Soft Computing (2022) 26:9069–9081
https://doi.org/10.1007/s00500-022-07305-3

FUZZY SYSTEMS AND THEIR MATHEMATICS

On the shortest path problem of uncertain random digraphs

Hao Li[1] · **Kun Zhang**[1]

Accepted: 13 June 2022 / Published online: 24 July 2022

Abstract

In the field of graph theory, the shortest path problem is one of the most significant problems. However, since varieties of indeterminated factors appear in complex networks, determining of the shortest path from one vertex to another in complex networks may be a lot more complicated than the cases in deterministic networks. To illustrate this problem, the model of uncertain random digraph will be proposed via chance theory, in which some arcs exist with degrees in probability measure and others exist with degrees in uncertain measure. The main focus of this paper is to investigate the main properties of the shortest path in uncertain random digraph. Methods and algorithms are designed to calculate the distribution of shortest path more efficiently. Besides, some numerical examples are presented to show the efficiency of these methods and algorithms.

Keywords Shortest path problem · Distance · Uncertain random digraph · Chance theory · Uncertainty theory

1 Introduction

In the field of graph and network optimization, the shortest path problem is a very important problem. The shortest path problem of graph aims to find the minimum of all path lengths from the starting point to the end point. The length of a path refers to the sum of the weights of all arcs on the path or the number of arcs. The shortest path problem has a wide range of applications, including the lowest cost problems, the shortest time problems, the minimum flow problems and so on. For certain graphs and networks, there are many classical algorithms for shortest path problems, such as: Dijkstra Algorithm, Bellman–Ford Algorithm, SPFA Algorithm and Floyd–Warshall Algorithm (Bondy and Murty 2008).

However, practical networks are always changing instead of keeping static. As a result, shortest path problems seem to become more complicated. To satisfy the need of researching shortest path problems under network changes, indeterministic factors such as random and uncertain factors have been taken into consideration in the research of graph theory.

To deal with the problems of indeterminacy, Kolmogoroff (1933) established the systems of probability theory in 1933. As we all know, probability needs a lot of historical data to make sure the statistical rules can be sufficiently generated. In 1959, two models of random graphs were proposed by Erdos and Renyi (1959). Of course, the theory of probability is never a master key to deal with all the problems in the field of indeterminacy. Just as what we mentioned above, probability theory is data-dependent, i.e., if there's no sufficient data to estimate the distribution, the theory will be invalid to solve the practical problems.

Besides probability theory, fuzzy theory also provides an important model framework for the study of the shortest path problem. Zadeh (1965) proposed the definition of fuzzy set in 1965. When using fuzzy theory to solve the shortest path problem, there are two main methods: direct solution methods and heuristic methods. In the work of using the direct solution methods, Deng et al. (2012) extended classical Dijkstra algorithm to deal with the fuzzy shortest path problems. Dubois and Prade (1980) found new methods to extended Floyd algorithms to solve fuzzy shortest path problems. Okada and Soper (2000) proposed a method to find all Pareto optimal paths from one certain node to all the other nodes with fuzzy numbers. As for heuristic methods, Chuang and Kung (2005) proposed a heuristic procedure to find the FSP length among all possible paths in a network. Hernandes et al. (2007) uses genetic algorithm to solve the fuzzy shortest path problem. Ebrahimnejad et al. (2016, 2015, 2020) proposed artificial bee colony (ABC) algorithm, particle swarm optimization algorithm, lexicographic optimization method to solve the fuzzy shortest problems with fuzzy arc weights.

✉ Kun Zhang
zk-math2020@ruc.edu.cn

Hao Li
hlimath@ruc.edu.cn

[1] School of Mathematics, Renmin University of China, Beijing 100872, China

Many excellent results have been obtained by using fuzzy theory to study the shortest path problem. However, fuzzy set theory is lack of duality axiom, which may lead to wrong results in practice. In order to overcome this defect of fuzzy theory, uncertainty theory was founded by Liu (2007). Shortest path problem with uncertain arc lengths was discussed by Gao (2011). In 2013, uncertain graph was proposed by Gao and Gao (2013). Some of uncertain graphs were soon discussed, such as Euler index (Zhang and Peng 2012), diameter index (Gao and Gao 2013), cycle index (Gao 2013), regularity index (Gao 2014), tree index (Gao 2016). In addition, some traditional problems of graph theory were discussed in uncertain graphs. More information can be found in Li et al. (2018), Rosyida et al. (2018), Zhou et al. (2014a), Gao et al. (2015), Gao and Qin (2016).

