\nu, a_\nu)\}$. By Lemma 3,3 we have $(a_\lambda, a_\lambda) \in \delta^{d_\lambda}$ for $\lambda = \alpha, \dots, \nu$. Since by Theorem 3,5 $[d_\alpha, \dots, d_\nu] = d$, we have $\bar{\varrho} \cap \Delta \subset \delta^d$.

On the other side if $(a_\lambda, a_\lambda) \in \delta^s = \varrho^{s(r+1)}$ (for $\lambda = \alpha, \dots, \nu$), then by Lemma 3,4 we have $d_\lambda \mid s$, hence $d \mid s$. This proves our Theorem.

We conclude this paragraph with the following

Theorem 3,7. *If $\bar{\varrho} \cap \Delta = z$, then $d(\varrho) = 1$ and $k(\varrho) \leqq n$.*

Proof. It follows by Theorem 3,3 that $\bar{\varrho} \cap \Delta = z$ if and only if $a_i\delta^{d_i} = a_i\delta^d = \emptyset$ for every $a_i \in \Omega$. Hence $\delta^d = z$, i.e. $\delta = z$, and $d(\varrho) = 1$. Further by Theorem 1,3 we have $\varrho^n = z$, which implies $k(\varrho) \leqq n$.

4. SOME ESTIMATES FOR $k(\varrho)$ AND $d(\varrho)$

Lemma 4,1. *If (a_i, a_i) is accessible by ϱ, then there is an integer h_i, $1 \leqq h_i \leqq n$, such that $a_i\varrho \subset a_i\varrho^{h_i+1}$.*

Proof. By supposition there is an integer h_i, $1 \leqq h_i \leqq n$, such that $a_i \in a_i\varrho^{h_i}$. Hence $a_i\varrho \subset a_i\varrho^{h_i+1}$.

Lemma 4,2. *If (a_i, a_i) is accessible by ϱ and $a_i\varrho$ contains $g_i \geqq 1$ different elements $\in \Omega$, then*

a) $k_i \leqq (n - g_i) h_i + 1$;

b) $d_i \mid h_i$, *hence* $d_i \leqq n$.

Proof. By Lemma 4,1 we have

$$a_i\varrho \subset a_i\varrho^{h_i+1} \subset a_i\varrho^{2h_i+1} \subset \dots \subset a_i\varrho^{(n-g_i)h_i+1} \subset a_i\varrho^{(n-g_i+1)h_i+1}.$$

Since $a_i\varrho$ contains $g_i \geqq 1$ elements $\in \Omega$, the set $a_i\varrho^{h_i+1}$ is either equal to $a_i\varrho$ or contains at least $g_i + 1$ different elements $\in \Omega$. Further, $a_i\varrho^{2h_i+1}$ is again either equal to $a_i\varrho^{h_i+1}$ or contains at least $g_i + 2$ elements $\in \Omega$. Etc. The chain cannot have more than $n - g_i + 1$ different members. There exists therefore an integer l, $0 \leqq l \leqq n - g_i$, such that $a_i\varrho^{lh_i+1} = a_i\varrho^{(l+1)h_i+1}$.

This implies:

a) $k_i \leqq lh_i + 1 \leqq (n - g_i) h_i + 1$;
b) $d_i \mid (l + 1) h_i + 1 - (lh_i + 1)$, i.e. $d_i \mid h_i$.

This proves Lemma 4,2.

Lemma 4,2 implies:

a) If $g_i \geqq 2$ and $1 \leqq h_i \leqq n$, we have $k_i \leqq (n - 2) n + 1 = (n - 1)^2$.
b) If $g_i = 1$ and $1 \leqq h_i \leqq n - 1$, we have $k_i \leqq (n - 1)(n - 1) + 1 = (n - 1)^2 + 1$.
c) If $g_i = 1$ and $h_i = n$, we have $k_i \leqq (n - 1) n + 1 = n^2 - n + 1$.

We show that the result sub c) can be sharpened.

We formulate this as a Lemma:

Lemma 4,3. *With the notations introduced above we have: If $g_i = 1$, $h_i = n$, then*

a) *either $k_i = 1$ and $d_i = n$,*
b) *or $k_i \leqq (n - 1)^2 + 1$ and $d_i \leqq n - 1$.*

Proof. Note that the supposition implies that in this case $a_i \in a_i\varrho^n$ and $a_i \notin a_i\varrho^h$ for $h < n$.

We first show that in this case $a_i\varrho^l$, $2 \leqq l \leqq n$, contains *exactly one element* $\in \Omega$ which is not contained in $L = a_i\varrho \cup \ldots \cup a_i\varrho^{l-1}$.

The set $a_i\varrho^l$ contains *at least one* element not contained in L. For $a_i\varrho^l \subset a_i\varrho \cup \ldots \cup a_i\varrho^{l-1}$ implies $a_i\varrho^{l+1} \subset a_i\varrho^2 \cup \ldots \cup a_i\varrho^l \subset a_i\varrho \cup \ldots \cup a_i\varrho^{l-1}$, and repeating this argument we get $a_i\varrho^n \subset a_i\varrho \cup \ldots \cup a_i\varrho^{l-1}$. Since $a_i \in a_i\varrho^n \subset L$, there is h, $1 \leqq h \leqq l - 1 \leqq n - 1$, such that $a_i \in a_i\varrho^h$. This is a contradiction to $a_i \notin a_i\varrho^h$ for $h < n$.

Note, by the way, that this implies that L contains at least $l - 1$ different elements $\in \Omega$.

Suppose now that $a_i\varrho^l$ contains *at least two* elements not contained in L. Then $L \cup a_i\varrho^l$ would contain at least $l + 1$ different elements $\in \Omega$. This would imply in an obvious manner that $a_i\varrho \cup \ldots \cup a_i\varrho^{n-1}$ contains all elements $\in \Omega$, a contradiction to $a_i \notin a_i\varrho^h$ for $h < n$. This proves our assertion.

Consider now the finite sequence $a_i\varrho, a_i\varrho^2, \ldots, a_i\varrho^n, a_i\varrho^{n+1}$.

1. Suppose that each of the sets $a_i\varrho^l$ $(l = 1, \ldots, n + 1)$ contains exactly one element $\in \Omega$. Then $a_i \in a_i\varrho^n$ implies $a_i = a_i\varrho^n$ and $a_i\varrho = a_i\varrho^{n+1}$, hence $k_i = 1$, $d_i = n$.

2. Suppose next that at least one of the sets $a_i\varrho^l$ $(l = 1, \ldots, n + 1)$ contains more than one element. Let l_0 be the least integer such that $a_i\varrho^{l_0}$ contains more than one element $\in \Omega$. By supposition $l_0 \geqq 2$. Further $l_0 \neq n + 1$ since (as we have just seen) the supposition that $a_i\varrho^n$ contains a unique element $\in \Omega$ implies $a_i\varrho = a_i\varrho^{n+1}$, i.e. $a_i\varrho^{n+1}$ contains a unique element $\in \Omega$, a contradiction. Hence $2 \leqq l_0 \leqq n$.

Write $a_i\varrho = \{a_\alpha\}$, $a_i\varrho^2 = \{a_\beta\}, \ldots, a_i\varrho^{l_0 - 1} = \{a_\nu\}$. Since $a_i\varrho^{l_0}$ contains at least two elements $\in \Omega$ and only one not contained in $\{a_\alpha, a_\beta, \ldots, a_\nu\}$, there is a $a_\lambda \in \in \{a_\alpha, a_\beta, \ldots, a_\nu\}$ such that $\{a_\lambda\} \in a_i\varrho^{l_0}$. Consequently there is an integer u, $1 \leqq \leqq u \leqq l_0 - 1$, such that $\{a_\lambda\} = a_i\varrho^{l_0 - u} \subset a_i\varrho^{l_0}$. This implies

$$a_i\varrho^{l_0 - u} \subset a_i\varrho^{l_0} \subset a_i\varrho^{l_0 + u} \subset \ldots \subset a_i\varrho^{l_0 + (n-1)u} .$$

This chain of $n + 1$ sets cannot have all members different one from the other. There is therefore an integer v, $-1 \leqq v \leqq n - 2$, such that

$$a_i\varrho^{l_0 + vu} = a_i\varrho^{l_0 + (v+1)u} .$$

This implies:

$$k_i \leqq l_0 + vu \leqq l_0 + (n - 2)(l_0 - 1) =$$
$$= l_0(n - 1) - (n - 2) \leqq n(n - 1) - (n - 2) = (n - 1)^2 + 1 ,$$

and

$$d_i \mid l_0 + (v + 1)u - (l_0 + vu) ,$$

i.e. $d_i \mid u$, where $u \leqq l_0 - 1 \leqq n - 1$; in particular, $d_i \leqq n - 1$. This proves Lemma 4,3.

Lemma 4,3 together with the cases a), b) considered above imply:

Theorem 4,1. *If (a_i, a_i) is accessible by ϱ we always have*

a) $k_i \leqq (n - 1)^2 + 1$,
b) $d_i \leqq n$.

It can be proved on examples that these results cannot be sharpened.

We next give an estimation for k_i if (a_i, a_i) is inaccessible. (Recall that card $\Omega \geqq 2$.)

Theorem 4,2. *If (a_i, a_i) is inaccessible by ϱ, then $k_i \leqq (n - 2)^2 + 2$.*

Proof. 1. We shall prove it by induction with respect to the number n. The Theorem holds if $n = 2$, since it is easy to see (by considering all possible cases) that in this case we have $k_i \leqq 2$.

Let card $\Omega = n \geqq 3$ and suppose that our Theorem holds for relations defined on any set Ω_1 with card $\Omega_1 \leqq n - 1$.

2. Suppose that (a_i, a_i) is inaccessible by ϱ. Denote $A_i = a_i\varrho \cup \ldots \cup a_i\varrho^n$ and $B_i = \Omega \setminus A_i$. Since $a_i \in B_i$, B_i is not empty.

We first state that the rectangle $A_i \times B_i$ does not belong to ϱ. For, if for some $a_\alpha \in A_i$ and some $a_\beta \in B_i$ there were $(a_\alpha, a_\beta) \in \varrho$, i.e. $a_\beta \in a_\alpha\varrho$, we would have

$$a_\beta \in a_\alpha\varrho \subset (a_i\varrho \cup \ldots \cup a_i\varrho^n)\,\varrho =$$
$$= a_i\varrho^2 \cup \ldots \cup a_i\varrho^{n+1} \subset a_i\varrho \cup \ldots \cup a_i\varrho^n = A_i\,,$$

contrary to the assumption.

Denote $a_i\varrho = \{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_s}\}$. By Lemma 3,1 we have $k_i \leqq 1 + \max\limits_{j=1,\ldots,s} k_{\alpha_j}$.

If a_λ is any element $\in A_i$, we have $a_\lambda\varrho \subset A_i\varrho = (a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^n)\,\varrho \subset A_i$. Consider the relation $\varrho_0 = \varrho \cap (A_i \times A_i)$. Since for any $a_\lambda \in A_i$ we have $a_\lambda\varrho \subset A_i$, we clearly have $a_\lambda\varrho = a_\lambda\varrho_0$ for all $a_\lambda \in A_i$. This implies that for any subset $A' \subset A_i$ we have $A'\varrho = A'\varrho_0$. In particular: $a_{\alpha_l}\varrho = a_{\alpha_l}\varrho_0$. Further $a_{\alpha_l}\varrho^2 = (a_{\alpha_l}\varrho)\,\varrho = (a_{\alpha_l}\varrho_0)\,\varrho = (a_{\alpha_l}\varrho_0)\,\varrho_0 = a_{\alpha_l}\varrho_0^2$ (we have used that $a_{\alpha_l}\varrho_0 \subset A_i$). Analogously $a_{\alpha_l}\varrho^u = a_{\alpha_l}\varrho_0^u$ holds for any integer $u \geqq 1$.

