

On the sharpness of a limiting case of the Sobolev imbedding theorem

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A refinement of the Sobolev imbedding theorem, due to Trudinger, is shown to be optimal in a natural sense.

Let Ω be a bounded domain in Euclidean n space, E^n . The Sobolev spaces $W_p^k(\Omega)$, where k is a non-negative integer and $p \geq 1$, consist of those functions in $L_p(\Omega)$ whose distributional derivatives of orders up to and including k are also in $L_p(\Omega)$ and are Banach spaces under the norm

$$(1) \quad \|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L_p(\Omega)} .$$

The Sobolev imbedding theorem (for the case $k = 1$) asserts that if Ω satisfies a cone condition and $p < n$, the space $W_p^1(\Omega)$ may be continuously imbedded in $L_q(\Omega)$ where $q = np/(n-p)$. If $p > n$, the functions in $W_p^1(\Omega)$ are continuous (after possible redefinition on a set of measure zero). A refinement, proved by Trudinger in [2], shows that the space $W_n^1(\Omega)$ may be continuously imbedded in the Orlicz space $L_\phi(\Omega)$ with defining N function

$$(2) \quad \phi(t) = e^{|t|^{\frac{n}{n-1}}} - 1 .$$

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The purpose of this note is to show that this result is optimal in the sense that the space $L_\phi(\Omega)$ above cannot be replaced by any smaller

Orlicz space. We let $W_p^{0,k}(\Omega) \subset W_p^k(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W_p^k(\Omega)$. The reader is referred to [2] for any other relevant definitions and notation.

THEOREM 1. *The space $W_n^1(\Omega)$ may not be continuously imbedded in any Orlicz space $L_\psi(\Omega)$ whose defining function ψ increases strictly more rapidly than the function ϕ given by (2).*

We remark here that ψ increases strictly more rapidly than ϕ if for every $\alpha > 0$, $\psi(\alpha\lambda)/\phi(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$. This happens if and only if $L_\psi(\Omega) \subsetneq L_\phi(\Omega)$. Theorem 1 is a consequence of the following two lemmas:

LEMMA 1. *Let ϕ and ψ be N functions, with ψ increasing strictly more rapidly than ϕ , and suppose that there exists a continuous imbedding of a normed, linear space X into $L_\psi(\Omega)$. Then*

$$J(u) = \int_\Omega \phi(u)dx \text{ is bounded on bounded subsets of } X.$$

Proof. We may identify X and its image $L_\psi(\Omega)$ under the imbedding map. Since this mapping is continuous, there is a constant $K \geq 0$ so that $\|u\|_\psi \leq K\|u\|_X$ for all u in X , where $\|u\|_\psi$ denotes the Luxemburg norm of u , that is

$$(3) \quad \|u\|_\psi = \inf \left\{ k > 0 ; \int_\Omega \psi \left(\frac{u}{k} \right) dx \leq 1 \right\}.$$

Since ψ increases strictly more rapidly than ϕ , there exists a non-decreasing function of $\lambda \geq 0$, $N(\lambda)$, satisfying

$$(4) \quad \phi(t) \leq \psi \left(\frac{t}{K\lambda} \right) \text{ for } t \geq N(\lambda).$$

Hence, for any u in X and $\lambda = \|u\|_X$,

$$(5) \quad \int_{\Omega} \phi(u) dx \leq \int_{\Omega} \phi(N(\lambda)) dx + \int_{\Omega} \psi\left(\frac{u}{\lambda K}\right) dx \leq \phi(N(\lambda)) |\Omega| + 1 .$$

The lemma is thus proved. //

LEMMA 2. Let $A = A(A)$ denote the set of Lipschitz continuous functions on the interval $[0, 1]$, vanishing at $x = 1$ and satisfying

$\int_0^1 x^{n-1} |u'|^n dx \leq A$. Then $J(u) = \int_0^1 x^{n-1} e^{|u|^{\frac{n}{n-1}}} dx$ is unbounded on A when $A > n^{n-1}$.