In recent years, networks became more and more complex. Different kinds of indeterminate factors appear at the same time. Random and uncertain factors in particular, appear in most cases. In order to solve this problem, Liu (2013b) proposed the chance theory in 2013 which includes concepts of uncertain random variable and chance measure. In addition, expected value and variance of an uncertain random variable were proposed. Chance theory has been applied to many optimization networks problems, such as uncertain random programming (Liu 2013a), goal programming (Qin 2018), multi-objective programming (Zhou et al. 2014b).

Uncertain random graph was proposed by Liu (2014). In an uncertain random graph, all edges are independent, and some edges exist with degrees in probability measure while other edges exist with degrees in uncertain measure. Liu (2014) discussed the diameter index of an uncertain random graph. Sheng and Gao (23016); Sheng and Mei (2020) gave the EVSPM, SPM and MCMSPM models of shortest path problems in uncertain random graph. In 2016, the Euler index of an uncertain random graph was discussed by Zhang et al. (2017). In 2018, the cycle index of an uncertain random graph was discussed by Chen et al. (2018). Li and Zhang (2020) discussed the edge-connectivity of uncertain random graph, and the vertex-connectivity of uncertain random graph was investigated by Li and Gao (2020).

The uncertain random graph model provides a new idea for us to study the shortest path problem of networks in reality. In real life, when we describe an uncertain event, we must first generate the distribution function of the event in advance. If this distribution is close enough to the frequency, we can safely use the probability model for modeling. If not, we will use uncertainty theory for modeling and analysis. However, the network in real life is often very complex. Some arcs can have enough historical data for probability modeling, while others may have a large difference between the distribution and the actual situation due to the lack of historical data, so the uncertain theory needs to be used. To sum up, we can find that the uncertain random graph model is more suitable for the description of real networks than the traditional probability model. It is a better research model of real networks and their shortest path problems.

In this paper, we will propose the model of uncertain random digraph, and discuss the shortest path problem. Definition of distance from one vertex to another in a random uncertain digraph will be given. Then, we will discuss the shortest path distribution and give an efficient algorithm to calculate the distribution. And we will use the method of this paper to establish the spreading model of COVID-19 in social networks. In addition, we will illustrate that our algorithm will greatly reduce the complexity compared with existing methods.

The remainder of the paper is organized as follows. In Sect. 2, we will introduce some necessary notations of digraph, and give a brief introduction of chance theory. In Sects. 3 and 4, we will discuss the shortest path problem in uncertain digraph and uncertain random digraph, respectively, including the form of the distribution of shortest path and algorithms of simplifying the process of calculation. Examples will be shown to illustrate the efficiency of these algorithms. The last section will conclude this paper with a brief summary.

2 Preliminaries

In this section, we first introduce some necessary definitions and notations of digraph. Then, we introduce some preliminary knowledge about chance theory.

2.1 Notations of digraphs

A digraph D is an ordered pair (V, A) consisting of a set V of vertices and a set A of arcs (directed edges). Without loss of generality, in the rest of this paper, we assume $V = \{1, 2, 3, \ldots, n\}$. An ordered pair $(i, j) \in A$ means there exist an arc from vertex i to j, where i and j are called the tail and the head of arc (i, j), respectively. Here, we will only consider simple digraphs.

A directed walk with length k is an alternating sequence $W = i_1 i_2 \cdots i_k i_{k+1}$ such that $\{i_1, i_2, \ldots, i_k\} \subset V$ and $(i_j, i_{j+1}) \in A$ for $j = 1, 2, \ldots, k$. If all the vertices of W are distinct, then W is called a directed path.

The adjacency matrix of digraph D, denoted by $M(D)$, is a $n \times n$ matrix such that

$$M(D) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \dots & \alpha_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn} \end{pmatrix},$$

where

$$\alpha_{ij} = \begin{cases} 1, & if\ (i,j) \in A \\ 0, & otherwise. \end{cases}$$

And if there is an arc from vertex i to j, then $\alpha_{ij} = 1$, otherwise $\alpha_{ij} = 0$. A well-known proposition is proposed in the following to find out whether there exists a directed walk of length $k(k = 1, 2, 3, \dots, n-1)$ from i to j.