Therefore $k_{\alpha_l}(\varrho) = k_{\alpha_l}(\varrho_0)$ and $d_{\alpha_l}(\varrho) = d_{\alpha_l}(\varrho_0)$.

Now ϱ_0 is a relation defined on $A_i \subset \Omega$ and card $A_i \leqq n - 1$.

If $a_\alpha \in a_i\varrho \subset A_i$ and (a_α, a_α) is accessible by ϱ, we have (by Theorem 4,1) $k_\alpha \leqq (n - 1 - 1)^2 + 1$.

If $a_\alpha \in A_i$ and (a_α, a_α) is inaccessible, we have by the inductive supposition $k_\alpha \leqq (n - 1 - 2)^2 + 2$. Hence

$$k_i \leqq 1 + \max_{j=1,\ldots,s} k_{\alpha_j} \leqq 1 + \max\{(n-2)^2 + 1, (n-3)^2 + 2\} = (n-2)^2 + 2\,,$$

for $n \geqq 3$. This proves our statement.

Remark. It can be shown on examples that the result of Theorem 4,2 cannot be sharpened.

Theorem 3,1, Theorem 4,1 and Theorem 4,2 imply:

Theorem 4,3. *For any binary relation ϱ on Ω (with* card $\Omega = n$*) we always have* $k(\varrho) \leqq (n - 1)^2 + 1$.

Remark. (Important.) If (a_i, a_i) is inaccessible, then $d_i(\varrho) \leqq n$ need not hold. This can be shown on the example of the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here (a_8, a_8) is inaccessible, and $d_8 = 10$. It is easy to compute (in simplest way by considering the corresponding oriented graph) that

$$\begin{array}{lll} a_8\varrho = \{a_2, a_7\} & a_8\varrho^5 = \{a_2, a_6\} & a_8\varrho^9 = \{a_2, a_5\} \\ a_8\varrho^2 = \{a_1, a_3\} & a_8\varrho^6 = \{a_1, a_7\} & a_8\varrho^{10} = \{a_1, a_6\} \\ a_8\varrho^3 = \{a_2, a_4\} & a_8\varrho^7 = \{a_2, a_3\} & a_8\varrho^{11} = \{a_2, a_7\} = a_8\varrho\,. \\ a_8\varrho^4 = \{a_1, a_5\} & a_8\varrho^8 = \{a_1, a_4\} & \end{array}$$

This shows that $k_8 = 1$ and $d_8 = [2, 5] = 10$. (This is in essential the simplest example of the kind required.)

5. REDUCIBLE AND IRREDUCIBLE RELATIONS

In the following we shall denote

$$\Pi(\varrho) = pr_1(\varrho \cup \bar{\varrho}^1) = pr_2(\varrho \cup \bar{\varrho}^1)\,.$$

Clearly we always have $\varrho \subset \Pi(\varrho) \times \Pi(\varrho)$ and $\operatorname{card} \Pi(\varrho) = n_\varrho \leqq n$.

Definition. A relation ϱ is called *a square* if $\Pi(\varrho) \times \Pi(\varrho) = \varrho$.

A square has always a non-empty intersection with the diagonal.

Definition. A relation ϱ is called *reducible* if $\Pi(\varrho)$ can be written as a union of two disjoint non-empty sets $\Pi(\varrho) = A \cup B$, $A \cap B = \emptyset$, and $\varrho \subset (A \times A) \cup (B \times A) \cup$ $\cup (B \times B)$. ϱ is called completely reducible if moreover $\varrho \subset (A \times A) \cup (B \times B)$.

A relation which is not reducible is called *irreducible.*

Remark. When speaking (in all what follows) about irreducible relations we shall always suppose that $\varrho \neq z$, hence card $\Pi(\varrho) \geqq 1$. If card $\Pi(\varrho) = 1$, then ϱ is irreducible.

A square is always irreducible. The diagonal is completely reducible.

Example 5,1. Consider the set $\Omega = \{a_1, a_2, a_3\}$ and the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here $\Pi(\varrho) = \{a_1, a_2\}$ and $\varrho = \Pi(\varrho) \times \Pi(\varrho)$. In the sense of our definition ϱ is irreducible. [In the sense of the theory of non-negative matrices the matrix $M(\varrho)$ is reducible. From our point of view there is a superfluous zero row and zero column.]

Example 5,2. Consider the relation ϱ on $\Omega = \{a_1, a_2, a_3, a_4\}$ with

$$M(\varrho) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $\Pi(\varrho) = \Omega$. Write $A = \{a_1, a_2, a_4\}$, $B = \{a_3\}$. Then $\varrho \subset (A \times A) \cup (B \times A)$, so that ϱ is reducible.

Lemma 5,1. *If ϱ is irreducible, then $pr_1(\varrho) = pr_2(\varrho) = \Pi(\varrho)$.*

Proof. For any $a_i \in \Pi(\varrho)$ we have $a_i\varrho \neq \emptyset$. For, $a_i\varrho = \emptyset$ would imply $\varrho \subset \{(\Pi(\varrho) \setminus a_i) \times (\Pi(\varrho) \setminus a_i)\} \cup \{(\Pi(\varrho) \setminus a_i) \times a_i\}$, contrary to the assumption of irreducibility. This proves $pr_1(\varrho) = \Pi(\varrho)$. Analogously $pr_2(\varrho) = \Pi(\varrho)$.

Clearly ϱ is irreducible if and only if $\bar{\varrho}^1$ is irreducible.

Lemma 5,2. *Let ϱ be irreducible. Let M be any proper subset of $\Pi(\varrho)$. Then $M\varrho$ contains at least one element not contained in M.*

Proof. Let $M = \{a_\alpha, a_\beta, \ldots, a_\nu\}$ be a non-empty proper subset of $\Pi(\varrho)$. Suppose for an indirect proof that

$$\{a_\alpha, a_\beta, \ldots, a_\nu\}\, \varrho \subset \{a_\alpha, a_\beta, \ldots, a_\nu\}\,.$$

Let be $(a_\varkappa, a_\lambda) \in \varrho$. If $a_\varkappa \in M$, then we necessarily have $a_\lambda \in M$. In other words: If $a_\varkappa \in M$, $a_\lambda \in \Pi(\varrho) \setminus M = B$, then $(a_\varkappa, a_\lambda) \notin \varrho$. Hence $\varrho \subset (M \times M) \cup (B \times M) \cup (B \times B)$. This says that ϱ is reducible, contrary to the assumption.

Remark. Lemma 5,2 need not be true if ϱ is reducible. Let be $\Omega = \{a_1, a_2, a_3, a_4\}$ and ϱ the relation with

$$M(\varrho) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Here $\Pi(\varrho) = \Omega$. Take $M = \{a_1, a_2\}$. Then $M\varrho = a_1\varrho \cup a_2\varrho = \{a_1\}$. Hence $M\varrho \subset \subset M$.

Put in Lemma 5,2 $M = a_i\varrho$. We then have the following important

Corollary. *Suppose that ϱ is irreducible, $a_i \in \Pi(\varrho)$. If $a_i\varrho \subsetneqq \Pi(\varrho)$, then $a_i\varrho^2$ contains at least one element $\in \Pi(\varrho)$ which is not contained in $a_i\varrho$.*

Lemma 5,3. *Suppose that ϱ is irreducible and $a_i \in \Pi(\varrho)$.*

a) *If $a_i\varrho$ contains $g_i \geqq 1$ elements, we have $a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^{n_\varrho - g_i + 1} = \Pi(\varrho)$.*

b) *In particular, for any $a_i \in \Pi(\varrho)$ we always have $a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^{n_\varrho} = \Pi(\varrho)$.*

c) *If $a_j \in \Pi(\varrho)$ and $i \neq j$, we have $a_j \in a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^{n_\varrho - 1}$.*

Proof. Note first: Since $a_i\varrho \subset \Pi(\varrho)$, we also have $a_i\varrho^2 = (a_i\varrho)\,\varrho \subset \bigcup_{a_\alpha \in a_i\varrho} a_\alpha\varrho \subset \subset \Pi(\varrho)$. Analogously $a_i\varrho^l \subset \Pi(\varrho)$ for any integer $l \geqq 1$.

a) By the last Corollary $a_i\varrho \cup a_i\varrho^2$ contains at least $\min(n_\varrho, g_i + 1)$ elements $\in \Pi(\varrho)$. Applying Lemma 5,2 we obtain that $(a_i\varrho \cup a_i\varrho^2) \cup (a_i\varrho \cup a_i\varrho^2)\,\varrho = a_i\varrho \cup a_i\varrho^2 \cup a_i\varrho^3$ contains at least $\min(n_\varrho, g_i + 2)$ different elements $\in \Pi(\varrho)$. Repeating this argument we obtain our result.

b) This follows from the fact that $a_i\varrho$ is not empty, so that $g_i \geqq 1$.

c) By Lemma 5,2 $a_i \cup a_i\varrho$ contains at least two elements $\in \Pi(\varrho)$. By the same argument as sub a) $a_i \cup a_i\varrho \cup a_i\varrho^2$ contains at least three elements $\in \Pi(\varrho)$. Finally we obtain $a_i \cup a_i\varrho \cup \ldots \cup a_i\varrho^{n_\varrho - 1} = \Pi(\varrho)$, from which our statement follows.

Lemma 5,4. *If ϱ is irreducible and $a_i \in \Pi(\varrho)$, then (a_i, a_i) is accessible by ϱ.*

Proof. Follows from $pr_1(\varrho) = \Pi(\varrho)$ and Lemma 5,3 (assertion b).

Lemma 5,5. *If ϱ is irreducible, then $\Pi(\varrho) = \Pi(\varrho^v)$ for any integer $v \geqq 1$.*

Proof. If ϱ is irreducible, then $\Pi(\varrho)\,\varrho = \bigcup_{a_i \in \Pi(\varrho)} a_i\varrho = pr_2(\varrho) = \Pi(\varrho)$. Hence $\Pi(\varrho)\,\varrho^v = \Pi(\varrho)$ for any integer $v \geqq 1$. This proves $pr_2(\varrho^v) = \Pi(\varrho)$ and a fortiori $pr_1[(\varrho^v)^{-1}] = \Pi(\varrho)$. Analogously $\varrho\,\Pi(\varrho) = \bigcup_{a_i \in \Pi(\varrho)} \varrho a_i = pr_1(\varrho) = \Pi(\varrho)$, hence $\varrho^v\,\Pi(\varrho) = \Pi(\varrho)$. This proves $pr_1(\varrho^v) = \Pi(\varrho)$. Now $pr_1[\varrho^v \cup (\varrho^v)^{-1}] = \Pi(\varrho)$, hence $\Pi(\varrho^v) = \Pi(\varrho)$.

Theorem 5,1. *A binary relation ϱ is irreducible if and only if*

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^{n_\varrho}$$

is a square, namely $\Pi(\varrho) \times \Pi(\varrho)$.