Proof. Let ρ satisfy $0 < \rho < 1$ and consider a sequence of piecewise linear functions, $u_k \in A$, $k = 1, 2, \dots$, satisfying $u'_k(x) = \alpha_{kj} > 0$ for $x \in (\rho^j, \rho^{j-1})$, $j = 1, 2, \dots, k$ and $u'_k(x) = 0$ for $x \in (0, \rho^k)$. Then

$$(6) \quad \int_0^1 x^{n-1} |u'_k|^n dx = \frac{\rho^{-n}-1}{n} \sum_{j=1}^k \rho^{jn} \alpha_{kj}^n = \frac{\nu^{n-1}}{n} \sum_{j=1}^k \alpha_{kj}^n$$

where $\nu = \rho^{-1}$ and $\alpha_{kj} = \rho^j \alpha_{kj}$.

Also for $x \leq \rho^k$, we have

$$(7) \quad u_k(x) = u_k(\rho^k) = -(\rho^{-1}-1) \sum_{j=1}^k \rho^j \alpha_{kj} = -(\nu-1) \sum_{j=1}^k \alpha_{kj} .$$

We now choose α_{kj} so that $\sum \alpha_{kj}$ is maximised subject to the constraint $(\nu^{n-1}) \sum \alpha_{kj}^n = An$; that is, we choose

$$\alpha_{k1} = \alpha_{k2} = \dots = \alpha_{kk} = \left(\frac{An}{\nu^{n-1}}\right)^{\frac{1}{n}} \frac{1}{k} \frac{1}{n}, \text{ so that}$$

$$(8) \quad u_k(\rho^k) = -(\nu-1) \left(\frac{A\nu}{\nu^{n-1}} \right)^{\frac{1}{n}} k^{\frac{n-1}{n}}.$$

Therefore, by (7) and (8),

$$\begin{aligned} J(u_k) &\geq \int_0^{\rho^k} x^{n-1} e^{|u|} \frac{n}{n-1} dx \\ &= \frac{1}{n} e^{\beta k} \end{aligned}$$

where $\beta = \beta(\nu) = (\nu-1)^{\frac{n}{n-1}} \left(\frac{A\nu}{\nu^{n-1}} \right)^{1/(n-1)} - n \log \nu$.

Since $\beta(\nu)/(\nu-1)$ approaches $A^{\frac{1}{n-1}} - n > 0$ as $\nu \rightarrow 1$, it is possible to choose $\nu > 1$ to guarantee $\beta > 0$. It then follows that $J(u_k)$ is unbounded. //

We remark here that if in the statement of Lemma 2, we assume $A < n^{n-1}$, then $J(u)$ is bounded on A . For then we have

$$\begin{aligned} |u(x)| &\leq \int_x^1 |u'(t)| dt \\ &\leq \left(\int_x^1 \frac{dt}{t} \right)^{1-\frac{1}{n}} \left(\int_0^1 t^{n-1} |u'(t)|^n dt \right)^{\frac{1}{n}} \text{ by Hölder's inequality,} \\ &\leq A^{\frac{1}{n}} \left(\log \frac{1}{x} \right)^{1-\frac{1}{n}} \end{aligned}$$

and consequently

$$J(u) \leq \int_0^1 x^{n-1-A^{\frac{1}{n-1}}} dx = \left(n-A^{\frac{1}{n-1}} \right)^{-1}.$$

To complete the proof of Theorem 1, we may without loss of generality take Ω as the unit sphere in E^n and consider spherically symmetric

functions, $u = u(r)$, only. Then

$$\int_{\Omega} |Du|^n dx = w_n \int_0^1 r^{n-1} |u_r|^n dr, \quad \int_{\Omega} e^{|u|^{\frac{n}{n-1}}} dx = w_n \int_0^1 r^{n-1} e^{|u|^{\frac{n}{n-1}}} dr,$$

and Theorem 1 consequently follows from Lemmas 1 and 2. As a consequence of Lemma 2, we also have

THEOREM 2. *The space $W_n^0(\Omega)$ may not be compactly imbedded in the Orlicz space $L_{\phi}(\Omega)$ where ϕ is given by (2).*

Note that Theorem 1 is also a consequence of Theorem 2. In view of Theorem 2 and Theorem 3 of [2], it would be of interest to study the unique solvability in $W_2^0(\Omega)$ of differential equations such as

$$\Delta u + ue^{u^2} = 0$$

in two dimensions. Hempel [1] has made some progress in this direction.

References

- [1] J.A. Hempel, "Superlinear variational boundary value problems and non-uniqueness", Doctoral thesis, University of New England, 1970.
- [2] N.S. Trudinger, "On imbedding into Orlicz spaces and some applications", *J. Math. Mech.* 17 (1967), 473-484.

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