Proposition 1 *Let D be a digraph and let X be the adjacency matrix of D. For two distinct vertices i, j, there exists a directed walk of length k from i to j if and only if* $a_{ij}(X^k) \geq 1$, *where* $a_{ij}(X^k)$ *is the* (i, j) *entry of matrix* X^k.

For any vertices $i, j \in V$, if there exist a directed path from vertex i to vertex j, then the distance from i to j, denoted by $d(i, j)$, is the minimal length of all the directed paths from i to j. If there is no directed path from i to j, which means i and j are in different components, then $d(i, j)$ is usually defined as ∞. In the field of graph theory, $d(i, j)$ can be calculated by Dijkstra Algorithm Bondy and Murty (2008), the complexity of which is $O(n^2)$. As all directed paths are directed walks, a proposition can be proposed in the following.

Proposition 2 *Let D be a digraph and X be the adjacency matrix of D. For two distinct vertices i, j and* $k = 1, 2, 3, \dots, n-1$,

$$d(i,j) \leq k\ if\ and\ only\ if\ \alpha_{ij}(I + X + X^2 + \cdots + X^k) > 0,$$

where α_{ij} *is the* (i, j)*-entry of matrix* $I + X + X^2 + \cdots + X^k$.

2.2 Chance theory

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ and $(\Omega, \mathcal{A}, \Pr)$ be an uncertainty space and a probability space, respectively. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a chance space. The product σ-algebra $\mathcal{L} \times \mathcal{A}$ is the smallest σ-algebra containing all measurable rectangles of the form $\Lambda \times A$, where $\Lambda \in \mathcal{L}$ and $A \in \mathcal{A}$. For each $\Theta \in \mathcal{L} \times \mathcal{A}$, Θ is called an event of the chance space. The chance measure of event Θ was defined by Liu (2013b) as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq x\} \mathrm{d}x.$$

An uncertain random variable is a function η from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ to the set of real numbers such that $\{\eta \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B. A random variable is called a Boolean random variable if it takes values from $\{0, 1\}$. Similarly, an uncertain variable or an uncertain random variable is called a Boolean uncertain variable or a Boolean uncertain random variable, respectively, if it takes values from $\{0, 1\}$. A function with n variables is called a Boolean function if it is a mapping from $\{0, 1\}^n$ to $\{0, 1\}$.

Theorem 1 (Liu 2013a) *Assume that* $\eta_1, \eta_2, \dots, \eta_m$ *are independent Boolean random variables, i.e.,*

$$\eta_i = \begin{cases} 1 & with\ probability\ measure\ a_i \\ 0 & with\ probability\ measure\ 1 - a_i \end{cases}$$

for $i = 1, 2, \dots, m$, *and the variables* $\tau_1, \tau_2, \dots, \tau_n$ *are independent Boolean uncertain variables, i.e.,*

$$\tau_j = \begin{cases} 1 & with\ uncertain\ measure\ b_j \\ 0 & with\ uncertain\ measure\ 1 - b_j \end{cases}$$

for $j = 1, 2, \dots, n$. *If f is a Boolean function, then* $\eta = f(\eta_1, \eta_2, \dots, \eta_m, \tau_1, \tau_2, \dots, \tau_n)$ *is a Boolean uncertain random variable such that*

$$Ch\{\eta = 1\} = \sum_{(x_1,\dots,x_m) \in \{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i) \right) f^*(x_1, \dots, x_m),$$

where

$$f^*(x_1, \dots, x_m) = \begin{cases} \sup\limits_{f(x_1,\dots,x_m,y_1,\dots,y_n)=1} \min\limits_{1 \leq j \leq n} \upsilon_j(y_j), \\ \quad if \sup\limits_{f(x_1,\dots,x_m,y_1,\dots,y_n)=1} \min\limits_{1 \leq j \leq n} \upsilon_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\dots,x_m,y_1,\dots,y_n)=0} \min\limits_{1 \leq j \leq n} \upsilon_j(y_j), \\ \quad if \sup\limits_{f(x_1,\dots,x_m,y_1,\dots,y_n)=1} \min\limits_{1 \leq j \leq n} \upsilon_j(y_j) \geq 0.5, \end{cases}$$

$$\mu_i(x_i) = \begin{cases} a_i & if\ x_i = 1 \\ 1 - a_i & if\ x_i = 0 \end{cases}$$

$$(i = 1, 2, \dots, m),\ \upsilon_j(y_j) = \begin{cases} b_j & if\ y_j = 1 \\ 1 - b_j & if\ y_j = 0 \end{cases}$$

$$(j = 1, 2, \dots, n).$$

3 The shortest path problem in uncertain digraphs

In this section, we will introduce the definition of uncertain digraph. Then, we will discuss the shortest path problem in uncertain digraphs and give the distribution of shortest path.