Proof. a) Suppose that ϱ is reducible and $\Pi(\varrho) = A \cup B$, $A \cap B = \emptyset$, (A, B non-empty), and

$$\varrho \subset (A \times A) \cup (B \times A) \cup (B \times B).$$

Then it is clear that no power of ϱ can contain (a_i, a_j) with $a_i \in A$, $a_j \in B$, so that $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^{n_\varrho}$ cannot be a square.

b) Denote $(a_i, M) = \bigcup_{a_j \in M} (a_i, a_j)$. If ϱ is irreducible, Lemma 5,3 (assertion b) implies

$$(a_i, a_i\varrho) \cup (a_i, a_i\varrho^2) \cup \ldots \cup (a_i, a_i\varrho^{n_\varrho}) = (a_i, \Pi(\varrho)).$$

Summing through $i = 1, 2, \ldots, n_\varrho$ we have:

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^{n_\varrho} = \Pi(\varrho) \times \Pi(\varrho).$$

A relation ϱ on Ω may have but need not have irreducible subrelations. We shall be interested in "*maximal*" irreducible subrelations of a given ϱ (if such exist).

Lemma 5,6. *If ϱ_1, ϱ_2 are two irreducible subrelations contained in ϱ, and $\varrho_1 \cap \cap \varrho_2 \neq z$, then $\varrho_1 \cup \varrho_2$ is an irreducible subrelation of ϱ.*

Proof. It is sufficient to prove that the transitive closure of $\varrho_1 \cup \varrho_2$ is a square, namely $(\Pi(\varrho_1) \cup \Pi(\varrho_2)) \times (\Pi(\varrho_1) \cup \Pi(\varrho_2)) = \tau$.

a) First it is obvious that $\overline{\varrho_1 \cup \varrho_2} \subset \tau$.

b) Let (a_i, a_j) be any element $\in \tau$. Suppose e.g. that $a_i \in \Pi(\varrho_1)$, $a_j \in \Pi(\varrho_2)$. (The remaining three cases can be handled analogously.) By supposition there is at least one couple (x_0, y_0) such that $(x_0, y_0) \in \varrho_1 \cap \varrho_2$. Since $a_i \in \Pi(\varrho_1)$, $x_0 \in \Pi(\varrho_1)$, there is an integer h such that $(a_i, x_0) \in \varrho_1^h$, $1 \leqq h \leqq n_{\varrho_1} \leqq n$. (We use Theorem 5,1.) Since $(x_0, y_0) \in \varrho_1$, we have $(a_i, x_0)(x_0, y_0) = (a_i, y_0) \in \varrho_1^{h+1}$. Now since $y_0 \in \Pi(\varrho_2)$, there is an integer l, $1 \leqq l \leqq n_{\varrho_2} \leqq n$ such that $(y_0, a_j) \in \varrho_2^l$. Therefore

$$(a_i, y_0)(y_0, a_j) = (a_i, a_j) \in \varrho_1^{h+1}\varrho_2^l \subset (\varrho_1 \cup \varrho_2)^{h+l+1} \subset$$
$$\subset (\varrho_1 \cup \varrho_2) \cup (\varrho_1 \cup \varrho_2)^2 \cup \ldots \cup (\varrho_1 \cup \varrho_2)^{n} = \overline{\varrho_1 \cup \varrho_2}.$$

Hence $\tau \subset \overline{(\varrho_1 \cup \varrho_2)}$. This proves $\tau = \overline{\varrho_1 \cup \varrho_2}$.

Definition. *An irreducible relation ϱ_0 is called maximal in ϱ*, if $\varrho_0 \subset \varrho$ and for any irreducible relation ϱ' for which $\varrho_0 \subset \varrho' \subset \varrho$ we have $\varrho_0 = \varrho'$.

Remark. If ϱ is irreducible, it is itself the unique maximal irreducible subrelation of ϱ.

Lemma 5,7. *The intersection of two maximal distinct irreducible subrelations of ϱ is the empty relation z.*

Proof. Suppose that ϱ_1 and ϱ_2 are two maximal irreducible subrelations of ϱ and $\varrho_1 \cap \varrho_2 \neq z$. We cannot have $\varrho_1 \subset \varrho_2$ (or $\varrho_2 \subset \varrho_1$) since otherwise one of them would not be maximal. Hence $\varrho_1 \cup \varrho_2$ is larger than ϱ_1 and ϱ_2 and by Lemma 5,6 irreducible, contrary to the assumption of maximality. Therefore $\varrho_1 \cap \varrho_2 = z$.

Theorem 5,2. *Any relation ϱ on a finite set Ω can be written as a union of disjoint relations in the form*

$$\varrho = \varrho_1 \cup \ldots \cup \varrho_v \cup \nu \,, \tag{5,0}$$

where ϱ_i $(i = 1, \ldots, v)$ are maximal irreducible subrelations of ϱ and ν is either z or a relation which does not contain irreducible subrelations. The decomposition (5,0) is (up to the order of summands) uniquely determined.

Proof. If ϱ does not contain an irreducible subrelation there is nothing to prove.

Let ϱ_1 be any maximal irreducible subrelation of ϱ. Write $\varrho = \varrho_1 \cup \sigma$, $\varrho_1 \cap \sigma = z$. If ϱ_2 is an other maximal subrelation we have by Lemma 5,7 $\varrho_2 \subset \sigma$ and we may write $\varrho = \varrho_1 \cup \varrho_2 \cup \sigma_1$ $(\varrho_2 \cap \sigma_1 = z)$. By this proceeding we obtain a decomposition into disjoint summands $\varrho = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_v \cup \nu$, where ν is either z or does not contain an irreducible subrelation. Suppose that there is an other such decomposition $\varrho = \varrho'_1 \cup \varrho'_2 \cup \ldots \cup \varrho'_u \cup \mu$, where ϱ'_i are again maximal and irreducible and μ has no irreducible subrelations. Taking the intersection with ϱ_1 we get $\varrho_1 = \bigcup_{i=1}^{u} (\varrho'_i \cap \varrho_1) \cup [\mu \cap \varrho_1]$. If ϱ_1 were different from all the ϱ'_i, then by Lemma 5,7 we would have $\varrho_1 = \mu \cap \varrho_1$, a contradiction with the supposition that μ has no irreducible subrelation. Hence ϱ_1 is equal to one of the ϱ'_i, say $\varrho_1 = \varrho'_1$. Repeating this argument we prove that the sets $\{\varrho_i\}$ and $\{\varrho'_i\}$ are identical, and moreover $\mu = \nu$.

Before proving Theorem 5,3 giving a criterium for the existence of irreducible subrelations we prove the following Lemma which will be needed also later.

Lemma 5,8. *The relation*

$$\varrho = \{(a_{i_1}, a_{i_2}), (a_{i_2}, a_{i_3}), (a_{i_3}, a_{i_4}), \ldots, (a_{i_{l-1}}, a_{i_l}), (a_{i_l}, a_{i_1})\}$$

is irreducible.

Proof. Clearly $\Pi(\varrho) = \{a_{i_1}, a_{i_2}, \ldots, a_{i_l}\}$. The relation ϱ^2 contains all couples with the subscripts $(i_1, i_3), (i_2, i_4), \ldots, (i_l, i_2)$. The relation ϱ^3 contains the couples with the subscripts $(i_1, i_4), (i_2, i_5), \ldots, (i_l, i_3)$, a.s.o. Therefore $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^l$ is the square $\Pi(\varrho) \times \Pi(\varrho)$, q.e.d.

Theorem 5,3. *A relation ϱ on Ω contains an irreducible subrelation if and only if the transitive closure $\bar{\varrho}$ of ϱ has a non-empty intersection with the diagonal Δ.*

Proof. The condition is necessary. If ϱ_0 is an irreducible subrelation of ϱ, then the transitive closure of ϱ_0 is a square, hence it has a non-empty intersection with Δ.

Suppose conversely that $\bar{\varrho} = \varrho \cup \varrho^2 \cup \ldots \cup \varrho^{n_\varrho}$ $[n_\varrho = \text{card}\, \Pi(\varrho)]$ contains at least one element $\in \Delta$, say (a_i, a_i). Then ϱ contains (at most n_ϱ) elements of the type $(a_i, a_{j_1}), (a_{j_1}, a_{j_2}), \ldots, (a_{j_v}, a_i)$ such that their product is (a_i, a_i). By Lemma 5,8 the relation $\varrho_0 = \{(a_i, a_{j_1}), (a_{j_1}, a_{j_2}), \ldots, (a_{j_v}, a_i)\}$ is irreducible. Hence ϱ contains an irreducible subrelation ϱ_0, q.e.d.

Theorem 5,4. *If ϱ does not contain an irreducible subrelation, then $\varrho^{n_\varrho} = z$, hence $d(\varrho) = 1$.*

Proof. Suppose that $\varrho^{n_\varrho} \neq z$. Then ϱ^{n_ϱ} contains a product of n_ϱ terms of the form

$$(a_{i_1}, a_{i_2})(a_{i_2}, a_{i_3}) \ldots (a_{i_{n_\varrho}}, a_{i_{n_\varrho+1}}) .$$

Since the set of integers $\{i_1, i_2, \ldots, i_{n_\varrho}, i_{n_\varrho+1}\}$ contains at most n_ϱ different numbers, there is an $\alpha < n_\varrho + 1$ such that $i_\alpha = i_\beta$ for some $\beta > \alpha$. But then $\varrho^{\beta-\alpha}$ (where $\beta - \alpha \leqq n_\varrho$) contains

$$(a_{i_\alpha}, a_{i_{\alpha+1}}) \ldots (a_{i_{\beta-1}}, a_{i_\beta}) = (a_{i_\alpha}, a_{i_\alpha}) \in \Delta .$$

The more the transitive closure $\bar{\varrho} = \varrho \cup \varrho^2 \cup \ldots \cup \varrho^{n_\varrho}$ has a non-empty intersection with Δ. This is a contradiction to Theorem 5,3.

The final goal of the next considerations is to prove Theorem 5,6.

Let ϱ be reducible, $\Pi(\varrho) = A \cup B$, $A \cap B = \emptyset$, A, B non-empty, and $\varrho \subset (A \times A) \cup (B \times A) \cup (B \times B)$.

Denote $\varrho \cap (A \times A) = \varrho_{AA}$, $\varrho \cap (B \times A) = \varrho_{BA}$, $\varrho \cap (B \times B) = \varrho_{BB}$, so that $\varrho = \varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB}$.

Denote further $k(\varrho_{AA}) = k_{11}$, $k(\varrho_{BB}) = k_{22}$, $d(\varrho_{AA}) = d_{11}$, $d(\varrho_{BB}) = d_{22}$.

Lemma 5,9. *If ϱ_{AA} is irreducible, then ϱ_{AA} is a maximal irreducible subrelation of ϱ.*

Proof. Let ϱ_0 be a subrelation of ϱ such that $\varrho_{AA} \subsetneqq \varrho_0$. It is sufficient to show that ϱ_0 is reducible.

Since $\varrho_0 \subset \varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB}$, we have either $\varrho_0 \cap \varrho_{BA} \neq z$ or $\varrho_0 \cap \varrho_{BB} \neq z$ (or both). In any case $\Pi(\varrho_0) \cap B \neq \emptyset$. Denote $\Pi(\varrho_0) \cap B = B_1$. By supposition $\Pi(\varrho_{AA}) \neq$

$\neq \emptyset$. Further $\varrho_{AA} \subset A \times A$ and $\varrho_{AA} \subset \varrho_0$ imply $\Pi(\varrho_{AA}) \subset A \cap \Pi(\varrho_0)$. Hence $\Pi(\varrho_0) \cap A \neq \emptyset$. Denote $\Pi(\varrho_0) \cap A = A_1$, so that $\Pi(\varrho_0) = A_1 \cup B_1$, where $A_1 \cap \cap B_1 = \emptyset$ and A_1, B_1 are non-empty. We have

$$\varrho_0 \subset (A_1 \cup B_1) \times (A_1 \cup B_1) =$$
$$= (A_1 \times A_1) \cup (A_1 \times B_1) \cup (B_1 \times A_1) \cup (B_1 \times B_1).$$

Since $\varrho \cap (A_1 \times B_1) \subset \varrho \cap (A \times B) = z$ and $\varrho_0 \subset \varrho$, we have $\varrho_0 \cap (A_1 \times B_1) = z$, so that $\varrho_0 \subset (A_1 \times B_1) \cup (B_1 \times A_1) \cup (B_1 \times B_1)$. This says that ϱ_0 is reducible.