3.1 Problem description

We first give the definition of uncertain digraph and some necessary notations.

Definition 1 Let $\mathcal{V}$ be a set of n vertices, and write $\mathcal{V} = \{1, 2, \ldots, n\}$. We call

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

an uncertain adjacency matrix, if α_{ij} represent the uncertain measures that the arcs (i, j) exist, $i, j = 1, 2, \ldots, n,$, respectively. The ordered pair $(\mathcal{V}, \mathcal{A})$ is called an uncertain digraph, which is denoted by $\mathbb{D}$.

Remark 1 Since $\mathbb{D}$ is simple, $\alpha_{ii} = 0,\ for\ i = 1, 2, \ldots, n$. Note that $\mathcal{A}$ is normally asymmetric, i.e., it's possible that $\alpha_{ij} \neq \alpha_{ji}$.

Write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

and

$$\mathbb{X} = \left\{ X \mid \begin{array}{l} x_{ij} = 0 \text{ or } 1, i, j = 1, 2, \ldots, n \\ x_{ii} = 0, i = 1, 2, \ldots, n \end{array} \right\}. \tag{1}$$

In $\mathbb{D} = (\mathcal{V}, \mathcal{A})$, all the arcs (i, j) exist with the uncertain measure α_{ij}. For any $X \in \mathbb{X}$ defined by (1), $x_{ij} = 1$ represents that the arc from i to j exists. By the definition of α_{ij}, the uncertain measure that the event $\{x_{ij} = 1\}$ happens is α_{ij}.

As $\mathbb{D}$ has n vertices, there are $n(n-1)$ uncertain arcs. Then, there are $2^{n(n-1)}$ possible realizations of arcs. Each realization of arcs and corresponding vertices form a deterministic digraph D. Since a digraph could be fully characterized by its adjacency matrix. Once a realization of arcs is given, there exists an unique matrix X in $\mathbb{X}$ satisfying such that X is the adjacency matrix of D. By the product axiom of uncertain measure (Liu 2007), the uncertain measure of the event that D appears is

$$\min_{1 \le i, j \le n} \mu_{ij}(X),$$

where

$$\mu_{ij}(X) = \begin{cases} \alpha_{ij} & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij} & \text{if } x_{ij} = 0. \end{cases}$$

For any two distinct vertices i and j in $\mathbb{D}$, the length of the shortest path from vertex i to j, denoted by $d(i, j)$, is a function of all x_{ij}. Since x_{ij} is an uncertain variable, $d(i, j)$ is an uncertain variable as well. Once X is given, we could calculate $d(i, j)$ by Dijkstra Algorithm (Bondy and Murty 2008). Therefore, we will naturally study the distribution and the properties of uncertain variable $d(i, j)$.

3.2 Distribution of shortest path in uncertain digraph

Aiming at dealing with the problem in the last subsection, we first give a lemma and some notations of uncertain theory.

Lemma 1 *(Operational Law of Boolean System) (Liu 2013a) Assume that the variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_j = \begin{cases} 1 & \text{with uncertain measure } b_j \\ 0 & \text{with uncertain measure } 1 - b_j \end{cases}$$

for $j = 1, 2, \ldots, n$. If f is a Boolean function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(y_1,\ldots,y_n)=1} \min\limits_{1 \le j \le n} \upsilon_j(y_j), \\ \quad if \sup\limits_{f(y_1,\ldots,y_n)=1} \min\limits_{1 \le j \le n} \upsilon_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(y_1,\ldots,y_n)=0} \min\limits_{1 \le j \le n} \upsilon_j(y_j), \\ \quad if \sup\limits_{f(y_1,\ldots,y_n)=1} \min\limits_{1 \le j \le n} \upsilon_j(y_j) \ge 0.5, \end{cases}$$

where y_j take values either 0 or 1, and υ_j are defined by

$$\upsilon_j(y_j) = \begin{cases} b_j & if\ y_j = 1 \\ 1 - b_j & if\ y_j = 0 \end{cases} \quad (j = 1, 2, \ldots, n)$$

for $i = 1, 2, \ldots, n$, respectively.

According to Proposition 2, the property $d(i, j) \le k$ can be fully characterized by the adjacency matrix of a digraph. So we give the characteristic function of this property in the following.

For any $n \times n$ Boolean matrix X and any integer k $(k = 1, 2, \ldots, n-1)$, we define

$$f_k(X) = \begin{cases} 1, & \text{if } \alpha_{ij}(I + X + \cdots + X^k) > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

where α_{ij} is the (i, j)-entry of matrix $I + X + X^2 + \cdots + X^k$. And by Lemma 1, we have the following Theorem.

Theorem 2 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph. For distinct vertices $i, j \in \mathcal{V}$, the uncertain measure that the* *distance from i to j is no more than k is denoted by $\eta_{ij}^k(\mathbb{D})$. And we have*

$$\eta_{ij}^k(\mathbb{D}) = \begin{cases} \sup\limits_{X\in\mathbb{X}, f_k(X)=1} \min\limits_{1\le i,j\le n} \mu_{ij}(X), \\ \quad \text{if} \sup\limits_{X\in\mathbb{X}, f_k(X)=1} \min\limits_{1\le i,j\le n} \mu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X\in\mathbb{X}, f_k(X)=0} \min\limits_{1\le i,j\le n} \mu_{ij}(X), \\ \quad \text{if} \sup\limits_{X\in\mathbb{X}, f_k(X)=0} \min\limits_{1\le i,j\le n} \mu_{ij}(X) \ge 0.5. \end{cases}$$

where

$$\mu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \tag{3}$$

and $\mathbb{X}$ is the set of matrices satisfying (1).

Remark 2 According to the duality axiom, the uncertain measure that i and j are in two different components is denoted by $\eta_{ij}^\infty(\mathbb{D})$. And we have $\eta_{ij}^\infty(\mathbb{D}) = 1 - \eta_{ij}^{n-1}(\mathbb{D})$.

According to Theorem 2, it is easy to calculate the distribution function of $d(i, j)$.

Theorem 3 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph. For two distinct vertices i and j, let $\Phi_{(i,j)}(x)$ be the distribution function of uncertain variable $d(i, j)$. Then,*

$$\Phi_{(i,j)}(x) = \begin{cases} 0, & \text{if } x < 1 \\ \eta_{ij}^{[x]}(\mathbb{D}), & \text{if } 1 \le x < n-1 \\ \eta_{ij}^{n-1}(\mathbb{D}), & \text{if } x \ge n-1. \end{cases}$$

3.3 Maximum d_{ij}^k-digraph

In $\mathbb{D}$, given distinct vertices i and j, the uncertain distribution function of $d(i, j)$ is given by Theorem 2 and 3. However, it's impractical to calculate the distribution function by Theorem 2 and 3, as we have to concern all the realization digraphs. For examples, if $\mathbb{D}$ has n uncertain arcs, then it has $2^{\frac{n(n-1)}{2}}$ uncertain arcs and $2^{\frac{n(n-1)}{2}}$ realization digraphs. The complexity of computing f_k is $O(n^3)$. So the complexity is $2^{\frac{n(n-1)}{2}} O(n^3)$ in total, which is not polynomial. In next part, we will propose a more efficient method to calculate $\eta_{ij}^k(\mathbb{D})$, and present a polynomial algorithm.

Definition 2 Let $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph. A digraph D is called a maximum d_{ij}^k digraph if all the following hold.

(1) $V(D) = \mathcal{V}$;
(2) $d(i, j) \le k$;
(3) $\min\limits_{(i,j)\in A(D)} \alpha_{ij}$ is maximum.

Maximum d_{ij}^k digraph given by Definition 2 could be used to calculated uncertain measure $\eta_{ij}^k(\mathbb{D})$, which is shown in the following Theorem.