Theorem 5,5. *With the notations introduced above, we have* $d(\varrho) = [d_{11}, d_{22}]$.

Proof. Note first that $(A \times A)(B \times A) = (A \times A)(B \times B) = (B \times A)(B \times A) = z$. The more $\varrho_{AA}\varrho_{BA} = \varrho_{AA}\varrho_{BB} = \varrho_{BA}\varrho_{BA} = z$.

Consider now the product

$$(5,1) \qquad \varrho^w = (\varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB})^w.$$

This set is a union of summands each of which contains w factors. Such a factor is $\neq z$ if and only if it is of the form

a) either ϱ_{AA}^w,

b) or ϱ_{BB}^w,

c) or $\varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1}$, $w_1 + w_2 = w - 1$.

Hence

$$(5,2) \qquad \varrho^w = \varrho_{AA}^w \cup \varrho_{BB}^w \cup \{\bigcup_{w_1 + w_2 = w - 1} \varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1}\}.$$

The last union is clearly contained in the rectangle $B \times A$ and we immediately get $\varrho^w \cap (A \times A) = \varrho_{AA}^w$, $\varrho^w \cap (B \times B) = \varrho_{BB}^w$.

1. By definition of the numbers $k = k(\varrho)$, $d = d(\varrho)$ we have $\varrho^k = \varrho^{k+d}$. Therefore $\varrho^k \cap (A \times A) = \varrho^{k+d} \cap (A \times A)$, i.e. $\varrho_{AA}^k = \varrho_{AA}^{k+d}$ and analogously $\varrho_{BB}^k = \varrho_{BB}^{k+d}$. This implies $d_{11} \mid d$, $d_{22} \mid d$, and denoting $d^* = [d_{11}, d_{22}]$, we have $d^* \mid d$.

2. Consider again (5,1) and (5,2).

a) Put $w = k^* = k_{11} + k_{22}$. Then we have either $w_1 \geqq k_{11}$ or $w_2 \geqq k_{22}$. (For $w_1 \leqq k_{11} - 1$, $w_2 \leqq k_{22} - 1$ would imply $w_1 + w_2 \leqq w - 2$, contrary to the assumption.) If $w_2 \geqq k_{22}$, then $\varrho_{BB}^{w_2} = \varrho_{BB}^{w_2+d^*}$, $\varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1} = \varrho_{BB}^{w_2+d^*}\varrho_{BA}\varrho_{AA}^{w_1}$. If $w_1 \geqq k_{11}$, then $\varrho_{AA}^{w_1} = \varrho_{AA}^{w_1+d^*}$, $\varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1} = \varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1+d^*}$. Hence

$$\varrho^{k^*} = (\varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB})^{k^*} \subset (\varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB})^{k^*+d^*} = \varrho^{k^*+d^*}.$$

This implies

$$(5,3) \qquad \varrho^{k^*+d^*} \subset \varrho^{k^*+2d^*}.$$

b) Put $w = k^* + 2d^*$. Then analogously as above we have either $w_1 \geqq k_{11} + d^*$ or $w_2 \geqq k_{22} + d^*$.

If $w_1 \geqq k_{11} + d^*$, then $\varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1} = \varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1-d^*}$. If $w_2 \geqq k_{22} + d^*$, $\varrho_{BB}^{w_2}\varrho_{BA}\varrho_{AA}^{w_1} = \varrho_{BB}^{w_2-d^*}\varrho_{BB}\varrho_{AA}^{w_1}$.

Hence

$$(5,4) \qquad \varrho^{k^*+2d^*} \subset \varrho^{k^*+d^*}.$$

The "inequalities" (5,3) and (5,4) imply $\varrho^{k^*+d^*} = \varrho^{k^*+2d^*}$. Hence $d(\varrho) \mid d^*$. Now $d \mid d^*$ and $d^* \mid d$ imply $d = d^*$, which proves our Theorem.

By the way we obtained the following Corollary

Corollary. *With the notations introduced above we have* $k(\varrho) \leqq k_{11} + k_{22} + [d_{11}, d_{22}]$.

Remark. The result of the Corollary is sharp in the sense that there are reducible relations ϱ for which $k(\varrho) = k_{11} + k_{22} + [d_{11}, d_{22}]$ holds. It is easy to derive from this result that for any reducible relation we always have $k(\varrho) < (n-1)^2 + 1$ but it is not possible to get immediately the result of Theorem 4,2.

Theorem 5,6. *If* $\varrho_1, \varrho_2, \ldots, \varrho_s$ *is the set of all maximal irreducible subrelations of* ϱ, *then*

$$d(\varrho) = [d(\varrho_1), d(\varrho_2), \ldots, d(\varrho_s)].$$

Proof. Suppose that ϱ is reducible and write in the sense of the foregoing Theorem $\varrho = \varrho_{AA} \cup \varrho_{BA} \cup \varrho_{BB}$. Then $d(\varrho) = [d(\varrho_{AA}), d(\varrho_{BB})]$.

If ϱ_{AA} does not contain an irreducible subrelation, we have $d(\varrho_{AA}) = 1$ [hence $d(\varrho) = d(\varrho_{BB})$].

If ϱ_{AA} is irreducible, then by Lemma 5,9, ϱ_{AA} is a maximal irreducible subrelation of ϱ, and it is sufficient to examine in the following ϱ_{BB}.

If ϱ_{AA} is reducible, we may write $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, (A_1, A_2 non-empty), and with $\varrho_{A_1A_1} = \varrho_{AA} \cap (A_1 \times A_1)$, $\varrho_{A_2A_1} = \varrho_{AA} \cap (A_2 \times A_1)$, $\varrho_{A_2A_2} = \varrho_{AA} \cap (A_2 \times A_2)$, we have $\varrho_{AA} = \varrho_{A_1A_1} \cup \varrho_{A_2A_1} \cup \varrho_{A_2A_2}$. Hence $d(\varrho_{AA}) = [d(\varrho_{A_1A_1}), d(\varrho_{A_2A_2})]$, therefore $d(\varrho) = [d(\varrho_{A_1A_1}), d(\varrho_{A_2A_2}), d(\varrho_{BB})]$.

Since, by Theorem 5,2, the set of all maximal irreducible subrelations of ϱ is uniquely determined, it is clear that by this proceeding we get in a finite number of steps Theorem 5,6.

6. IRREDUCIBLE RELATIONS

In this section we shall suppose that ϱ is irreducible and without loss of generality we shall suppose that $\Pi(\varrho) = \Omega$. Recall the notation $\Omega \times \Omega = \omega$.

We have seen that in this case $pr_1(\varrho^v) = pr_2(\varrho^v) = \Omega$ for any integer $v \geqq 1$, and every (a_i, a_i) is accessible by ϱ. Moreover by Lemma 1,5 and 1,6 $\Delta \subset \varrho^r$. Note also that $\omega\varrho = \varrho\omega = \omega$.

By Theorem 5,1 we have

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = \omega .$$

The more

$$\varrho \cup \varrho^2 \cup \ldots \cup \varrho^{k-1} \cup \varrho^k \cup \ldots \cup \varrho^{k+d-1} = \omega .$$

Multiply this relation by ϱ^r. On the right hand side we get $\varrho^r\omega = \omega$. On the left hand side we get $\varrho^r \cup \varrho^{r+1} \cup \ldots \cup \varrho^{r+d-1}$ or – what is the same – $\varrho^k \cup \varrho^{k+1} \cup \ldots \ldots \cup \varrho^{k+d-1}$. Therefore

$$\varrho^k \cup \varrho^{k+1} \cup \ldots \cup \varrho^{k+d-1} = \omega . \tag{6,1}$$

Conversely, if (6,1) holds, then by Lemma 1,1 we have

$$\omega = \varrho^k \cup \ldots \cup \varrho^{k+d-1} \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n .$$

We have proved:

Theorem 6,1. *A binary relation ϱ with $\Pi(\varrho) = \Omega$ is irreducible if and only if*

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d = \omega . \tag{6,2}$$

Remark. (Compare with Theorem 1,3.) For any relation we always have

$$\delta \cup \delta^2 \cup \ldots \cup \delta^d \subset \varrho \cup \varrho^2 \cup \ldots \cup \varrho^n .$$

But the sign of equality here may hold even if ϱ is reducible. Take, e.g., the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then ϱ^2 is idempotent,

$$M(\varrho^2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We have $\delta = \varrho^2$, $d = 1$, $k = 2$ and $\delta = \varrho^2 = \varrho \cup \varrho^2 \cup \varrho^3$.

Since $a_i\omega \neq \Omega$, (6,2) implies

(6,3) $$a_i\delta \cup a_i\delta^2 \cup \ldots \cup a_i\delta^d = \Omega$$

for $i = 1, 2, \ldots, n$. By definition of the number d_i we have

(6,4) $$a_i\delta \cup a_i\delta^2 \cup \ldots \cup a_i\delta^{d_i} = \Omega\,.$$

By Lemma 3,3 we have $a_i \in a_i\delta^{d_i}$, $a_j \in a_j\delta^{d_j}$. By (6,4) there are integers α, β such that $a_i \in a_j\delta^\alpha$, $a_j \in a_i\delta^\beta$. Hence

$$a_i \in a_j\delta^\alpha \subset a_i\delta^{\alpha+\beta}\,, \quad a_j \in a_i\delta^\beta \subset a_j\delta^{\alpha+\beta}\,,$$
$$a_i \in a_j\delta^\alpha \subset a_j\delta^{\alpha+d_j} \subset a_i\delta^{\alpha+\beta+d_j}\,,$$
$$a_j \in a_i\delta^\beta \subset a_i\delta^{\beta+d_i} \subset a_j\delta^{\alpha+\beta+d_i}\,.$$

This implies

$$d_i \mid \alpha + \beta\,, \qquad d_j \mid \alpha + \beta\,,$$

and

$$d_i \mid \alpha + \beta + d_j\,,\ d_j \mid \alpha + \beta + d_i\,.$$

Hence $d_i \mid d_j$ and $d_j \mid d_i$, i.e. $d_i = d_j$. Since moreover $d = [d_1, d_2, \ldots, d_n]$, we have:

Theorem 6,2. *For an irreducible relation ϱ we have*

$$d_1(\varrho) = d_2(\varrho) = \ldots = d_n(\varrho) = d(\varrho)\,.$$

Corollary. *If ϱ is irreducible, we have $\Delta \subset \delta^d$, while $\Delta \cap (\delta \cup \delta^2 \cup \ldots \cup \delta^{d-1}) = z$.*

Lemma 6,1. *If ϱ is irreducible, then $\varrho \subset \delta$, $\varrho^2 \subset \delta^2, \ldots, \varrho^d \subset \delta^d$.*

Proof. Since $\Delta \subset \delta^d$, we have $\varrho = \varrho\Delta \subset \varrho\,.\,\varrho^r = \varrho^{r+1} = \delta$. This implies $\varrho^2 \subset \varrho^{2(r+1)} = \delta^2, \ldots, \varrho^d \subset \varrho^{d(r+1)} = \delta^d$.

Theorem 6,3. *If ϱ is irreducible, the relations $\delta, \delta^2, \ldots, \delta^d$ are pairwise disjoint.*

Proof. Suppose for an indirect proof that there is a couple δ^i, δ^j, $1 \leqq i < j \leqq d$ such that $\delta^i \cap \delta^j \neq z$.