Theorem 4 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph and i, j be the two vertices in $\mathcal{V}$. Let $D = (V, A)$ be a maximum d_{ij}^k digraph of $\mathbb{D}$. Then,*

$$\eta_{ij}^k(\mathbb{D}) = \min_{(i,j)\in A(D)} \alpha_{ij}.$$

Proof We assume $\min\limits_{(i,j)\in A(D)} \alpha_{ij} = a_0$. Let $\mathbb{D}_0 = (\mathcal{V}, \mathcal{A}_0)$ be an uncertain digraph, where $\mathcal{A}_0 = (b_{ij})_{n\times n}$ and

$$b_{ij} = \begin{cases} \alpha_{ij}, & \text{if } (i,j) \in A(D) \\ 0, & \text{if } (i,j) \notin A(D) \end{cases} \quad for\ 1 \le i, j \le n\ .$$

We will first prove $\eta_{ij}^k(\mathbb{D}) \ge a_0$. Note that D could be viewed as a realization digraph of $\mathbb{D}_0$, and the uncertain measure that it appears is a_0. We have $\eta_{ij}^k(\mathbb{D}) \ge \min\limits_{(i,j)\in A(D)} \alpha_{ij} = a_0$. As $\mathbb{D}$ has more arcs than $\mathbb{D}_0$, it's easier to find a directed path from i to j. Then, we have

$$\eta_{ij}^k(\mathbb{D}) \ge \eta_{ij}^k(\mathbb{D}_0) \ge \min_{(i,j)\in A(D)} \alpha_{ij} = a_0. \tag{4}$$

We then show that $\eta_{ij}^k(\mathbb{D}) \le a_0$. As the number of realization digraphs is finite, there exists a realization digraph D_1 of $\mathbb{D}$ whose adjacency matrix X_1 satisfies $f_k(X_1) = 1$ and

$$\sup_{X\in\mathbb{X}, f_k(X)=1} \min_{1\le i,j\le n} \mu_{ij}(X) = \min_{1\le i,j\le n} \mu_{ij}(X_1).$$

f_k and μ_{ij} are defined by (2) and (3), respectively. By the definition of D, $\min\limits_{(i,j)\in A(D_1)} \mu_{ij} \le a_0$. Thus,

$$\begin{aligned} &\sup_{X\in\mathbb{X}, f_k(X)=1} \min_{1\le i,j\le n} \mu_{ij}(X) \\ &= \min_{1\le i,j\le n} \mu_{ij}(X_1) \\ &\le \min_{(i,j)\in A(D_1)} \mu_{ij} \\ &= \left(\min_{(i,j)\in A(D_1)} \alpha_{ij}\right) \wedge \left(\min_{(i,j)\notin A(D_1)} (1-\alpha_{ij})\right) \\ &\le \min_{(i,j)\in A(D_1)} \alpha_{ij} \\ &\le a_0. \end{aligned} \tag{5}$$

Let $D_2 = (V, A_2)$ be a realization digraph of $\mathbb{D}$ such that $A_2 = A(D) \cup \{(i,j) | \alpha_{ij} > a_0\} \backslash \{(i,j) | \alpha_{ij} = a_0\}$. Let X_2 be the adjacency matrix of D_2. Therefore, $\min\limits_{(i,j)\in A(D_2)} \alpha_{ij} > a_0$. By the choice of digraph D_2, $d(i, j) > k$ holds in D_2, i.e., $f_k(X_2) = 0$.

If $\sup_{X\in\mathbb{X}, f_k(X)=1} \min_{1\le i,j\le n} \alpha_{ij}(X) < 0.5$, according to Theorem 2 and (5), then we have

$$\eta_{ij}^k(\mathbb{D}) = \sup_{X\in\mathbb{X}, f_k(X)=1} \min_{1\le i,j\le n} \mu_{ij}(X) \le a_0. \tag{6}$$

If $\sup_{X\in\mathbb{X}, f_k(X)=1} \min_{1\le i,j\le n} \alpha_{ij}(X) \ge 0.5$, then by (5), $a_0 \ge 0.5$. According to Theorem 2,

$$\begin{aligned}\eta_{ij}^k(\mathbb{D}) &= 1 - \sup_{X\in\mathbb{X}, f_k(X)=0} \min_{1\le i,j\le n} \mu_{ij}(X)\\ &\le 1 - \min_{1\le i,j\le n} \mu_{ij}(X_2)\\ &= 1 - \left(\min_{(i,j)\in A(D_2)} \alpha_{ij}\right) \wedge \left(\min_{(i,j)\notin A(D_2)} (1-\alpha_{ij})\right)\\ &\le 1 - \left(\min_{(i,j)\in A(D_2)} \alpha_{ij}\right) \wedge (1-a_0)\\ &\le 1 - a_0 \wedge (1-a_0).\end{aligned}$$