Consider the relation

$$\tau = \bigcup_{\alpha<\beta} (\delta^\alpha \cap \delta^\beta)\,, \quad \alpha = 1, 2, \ldots, d-1\,; \quad \beta = 2, 3, \ldots, d\,.$$

By supposition $\tau \neq z$. Clearly

$$\tau \subset \delta \cup \delta^2 \cup \ldots \cup \delta^{d-1},$$

so that $\tau \cap \Delta = z$. Further (since $\delta^d \cap \delta^{d+1} = \delta^d \cap \delta^1$)

$$\delta\tau = \delta \bigcup_{\alpha<\beta} [\delta^\alpha \cap \delta^\beta] \subset \bigcup_{\alpha<\beta} [\delta^{\alpha+1} \cap \delta^{\beta+1}] \subset \tau .$$

This implies

$$\tau \supset \delta\tau \supset \delta^2\tau \supset \ldots \supset \delta^d\tau \supset \Delta\tau = \tau .$$

Therefore $\tau = \delta\tau = \delta^2\tau = \ldots \delta^d\tau$ and

$$\tau = \delta\tau \cup \delta^2\tau \cup \ldots \cup \delta^d\tau = (\delta \cup \ldots \cup \delta^d)\,\tau = \omega\tau .$$

Since $\tau \neq z$, there is a couple $(a_i, a_j) \in \tau$. Hence

$$\tau = \omega\tau \supset \omega(a_i, a_j) \ni (a_j, a_i)(a_i, a_j) = (a_j, a_j) .$$

This is a contradiction to $\Delta \cap \tau = z$, and this contradiction completes the proof of our Theorem.

Corollary. *If ϱ is irreducible and $a_i \in \Pi(\varrho)$, then the sets $a_i\delta, a_i\delta^2, \ldots, a_i\delta^d$, are pairwise disjoint.*

Theorem 6,3 and Lemma 6,1 imply:

Theorem 6,4. *If ϱ is irreducible, then $\varrho, \varrho^2, \ldots, \varrho^d$ are pairwise disjoint. More generally: Any d consecutive powers $\varrho^l, \varrho^{l+1}, \ldots, \varrho^{l+d-1}$ are pairwise disjoint.*

Multiply $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = \omega$ by ϱ^r. We have $\varrho^{r+1} \cup \varrho^{r+2} \cup \ldots \cup \varrho^{r+n} = \omega$, i.e. $\delta \cup \delta^2 \cup \ldots \cup \delta^n = \omega$. Comparing with $\delta \cup \ldots \cup \delta^d = \omega$, in which no summand can be deleted, we have:

Theorem 6,5. *For an irreducible relation with* $\operatorname{card} \Pi(\varrho) = n$ *we always have* $1 \leqq d(\varrho) \leqq n$.

Remark 1. It can be proved (see Corollary 2 to Theorem 7,2) that there is an irreducible relation ϱ with $d(\varrho) = l$ for any l satisfying $1 \leqq l \leqq n$.

Remark 2. Theorem 6,5 follows also from Theorem 6,2 and Theorem 4,1, taking account of the fact that for an irreducible ϱ all couples (a_i, a_i) are accessible by ϱ.

Remark 3. By the argument used in Lemma 6,1 we have $\varrho^{\alpha d+\beta} \subset \delta^\beta$ for any integers $\alpha \geqq 0$, $\beta \geqq 1$. This implies

$$\begin{aligned}
&\varrho \cup \varrho^{d+1} \cup \varrho^{2d+1} \cup \ldots \subset \delta\,,\\
&\varrho^2 \cup \varrho^{d+2} \cup \varrho^{2d+2} \cup \ldots \subset \delta^2\,,\\
&\vdots\\
&\varrho^d \cup \varrho^{2d} \cup \varrho^{3d} \cup \ldots \subset \delta^d\,.
\end{aligned}$$

This can be sharpened, using the fact that $\bar{\varrho} = \varrho \cup \ldots \cup \varrho^n = \delta \cup \ldots \cup \delta^d$ in the following way: Write $n = \alpha d + s$, where $\alpha \geqq 1$ is an integer and $0 \leqq s < d$. We then have

$$\begin{aligned}
(6,5)\qquad &\varrho \cup \varrho^{d+1} \cup \varrho^{2d+1} \cup \ldots \cup \varrho^{(\alpha-1)d+1} \cup \varrho^{\alpha d+1} = \delta\,,\\
&\vdots\\
&\varrho^s \cup \varrho^{d+s} \cup \varrho^{2d+s} \cup \ldots \cup \varrho^{(\alpha-1)d+s} \cup \varrho^{\alpha d+s} = \delta^s\,,\\
&\varrho^{s+1} \cup \varrho^{d+s+1} \cup \varrho^{2d+s+1} \cup \ldots \cup \varrho^{(\alpha-1)d+s+1} = \delta^{s+1}\,,\\
&\vdots\\
&\varrho^d \cup \varrho^{2d} \cup \varrho^{3d} \cup \ldots \cup \varrho^{\alpha d} = \delta^d\,.
\end{aligned}$$

Hereby, if $s = 0$, then ϱ^0 means ϱ^d and δ^0 means δ^d.

It follows from the Corollary to Theorem 6,2 that Δ may have a non-empty intersection only with some of the powers $\varrho^d, \varrho^{2d}, \ldots, \varrho^{(\beta-1)d}$, where $\beta d = r$, and Δ is contained in $\varrho^r, \varrho^{r+d}, \varrho^{r+2d}, \ldots$ Since $r = \beta d$, $r + d = (\beta + 1)\,d$, $r + 2d = (\beta + 2)\,d, \ldots$, the integer d is clearly the greatest common divisor of all exponents just mentioned. We have proved:

Theorem 6,6. *If ϱ is irreducible, then $d(\varrho)$ is the greatest common divisor of all integers $\alpha \geqq 1$ for which $\Delta \cap \varrho^\alpha \neq z$.*

We shall next deal with *proper irreducible subrelations* of a given irreducible relation ϱ.

Theorem 6,7. *If an irreducible relation ϱ contains an irreducible subrelation σ, then $d(\varrho) \mid d(\sigma)$.*

Proof. Consider the sequences

$$\varrho, \varrho^2, \ldots, \varrho^{k(\varrho)-1} \mid \varrho^{k(\varrho)}, \ldots, \varrho^{k(\varrho)+d(\varrho)-1} \mid \varrho^{k(\varrho)}, \ldots,$$
$$\sigma, \sigma^2, \ldots, \sigma^{k(\sigma)-1} \mid \sigma^{k(\sigma)}, \ldots, \sigma^{k(\sigma)+d(\sigma)-1} \mid \sigma^{k(\sigma)}, \ldots$$

Put $l = \max\,[k(\varrho), k(\sigma)]$. By supposition

$$\sigma^l \subset \varrho^l,\ \sigma^{l+1} \subset \varrho^{l+1}, \ldots, \sigma^{l+d(\varrho)-1} \subset \varrho^{l+d(\varrho)-1}, \ldots$$

Since $\varrho^l, \varrho^{l+1}, \ldots, \varrho^{l+d(\varrho)-1}$ are pairwise disjoint, we conclude that $\sigma^l, \sigma^{l+1}, \ldots$ $\ldots, \sigma^{l+d(\varrho)-1}$ are pairwise disjoint, so that $d(\sigma) \geqq d(\varrho)$. Now $\sigma^l = \sigma^{l+d(\sigma)} \subset \varrho^{l+d(\sigma)} \cap$ $\cap \varrho^l$ implies $\varrho^{l+d(\sigma)} = \varrho^l$, hence $d(\varrho) \mid d(\sigma)$, q.e.d.

The one point relation $\sigma = \{(a_i, a_i)\}$ is irreducible with $d(\sigma) = 1$. This implies:

Corollary. *If an irreducible relation ϱ contains at least one element (a_i, a_i) of the diagonal Δ, then $d(\varrho) = 1$.*

Definition. Let ϱ be irreducible. *An irreducible subrelation ϱ_0, $z \neq \varrho_0 \subset \varrho$, is called minimal* if for any irreducible σ, $z \neq \sigma \subset \varrho_0$, we have $\sigma = \varrho_0$.

Definition. A relation of the form

$$\varrho_0 = \{(a_{i_1}, a_{i_2}), (a_{i_2}, a_{i_3}), \ldots, (a_{i_l}, a_{i_1})\}$$

is called a *cyclic relation of the length l.*

If all elements in $\{i_1, i_2, \ldots, i_l\}$ are different one from the other, ϱ_0 is called *an elementary cycle.*

Clearly the length of any elementary cycle is $\leqq n$.

Lemma 6,2. *Let ϱ be irreducible. Then $\varrho_0 \subset \varrho$ is a minimal irreducible subrelation of ϱ if and only if ϱ_0 is an elementary cycle.*

Proof. a) Suppose that ϱ_0 is a minimal irreducible subrelation of ϱ. Take any $a_i \in \Pi(\varrho_0)$. Denote by h_i the least integer $h_i \geqq 1$ such that $a_i \in a_i\varrho_0^{h_i}$, i.e. $(a_i, a_i) \in \varrho_0^{h_i}$. Then there exist elements $x_2, \ldots, x_{h_i-1}, x_{h_i} \in \Pi(\varrho_0)$ such that

$$(a_i, x_2)(x_2, x_3)(x_3, x_4) \ldots (x_{h_i}, a_i) = (a_i, a_i),$$

and the elements $a_i, x_2, x_3, \ldots, x_{h_i}$ are all different one from the other.

The relation

$$\varrho_{00} = \{(a_i, x_2), (x_2, x_3), \ldots, (x_{h_i}, a_i)\}$$

is clearly an elementary cyclic relation. By Lemma 5,8 ϱ_{00} is irreducible. The inclusion $\varrho_{00} \subset \varrho_0$, with respect to the minimality of ϱ_0, implies $\varrho_{00} = \varrho_0$.

b) Let be conversely $\varrho_0 = \{(y_1, y_2), (y_2, y_3), \ldots, (y_l, y_1)\}$, $y_i \in \Omega$, any elementary cyclic relation of length l. Let ϱ_{00} be a proper subset of ϱ_0. Suppose, e.g., $(y_l, y_1) \notin$ $\notin \varrho_{00}$. Then $\varrho_{00}^l = z$ so that ϱ_{00} cannot be irreducible. For ϱ_{00}^l is a union of products of length l with factors chosen from the set $\{(y_1, y_2), \ldots, (y_{l-1}, y_l)\}$. Every such product contains at least one factor twice and one verifies easily that this implies that each summand of ϱ_{00}^l is equal to z. This proves Lemma 6.2.

Our next goal is to prove Theorem 6,8.

Let ϱ be irreducible, $\Pi(\varrho) = \Omega$, and h_i the least integer such that $a_i \in a_i\varrho^{h_i}$. It follows from the preceding proof that the set of all integers $\{h_1, h_2, \ldots, h_n\}$ is exactly the set of lengths of all elementary cycles contained in ϱ.

If l is the length of any cycle contained in ϱ, then by Theorem 6,7 $d(\varrho) \mid l$. In particular $d(\varrho) \mid d_0$, where $d_0 = g.c.d(h_1, h_2, \ldots, h_n)$. We shall show that $d(\varrho) = d_0$.

Recall that (a_i, a_i) $(i = 1, 2, \ldots, n)$ is contained in $\varrho^{\lambda d}$ for any integer λ such that $d\lambda \geqq k(\varrho)$. Take a fixed λ such that $l = \lambda d > \max(k(\varrho), n + 1)$. Then there exist elements $x_1^{(i)}, x_2^{(i)}, \ldots, x_{l-1}^{(i)}$ all $\in \Omega$ such that

$$(a_i, x_1^{(i)})(x_1^{(i)}, x_2^{(i)})(x_2^{(i)}, x_3^{(i)}) \ldots (x_{l-1}^{(i)}, a_i) = (a_i, a_i),$$

and each of the couples to the left is contained in ϱ. Since $l - 1 > n$, there exist necessarily two integers $\alpha, \beta \geqq 1$ such that $x_\alpha^{(i)} = x_{\alpha+\beta}^{(i)}$. The relation

$$\varrho_{00} = \{(a_i, x_1^{(i)}), (x_1^{(i)}, x_2^{(i)}), \ldots, (x_{l-1}^{(i)}, a_i)\}$$

can be decomposed into two relations of lengths $l - \beta$ and β, $\varrho_{00} = \varrho_1 \cup \varrho_2$, where

$$\varrho_1 = \{(a_i, x_1^{(i)}), \ldots, (x_{\alpha-1}^{(i)}, x_\alpha^{(i)}), (x_{\alpha+\beta}^{(i)}, x_{\alpha+\beta+1}^{(i)}), \ldots, (x_{l-1}^{(i)}, a_i)\},$$
$$\varrho_2 = \{(x_\alpha^{(i)}, x_{\alpha+1}^{(i)}), \ldots, (x_{\alpha+\beta-1}^{(i)}, x_{\alpha+\beta}^{(i)})\}.$$

If ϱ_1 or ϱ_2 (or both) is not an elementary cycle we can decompose the appropriate one (or both) further, a.s.o. We finally obtain

$$l = \lambda d = c_1^{(\lambda)} h_1 + c_2^{(\lambda)} h_2 + \ldots + c_n^{(\lambda)} h_n,$$

where $c_j^{(\lambda)}$ are non-negative integers. This implies that $d_0 \mid \lambda d$ for any integer λ which is larger than some λ_0. Hence $d_0 \mid d$. This, together with $d \mid d_0$, implies $d = d_0$. We have proved:

Theorem 6,8. *If ϱ is irreducible and h_i is the least integer such that $a_i \in a_i\varrho^{h_i}$, then $d(\varrho) = g.c.d(h_1, h_2, \ldots, h_n)$.*

Remark. In Theorem 4,3 we have proved that for any binary relation ϱ with card $\Pi(\varrho) = n$, we have $k(\varrho) \leqq (n-1)^2 + 1$. If ϱ is irreducible and card $G(\varrho) = d$ it is proved in a forthcomming paper ([31]) [by means of the formulae (6,5)] that $k(\varrho) \leqq (n-d)^2/d + d$. And this is the best possible result.

7. POWERS OF AN IRREDUCIBLE RELATION

We shall now study the behaviour of ϱ^v for various $v > 1$.

In all of this section we suppose again that $\Pi(\varrho) = \Omega$.

Note first: If ϱ is irreducible, then so is δ and $\Pi(\delta) = \Pi(\varrho)$. [For by Theorem 6,1 $\delta \cup \delta^2 \cup \ldots \cup \delta^d = \omega$ and by Lemma 5,5 $\Pi(\varrho) = \Pi(\varrho^{r+1}) = \Pi(\delta)$.]

In general the converse statement need not hold. Let, e.g., ϱ be the binary relation with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$M(\varrho^2) = M(\delta) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here ϱ is reducible with $\Pi(\varrho) = \{a_1, a_2, a_3\}$ while δ is irreducible and $\Pi(\delta) = \{a_3\}$. Hence δ may be irreducible even if ϱ is reducible.

But if we suppose that δ is irreducible and $\Pi(\delta) = \Pi(\varrho)$, then ϱ is also irreducible. For, suppose that ϱ is reducible $\Pi(\varrho) = \Omega$ and $\varrho \subset (A \times A) \cup (B \times A) \cup (B \times B) = \tau$, where A, B are non-empty, $A \cap B = \emptyset$, $A \cup B = \Omega$. Since $\tau = \tau^2$, we have $\delta = \varrho^{r+1} \subset \tau$. Since $\Pi(\delta) = \Pi(\varrho)$, we have $A \cap \Pi(\delta) = A \cap \Pi(\varrho) \doteq A \neq \emptyset$, $B \cap \Pi(\delta) = B \neq \emptyset$, so that δ is reducible.

Hence: If $\Pi(\varrho) = \Pi(\delta)$, then ϱ is irreducible if and only if δ is irreducible.

If ϱ is irreducible, then some power $\varrho^v (v > 1)$ may be reducible.

We first prove:

Lemma 7,1. *If ϱ is irreducible and $d(\varrho) \geqq 1$, then there exist d pairwise disjoint non-empty subsets $A_1, A_2, \ldots, A_d$ such that $\Omega = A_1 \cup \ldots \cup A_d$ and*

$$\delta = (A_1 \times A_2) \cup (A_2 \times A_3) \cup \ldots \cup (A_d \times A_1).$$

More generally: For every $u \geqq 1$

$$\delta^u = (A_1 \times A_{1+u}) \cup (A_2 \times A_{2+u}) \cup \ldots \cup (A_d \times A_{d+u}),$$

the subscripts taken (mod d).

Proof. Denote $A_1 = a_1\delta$, $A_2 = a_1\delta^2, \ldots, A_d = a_1\delta^d$. By Corollary to Theorem 6,3 we have $\Omega = A_1 \cup \ldots \cup A_d$ and the A_i are pairwise disjoint.

Let be $(a_i, a_j) \in \delta$, i.e. $a_j \in a_i\delta$. Now there is an integer s, $1 \leqq s \leqq d$, such that $a_i \in a_1\delta^s = A_s$. Hence $a_j \in (a_1\delta^s)\,\delta = A_{s+1}$ so that $(a_i, a_j) \in A_s \times A_{s+1}$ [the subscripts taken (mod d)]. This proves

(7,1) $$\delta \subset (A_1 \times A_2) \cup (A_2 \times A_3) \cup \ldots \cup (A_d \times A_1).$$

For further purposes remark that $(A_i \times A_j)(A_l \times A_m)$ is z if $j \neq l$, and $(A_i \times A_m)$ if $j = l$.

Let now u be any integer $\geqq 1$ and consider the power

$$(7,2) \qquad \delta^u \subset [(A_1 \times A_2) \cup (A_2 \times A_3) \cup \ldots \cup (A_d \times A_1)]^u .$$

The right hand side of (7,2) is a union of products each of which contains u factors. Such a factor is z unless the product is of the form

$$(A_i \times A_{i+1})(A_{i+1} \times A_{i+2}) \ldots (A_{i+u-1} \times A_{i+u}),$$

in which case it is equal to $A_i \times A_{i+u}$. Hence we have

$$(7,3) \qquad \begin{aligned} \delta^2 &\subset (A_1 \times A_3) \cup (A_2 \times A_4) \cup \ldots \cup (A_d \times A_2), \\ \delta^3 &\subset (A_1 \times A_4) \cup (A_2 \times A_5) \cup \ldots \cup (A_d \times A_3), \\ &\vdots \\ \delta^d &\subset (A_1 \times A_1) \cup (A_2 \times A_2) \cup \ldots \cup (A_d \times A_d) . \end{aligned}$$

Now since $\delta \cup \ldots \cup \delta^d = \omega$, and all summands $A_i \times A_k$ in (7,1) and (7,3) are disjoint, we have for every integer u, $1 \leqq u \leqq d$,

$$(7,4) \qquad \delta^u = (A_1 \times A_{1+u}) \cup (A_2 \times A_{2+u}) \cup \ldots \cup (A_d \times A_{d+u}) .$$

This proves Lemma 7,1 for $1 \leqq u \leqq d$. For $u > d$ the same holds since the sets A_l, where the subscripts are taken (mod d), periodically repeat.

Corollary. *Let be $\Omega = \{a_1, a_2, \ldots, a_n\}$ and l any integer, $1 \leqq l \leqq n$. Then there exists an irreducible relation σ such that $d(\sigma) = l$.*

Proof. Write Ω in an arbitrary manner as a union of l pairwise disjoint non-empty subsets $\Omega = A_1 \cup \ldots \cup A_l$ and put $\sigma = (A_1 \times A_2) \cup (A_2 \times A_3) \cup \ldots \cup (A_l \times A_1)$. Then analogously as above

$$\begin{aligned} \sigma^2 &= (A_1 \times A_3) \cup \ldots \cup (A_l \times A_2), \\ &\vdots \\ \sigma^l &= (A_1 \times A_1) \cup \ldots \cup (A_l \times A_l) . \end{aligned}$$

This implies that $\sigma \cup \sigma^2 \cup \ldots \cup \sigma^l = \omega$, hence σ is irreducible. Moreover $\sigma^{l+1} = \sigma$ and $\sigma^i \neq \sigma^j$ for $i \neq j$, so that $d(\sigma) = l$, q.e.d.

Lemma 7,2. *Suppose that ϱ is irreducible. Denote $u_0 = (d, u)$. Then there are u_0 pairwise disjoint non-empty subsets $T_1, T_2, \ldots, T_{u_0}$, such that $T_1 \cup T_2 \cup \ldots \cup T_{u_0} = \Omega$ and*

$$\delta^u \cup \delta^{2u} \cup \ldots \cup \delta^{du} = (T_1 \times T_1) \cup (T_2 \times T_2) \cup \ldots \cup (T_{u_0} \times T_{u_0}) .$$

Proof. Denote $d_0 = d/u_0$ and consider the relation

$$\sigma = \delta^u \cup \delta^{2u} \cup \ldots \cup \delta^{du} = \delta^u \cup \delta^{2u} \cup \ldots \cup \delta^{d_0u} = \delta^{u_0} \cup \delta^{2u_0} \cup \ldots \cup \delta^{d_0u_0}.$$

By (7,4) we have

$$\sigma = \bigcup_{j=1}^{d_0} \bigcup_{i=1}^{d} (A_i \times A_{i+ju_0}) = \bigcup_{i=1}^{d} [A_i \times \bigcup_{j=1}^{d_0} A_{i+ju_0}].$$

Denote

$$T_i = A_{i+u_0} \cup A_{i+2u_0} \cup \ldots \cup A_{i+d_0u_0} = A_i \cup A_{i+u_0} \cup \ldots \cup A_{i+(d_0-1)u_0}.$$

Clearly $T_i = T_{i+u_0} = T_{i+2u_0} = \ldots$, and we have

$$\sigma = \bigcup_{i=1}^{d} (A_i \times T_i) = \bigcup_{i=1}^{u_0} [(A_i \cup A_{i+u_0} \cup \ldots \cup A_{i+(d_0-1)u_0}) \times T_i] = \bigcup_{i=1}^{u_0} (T_i \times T_i).$$

This proves our Lemma.

Lemma 7,2 implies in particular: $\delta^u \subset (T_1 \times T_1) \cup \ldots \cup (T_{u_0} \times T_{u_0})$. Denote $\delta^u \cap (T_i \times T_i) = \delta_i$. We then have

(7,5) $$\delta^u = \delta_1 \cup \delta_2 \cup \ldots \cup \delta_{u_0}.$$

Since $\Pi(\delta^u) = \Pi(\varrho) = \Omega$, we have $\delta_i \neq z$ and $\Pi(\delta_i) = T_i$. Since $\delta_i\delta_j = z$ for $i \neq j$, (7,5) implies

(7,6) $$\begin{aligned} \delta^{2u} &= \delta_1^2 \cup \delta_2^2 \cup \ldots \cup \delta_{u_0}^2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta^{du} &= \delta_1^d \cup \delta_2^d \cup \ldots \cup \delta_{u_0}^d. \end{aligned}$$

Summing (7,5) and (7,6) we have

$$\delta^u \cup \delta^{2u} \cup \ldots \cup \delta^{du} = \bigcup_{i=1}^{u_0} \{\delta_i \cup \delta_i^2 \cup \ldots \cup \delta_i^d\}.$$

Comparing with Lemma 7,2 we have $\delta_i \cup \delta_i^2 \cup \ldots \cup \delta_i^d = T_i \times T_i$. This says that every δ_i is irreducible.