Since $a_0 \ge 0.5 \ge 1 - a_0$, we have

$$\eta_{ij}^k(\mathbb{D}) \le 1 - (1 - a_0) = a_0. \tag{7}$$

By inequalities (6) and (7), $\eta_{ij}^k(\mathbb{D}) \le a_0$. And by inequalities (4), $\eta_{ij}^k(\mathbb{D}) = a_0$ is proved to be correct. □

3.4 Algorithm and example

By Theorem 4, in an uncertain digraph $\mathbb{D} = (\mathcal{V}, \mathcal{A})$, once i, j, k are given, the uncertain measure $\eta_{ij}^k(\mathbb{D})$ can be calculated by finding a maximum d_{ij}^k digraph of $\mathbb{D}$. Although such a maximum digraph may not be unique, we can always use a greedy algorithm to find a maximum d_{ij}^k digraph of $\mathbb{D}$.

Algorithm 1 Greedy Algorithm for calculating $\eta_{ij}^k(\mathbb{D})$.
Note that $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph. Let $A(\mathbb{D})$ be the set of uncertain arcs. Let i and j be the vertices in $\mathcal{V}$ and k be a constant taking values from $\{1, 2, \ldots n-1\}$.
Step 1. *Set $A = \emptyset$.*
Step 2. *Choose $(i, j) \in A(\mathbb{D})$ such that α_{ij} is maximum. Set $A = A \cup \{(i, j)\}$, and $A(\mathbb{D}) = A(\mathbb{D})\backslash\{(i, j)\}$. Calculate $d(i, j)$ using Dijkstra Algorithm(if there's no path from i to j, return to n) in $D = (V, A)$.*
Step 3. *If $d(i, j) \le k$, stop the iteration. The digraph $D = (\mathcal{V}, A)$ is a maximum d_{ij}^k digraph. Thus, by Theorem 4, $\eta_{ij}^k(\mathbb{D}) = \alpha_{ij}$. Otherwise, go to Step 4.*
Step 4. *If $A(\mathbb{D}) \ne \emptyset$, then go to Step 2. If $A(\mathbb{D}) = \emptyset$, then stop and $\eta_{ij}^k(\mathbb{D}) = 0$.*

Using algorithm 1 and Theorem 3, distribution function $\Phi_{(i,j)}(x)$ could be calculated easily by the following Algorithm.

Algorithm 2 Maximum Digraph Algorithm(MDA) for calculating $\Phi_{(i,j)}(x)$.
Note that $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be an uncertain digraph. Let $A(\mathbb{D})$ be the set of uncertain arcs. Let i and j be the vertices in $\mathcal{V}$.
Step 1. *Set $A = \emptyset$, $k = n - 1$.*
Step 2. *Choose $(i, j) \in A(\mathbb{D})$ such that α_{ij} is maximum. Set $A = A \cup \{(i, j)\}$, and $A(\mathbb{D}) = A(\mathbb{D})\backslash\{(i, j)\}$. Calculate $d(i, j)$ using Dijkstra Algorithm(if there's no path from i to j, return to n) in $D = (V, A)$.*
Step 3. *If $d(i, j) \le k$, then the digraph $D = (\mathcal{V}, A)$ is a maximum d_{ij}^k digraph. Thus, by Theorem 4, $\eta_{ij}^k(\mathbb{D}) = \alpha_{ij}$. Let $k = k - 1$ and then go to Step 3; otherwise, go to Step 4.*
Step 4. *If $A(\mathbb{D}) \ne \emptyset$, then go to Step 2. If $A(\mathbb{D}) = \emptyset$, then stop and $\eta_{ij}^k(\mathbb{D}) = 0$.*
Step 5. *According to Theorem 3, we could calculate the distribution function $\Phi_{(i,j)}(x)$.*

Next, we will discuss the complexity of Algorithm 2. Let m be the number of different uncertain values with which arcs exist. In Step 2, we need to calculate $d(i, j)$ by Dijkstra Algorithm, which the complexity is $O(n^2)$. As the Algorithm has m interactions of Step 2, the complexity of Algorithm 1 is $O(mn^2)$. When the number of vertices is small, the algorithm is quite efficient. Here is an example.