Now $\varrho^u \subset \delta^u$ implies $\varrho^u \subset (T_1 \times T_1) \cup \ldots \cup (T_{u_0} \times T_{u_0})$. Denote $\varrho^u \cap (T_i \times T_i) = \varrho_i$. Since again $\Pi(\varrho^u) = \Omega$, we have $\varrho_i \neq z$, $\Pi(\varrho_i) = T_i$ and

$$\varrho^u = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_{u_0}.$$

Since again $\varrho_i\varrho_j = z$ for $i \neq j$, we have [by raising the last equality to the $(r+1)$-th power]:

$$\delta^u = \varrho^{u(r+1)} = \varrho_1^{r+1} \cup \varrho_2^{r+1} \cup \ldots \cup \varrho_{u_0}^{r+1}.$$

Comparing with (7,5) we have $\varrho_i \subset \delta_i$ and $\varrho_i^{r+1} = \delta_i$. Since $\Pi(\varrho_i) = \Pi(\delta_i)$ and δ_i is irreducible, ϱ_i is irreducible. We have proved:

Theorem 7,1. *Let ϱ be irreducible. Denote $(d, u) = u_0$. Then ϱ^u is completely reducible into u_0 disjoint irreducible relations.*

Suppose now that $u = d$. We then have $u_0 = d$, $d_0 = 1$, $\sigma = \delta^d$ and

$$\varrho^d \subset \delta^d = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \ldots \cup (A_d \times A_d). \tag{7,7}$$

Further

$$\varrho_i = \varrho^d \cap (A_i \times A_i) \subset \delta_i = \delta^d \cap (A_i \times A_i) = A_i \times A_i = \varrho_i^{r+1}.$$

The irreducible relation ϱ_i has the *special property* that ϱ_i^{r+1} is a square.

Definition. An irreducible relation ϱ is called *primitive* if there is an integer $w \geqq 1$ such that ϱ^w is a square [namely $\Pi(\varrho) \times \Pi(\varrho)$].

Thus $\varrho^d = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_d$, where each ϱ_i is a primitive relation.

If $u < d$, we have

$$\varrho^u = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_{u_0}, \quad u_0 = (d, u) < d. \tag{7,8}$$

We show that not all ϱ_i at the right side of (7,8) can be primitive. Suppose the contrary. Then there is a w_0 such that for $w \geqq w_0$ each ϱ_i^w in (7,8) is a square. Put for w a multiple of d, say $w = w_0 d$. Then (7,8) implies

$$\varrho^{uw_0d} = \varrho_1^{w_0d} \cup \ldots \cup \varrho_{u_0}^{w_0d}.$$

If moreover w_0 is such that $uw_0d \geqq r$, we have on the left hand side a union of d disjoint squares while on the right hand side we have only u_0 squares. This constitutes a contradiction. We have proved:

Theorem 7,2. *Let ϱ be irreducible. Then the number $d(\varrho)$ is the least positive integer u such that ϱ^u is completely reducible into a union of disjoint primitive relations. The number of these primitive relations is exactly d.*

We return for a while to (7,7). It is immediately to seen that δ^d is a symmetric relation. Also $\Delta \subset \delta^d$, hence δ^d is reflexive. Finally $\delta^d = \varrho^r$ is transitive. We have proved:

Theorem 7,3. *To any irreducible relation ϱ there is an integer $s > 0$ such that ϱ^s is an equivalence relation on $\Pi(\varrho)$. The number $s = r(\varrho)$ is the least such integer.*

Remark. This assertion is not necessarily true for any relation. E.g., if ϱ is the relation with $M(\varrho) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, it is clear that no power of ϱ is an equivalence relation.

Let ϱ be any relation. It is quite natural to ask about the *primitive subrelations* of ϱ (if such exist).

Lemma 8,1. *If ϱ_1, ϱ_2 are two primitive subrelations of ϱ and $\varrho_1 \cap \varrho_2 \neq z$, then $\varrho_1 \cup \varrho_2$ is a primitive subrelation of ϱ.*

Proof. Denote $\Pi(\varrho_1) \cup \Pi(\varrho_2) = \Omega_0$. First, since a primitive relation is irreducible, $\varrho_1 \cup \varrho_2$ is irreducible and the transitive closure $\overline{\varrho_1 \cup \varrho_2}$ is equal to $\Omega_0 \times \Omega_0$. (See Lemma 5,6.)

Since ϱ_1, ϱ_2 are primitive, there exist two positive integers w_1, w_2 such that $\varrho_1^{w_1} = \Pi(\varrho_1) \times \Pi(\varrho_1)$, $\varrho^{w_2} = \Pi(\varrho_2) \times \Pi(\varrho_2)$. Denote $w = \max(w_1, w_2)$.

By supposition there is at least one couple $(x_0, y_0) \in \varrho_1 \cap \varrho_2$. Let (a_i, a_j) be any element $\in \Omega_0 \times \Omega_0$. Suppose, e.g., $a_i \in \Pi(\varrho_1)$, $a_j \in \Pi(\varrho_2)$. Write $(a_i, a_j) = (a_i, x_0) \cdot (x_0, y_0)(y_0, a_j)$. Since $x_0 \in \Pi(\varrho_1)$, we have $(a_i, x_0) \in \varrho_1^w$ and since $y_0 \in \Pi(\varrho_2)$, we have $(y_0, a_j) \in \varrho_2^w$. Further $(x_0, y_0) \subset \varrho_1$, so that $(a_i, a_j) \in \varrho_1^w \varrho_1 \varrho_2^w = \varrho_1^w \varrho_2^w$ (since $\varrho_1^{w+1} = \varrho_1^w$).

If $a_i \in \Pi(\varrho_2)$, $a_j \in \Pi(\varrho_1)$, we have $(a_i, a_j) \in \varrho_2^w \varrho_1^w$. If a_i and $a_j \in \Pi(\varrho_1)$, we have $(a_i, a_j) \in \varrho_1^w$. If a_i and $a_j \in \Pi(\varrho_2)$, we have $(a_i, a_j) \in \varrho_2^w$.

Hence:

$$\Omega_0 \times \Omega_0 \subset \varrho_1^w \cup \varrho_2^w \cup \varrho_1^w \varrho_2^w \cup \varrho_2^w \varrho_1^w =$$
$$= \varrho_1^{2w} \cup \varrho_2^{2w} \cup \varrho_1^w \varrho_2^w \cup \varrho_2^w \varrho_1^w \subset (\varrho_1 \cup \varrho_2)^{2w}.$$

Therefore $(\varrho_1 \cup \varrho_2)^{2w} = \Omega_0 \times \Omega_0$. This says that $\varrho_1 \cup \varrho_2$ is primitive.

It is clear what is meant by a *maximal primitive subrelation* of a given relation ϱ.

Analogously as in Lemma 5,7 and Theorem 5,2 we can prove:

Lemma 8,2. *The intersection of two different maximal primitive subrelations of ϱ is the zero relation z.*

Lemma 8,3. *The set of all maximal primitive subrelations contained in a given relation ϱ is uniquely determined.*

A characterization of the set of all maximal primitive subrelations of a given ϱ is possible on the base of the following

Lemma 8,4. *An irreducible relation ϱ contains a primitive subrelation if and only if ϱ is itself primitive.*

Proof. If σ is primitive and $\sigma \subset \varrho$, then Theorem 6,7 implies $d(\varrho) \mid d(\sigma)$ and since $d(\sigma) = 1$, we have $d(\varrho) = 1$, i.e. ϱ is primitive. (For then by Theorem 6,1 $\varrho^{r+1} = \delta = \Pi(\varrho) \times \Pi(\varrho)$.)

This implies:

Theorem 8,1. *To find all maximal primitive subrelations of a given relation ϱ we proceed as follows. We find all maximal irreducible subrelations of ϱ and delete from them all which are not primitive. The remaining one's (if such exist) give exactly all maximal primitive subrelations of ϱ.*

The notion of a primitive relation is useful also for the following generalization of Theorem 7,3.

Theorem 8,2. *Let ϱ be any relation, $\Pi(\varrho) = \Omega$. Then some power of ϱ is an equivalence relation on Ω if and only if ϱ^d is either primitive or completely reducible into primitive relations.*

Proof. If ϱ^l is an equivalence relation on Ω, every (a_i, a_i) is accessible, so that $\Delta \subset \varrho \cup \ldots \cup \varrho^n$. Since ϱ^l is transitive, we have by Lemma 1,6 $\varrho^l = \varrho^r$ and by Lemma 1,5 $\Delta \subset \varrho^r$. This implies $\Delta\varrho = \varrho \subset \varrho^{r+1} = \delta$, $\varrho^2 \subset \delta^2, \ldots, \varrho^d \subset \delta^d = \varrho^r$.

Since $\delta^d = \varrho^r$ is an equivalence relation, we necessarily have

$$\delta^d = (A_1 \times A_1) \cup \ldots \cup (A_s \times A_s), \tag{8,1}$$

where $s \geqq 1$, $\bigcup_{i=1}^{s} A_i = \Omega$, $A_i \cap A_j = \emptyset$ (for $i \neq j$). Therefore

$$\varrho^d \subset (A_1 \times A_1) \cup \ldots \cup (A_s \times A_s).$$

Denoting $\varrho_i = \varrho^d \cap (A_i \times A_i)$, we have $\varrho^d = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_s$. This implies $\delta^d = \varrho^{d(r+1)} = \varrho_1^{r+1} \cup \varrho_2^{r+1} \cup \ldots \cup \varrho_s^{r+1}$. Comparing with (8,1) we have $\varrho_i^{r+1} = A_i \times A_i$, so that ϱ_i is primitive. Hence ϱ^d is either primitive or a union of disjoint primitive relations.

Conversely: If ϱ^d is completely reducible into primitive relations, then clearly a sufficiently high power of ϱ is an equivalence relation. This proves our Theorem.

9. SOME SPECIAL CASES

A relation is called *symmetric* if $\varrho = \bar{\varrho}^1$. We first give sharp estimations for $k(\varrho)$ and $d(\varrho)$ in the case of a symmetric relation.

Lemma 9,1. *For any relation ϱ we always have $\varrho \subset \varrho\bar{\varrho}^1\varrho$.*

Proof. If $(a_i, a_j) \in \varrho$, then $(a_i, a_j) = (a_i, a_j)(a_j a_i)(a_i, a_j) \subset \varrho\bar{\varrho}^1\varrho$.

Corollary. *For any symmetric relation ϱ we have $\varrho \subset \varrho^3$.*

The last Corollary implies

$$(9,1) \qquad \varrho \subset \varrho^3 \subset \varrho^5 \subset \dots$$

$$(9,2) \qquad \varrho^2 \subset \varrho^4 \subset \varrho^6 \subset \dots$$

Hence for the transitive closure of ϱ we have

$$\bar{\varrho} = \varrho \cup \varrho^2 \cup \dots \cup \varrho^n = \varrho^{n-1} \cup \varrho^n .$$

The finiteness implies that the chains (9,1) and (9,2) have only a finite number of different elements, so that there are integers l_1 and l_2 such that

$$\varrho \subset \varrho^3 \subset \dots \subset \varrho^{l_1 - 2} \subsetneqq \varrho^{l_1} = \varrho^{l_1 + 2} = \dots$$
$$\varrho^2 \subset \varrho^4 \subset \dots \subset \varrho^{l_2 - 2} \subsetneqq \varrho^{l_2} = \varrho^{l_2 + 2} = \dots$$

Therefore the group $G(\varrho)$ contains at most two different elements, so that either $G(\varrho) = \{\varrho^k, \varrho^{k+1}\}$ or $G(\varrho) = \{\varrho^k\}$.

We have proved:

Theorem 9,1. *For any symmetric relation ϱ on Ω we always have $d(\varrho) \leqq 2$.*

If a symmetric relation is reducible, it is clearly completely reducible into a union of irreducible summands. Hence we shall first deal with the case of an *irreducible* relation and suppose $\Pi(\varrho) = \Omega$.

In the next Lemma we treat simultaneously both cases $d(\varrho) = 1$ and $d(\varrho) = 2$.

Lemma 9,2. *If ϱ is symmetric and irreducible and $n \geqq 3$, we have $\omega = \varrho^{n-2} \cup \cup \varrho^{n-1} = \varrho^k \cup \varrho^{k+1}$.*

Proof. By Lemma 5,3 (assertion c) we have $a_j \in a_i\varrho \cup \dots \cup a_i\varrho^{n-1}$ for any $j \neq i$. Hence all couples (a_i, a_j) with $j \neq i$ are contained in $\varrho \cup \varrho^2 \cup \dots \cup \varrho^{n-1}$. Since ϱ^2 clearly contains the whole diagonal Δ, we have $\omega = \varrho \cup \dots \cup \varrho^{n-1} = \varrho^{n-2} \cup \varrho^{n-1}$. Since $d(\varrho)$ is at most 2, we also have $\omega = \varrho^k \cup \varrho^{k+1}$. This proves our Lemma.

Theorem 9,2. *Suppose that ϱ is symmetric and irreducible and* $\operatorname{card} \Pi(\varrho) = \operatorname{card} \Omega = n \geqq 3$.

a) *If $d(\varrho) = 1$, then $k(\varrho) \leqq 2n - 2$.*

b) *If $d(\varrho) = 2$, then $k(\varrho) \leqq n - 2$.*

Proof. a) If $d(\varrho) = 1$, then ϱ is primitive, and ϱ^2 is also primitive. The relation (9,2) implies (with $a_i \in \Omega$)

$$a_i\varrho^2 \subset a_i\varrho^4 \subset \ldots \subset a_i\varrho^{2n-2} \subset a_i\varrho^{2n}.$$

In this case ϱ^2 contains the whole diagonal and in each row at least one further element $\in \Omega$. For otherwise ϱ^2 would be reducible, contrary to the primitivity of ϱ^2. This implies that $a_i\varrho^2$ contains at least 2 elements $\in \Omega$, $a_i\varrho^4$ contains at least 3 different elements $\in \Omega$; a.s.o. Finally $a_i\varrho^{2n-2}$ contains at least $(n-1)+1 = n$ elements. Hence $a_i\varrho^{2n-2} = a_i\varrho^{2n}$. This proves $k_i \leqq 2n - 2$ (for $i = 1, \ldots, n$). By Theorem 3,1 we have $k(\varrho) \leqq 2n - 2$.

b) If $d(\varrho) = 2$, we have either α) $\varrho^{n-2} \subset \varrho^k$, $\varrho^{n-1} \subset \varrho^{k+1}$, or β) $\varrho^{n-2} \subset \varrho^{k+1}$, $\varrho^{n-1} \subset \varrho^k$.

Now (by Theorem 6,4) $\varrho^k \cap \varrho^{k+1} = z$ and this implies $\varrho^{n-2} \cap \varrho^{n-1} = z$. In the case α) we have (by Lemma 9,2) $\varrho^{n-2} = \varrho^k$ so that $k(\varrho) \leqq n - 2$. In the case β) we have $\varrho^{n-1} = \varrho^k$ and $\varrho^{n-2} = \varrho^{k+1}$. Hence $\varrho^n = \varrho^{k+1}$, i.e. $\varrho^{n-2} = \varrho^n$. Therefore $k(\varrho) \leqq$ $\leqq n - 2$. This proves our assertion.

Remark 1. It is easy to see directly that for $n = 2$ and $d(\varrho) = 1$ we have $k(\varrho) \leqq 2$. If $n = 2$ and $d(\varrho) = 2$, we have $k(\varrho) = 1$.

Remark 2. The result of Theorem 9,2 is sharp in the sense that there are relations with $k(\varrho) = 2n - 2$ and $k(\varrho) = n - 2$ respectively. For the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

we have $d(\varrho) = 1$ and $k(\varrho) = 4$. For the relation ϱ with

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

we have $d(\varrho) = 2$ and $k(\varrho) = 1$.

Suppose now that ϱ is *reducible* and $\varrho = \varrho_1 \cup \varrho_2 \cup \ldots \cup \varrho_s$, $s \geqq 2$, where ϱ_i are irreducible. Denote $\Pi(\varrho_i) = A_i$, card $A_i = n_i$, so that $n_i + \ldots + n_s = n$.

If $d(\varrho) = 1$, $n > 3$, then $d(\varrho_i) = 1$ for $i = 1, 2, \ldots, s$. Hence $k(\varrho_i) \leqq 2n_i - 2$. Therefore $k(\varrho) \leqq \max_{i=1,\ldots,s} (2n_i - 2)$. Since $\max n_i \leqq n - s + 1$, $k(\varrho) \leqq 2(n - s + 1) -$ $- 2 = 2n - 2s \leqq 2n - 4$.

If $d(\varrho) = 2$ and $n > 3$, there is at least one ϱ_i, say ϱ_1, such that $2 \leqq n_1 \leqq n - 1$, $d(\varrho_1) = 2$, so that $k(\varrho_1) \leqq n_1 - 2 \leqq n - 3$. Further

$$\max_{i=2,\ldots,s} n_i \leqq (n - n_1) - (s - 1) + 1 = n - n_1 - s + 2,$$

and

$$\max_{i=2,\dots,s} k(\varrho_i) \leqq 2(n - n_1 - s + 2) - 2 \leqq 2(n - 2 - 2 + 2) - 2 = 2n - 6\,,$$

$$k(\varrho) \leqq \max(n - 3,\ 2n - 6) \leqq 2n - 6 \quad (\text{for } n > 3)\,.$$

If $n = 3$ and $d(\varrho) = 2$, we have $s = 2$, $\varrho = \varrho_1 \cup \varrho_2$, card $\Pi(\varrho_1) = 2$, card $\Pi(\varrho_2) = 1$. Further $k(\varrho_1) \leqq 2$, $k(\varrho_2) = 1$, so that $k(\varrho) \leqq 2$. If $n = 3$ and $d(\varrho) = 1$, we get again $k(\varrho) \leqq 2$. If $n = 2$, we have $k(\varrho) = 1$.

Sumarily:

Theorem 9,3. *Suppose, that ϱ is symmetric and reducible with* card $\Omega = n$. *We have:*

a) *If $n > 3$ and $d(\varrho) = 1$, then $k(\varrho) \leqq 2n - 4$.*
b) *If $n > 3$ and $d(\varrho) = 2$, then $k(\varrho) \leqq 2n - 6$.*
c) *If $n = 3$, then $k(\varrho) \leqq 2$.*
d) *If $n = 2$, then $k(\varrho) = 1$.*

A relation ϱ is called *reflexive* if $\Delta \subset \varrho$. The set of all reflexive relations on Ω forms a subsemigroup R_Ω of B_Ω. R_Ω contains $2^{n^2 - n}$ elements.

Theorem 9,4. *For any reflexive relation ϱ on Ω (with* card $\Omega \geqq 2$) *we have $d(\varrho) = 1$ and $k(\varrho) \leqq n - 1$.*

Proof. $\Delta \subset \varrho$ implies

$$(9,3) \qquad \varrho \subset \varrho^2 \subset \dots \subset \varrho^n \subset \varrho^{n+1} \subset \dots$$

The transitive closure of ϱ is $\bar{\varrho} = \varrho^n$. In particular $\varrho^{n+1} \subset \varrho^n$, and therefore $\varrho^n = \varrho^{n+1}$. This proves $d(\varrho) = 1$ and $k(\varrho) \leqq n$. To prove $k(\varrho) \leqq n - 1$ it is sufficient to show that $\varrho^n \subset \varrho^{n-1}$.

Any element $(a_i, a_j) \in \varrho^n$ can be written in the form

$$(9,4) \qquad (a_i, a_j) = (a_{i_1}, a_{i_2})(a_{i_2}, a_{i_3}) \dots (a_{i_l}, a_{i_{l+1}}) \dots (a_{i_t}, a_{i_{t+1}}) \dots (a_{i_n}, a_{i_{n+1}})\,,$$

where we denote $i_1 = i$ and $i_{n+1} = j$. Since $\Delta \subset \varrho \subset \varrho^{n-1}$, we may suppose $i \neq j$. The integers $i_1, i_2, \dots, i_{n+1}$ cannot be all different and there is at least one of them which occurs more than once. If $i_l = i_t$ $(1 \leqq l < t \leqq n)$, we cann omit in (9,4) $(a_{i_l}, a_{i_{l+1}}) \dots (a_{i_{t-1}}, a_{i_t})$, so that $(a_i, a_j) \in \varrho^s$, where $s \leqq n - 1$, and hence (with respect to (9,3)) $(a_i, a_j) \in \varrho^{n-1}$. Analogously, if $i_l = i_t$ $(2 \leqq l < t \leqq n + 1)$ we obtain by the same argument $(a_i, a_j) \in \varrho^{n-1}$. Therefore $\varrho^n \subset \varrho^{n-1}$. This proves our statement.

Corollary 1. *A reflexive relation ϱ is irreducible if and only if there is an integer $s \leqq n - 1$ such that $\varrho^s = \omega$ (i.e. iff ϱ is primitive).*

Corollary 2. *If ϱ is reflexive, then ϱ^t si transitive if and only if it is an idempotent.*

For if ϱ^t is transitive, we have $\varrho^{2t} \subset \varrho^t$. On the other side we always have $\varrho^t \subset \varrho^{2t}$, so that $\varrho^t = \varrho^{2t}$.

Without going into details concerning the *structure* of R_Ω it seems to be worth to make the following comment.

If ϱ and σ are two primitive relations $\in B_\Omega$ with $\Pi(\varrho) = \Pi(\sigma) = \Omega$, their product $\varrho\sigma$ need not be primitive (it may be even reducible). In R_Ω the situation is simpler. We have:

Theorem 9,5. *The set of all primitive relations $\in R_\Omega$ is a two-sided ideal of R_Ω.*

Proof. It is sufficient to prove: If $\varrho \in R_\Omega$ is primitive and σ any relation $\in R_\Omega$, then $\varrho\sigma$ and $\sigma\varrho$ are primitive. Now $\Delta \subset \sigma$ implies $\varrho \subset \varrho\sigma$, and $\Delta \subset \varrho$ implies $\sigma \subset \varrho\sigma$, so that $\varrho \cup \sigma \subset \varrho\sigma$. If ϱ is primitive, i.e. there is an integer m, $1 \leqq m \leqq n - 1$, such that $\varrho^m = \omega$, we have $(\varrho\sigma)^m \supset (\varrho \cup \sigma)^m \supset \varrho^m = \omega$. Hence $(\varrho\sigma)^m = \omega$, and analogously $(\sigma\varrho)^m = \omega$. This proves our assertion.

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