Example 1 Let $\mathbb{D} = \{\mathcal{V}, \mathcal{A}\}$ be an uncertain digraph. $\mathcal{V} = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 0.9 & 0.4\\ 1 & 0 & 0.9 & 0 & 0.3\\ 1 & 0.3 & 0 & 0.6 & 0\\ 0 & 0 & 0 & 0 & 0.8\\ 0 & 0 & 0 & 0.3 & 0 \end{pmatrix}.$$

We calculate the value of η_{25}^k, $k \in \{1, 2, 3, 4\}$ using Algorithm 1.

The uncertain measures of all arcs are listed from high to low as follows: 1, 0.9, 0.8, 0.6, 0.4, 0.3 (Fig. 1).

In the first iteration, $k = 4$, and then we add the arcs with uncertain measure 1. It can be easily verified that $d(2, 5) = +\infty > 4$. According to the algorithm, we continue the iteration.

In the second iteration, $k = 4$, then we add the arcs with uncertain measure 0.9. It can be found that $d(2, 5) = +\infty > 4$. We continue the iteration.

In the third iteration, $k = 4$, then we add the arcs with uncertain measure 0.8. We find it that $d(2, 5) = 3 \le 4$. D is the maximum d_{25}^4 digraph, thus $\eta_{25}^4 = 0.8$. Let $k = 3$ and we find that $d(2, 5) \le 3$ still holds. So $\eta_{25}^3 = 0.8$. Let $k = 2$, and we continue the iteration.

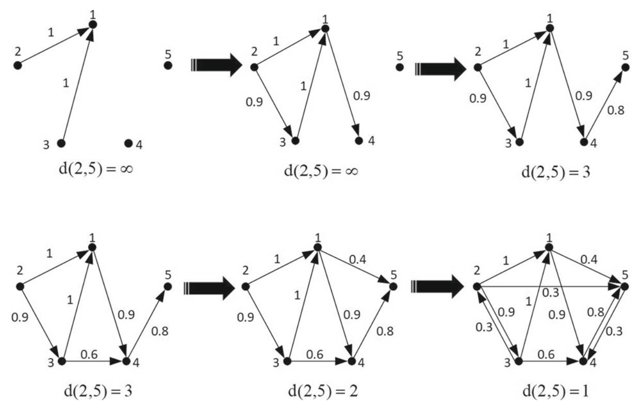


Fig. 1 Iterations in Example 1

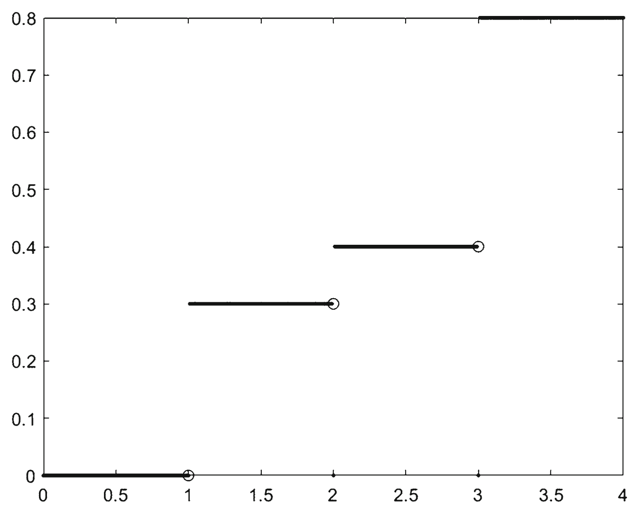


Fig. 2 Distribution function of $d(2,5)$

In the fourth iteration, $k = 2$, then we add the arcs with uncertain measure 0.6. By Dijkstra Algorithm, $d(2,5) = 3 > 2$, so we continue.

In the fifth iteration, $k = 2$, then we add the arcs with uncertain measure 0.4. We find that $d(2,5) = 2 \leq 2$, so $\eta_{25}^2 = 0.4$. Let $k = 1$, and we continue the iteration.

In the sixth iteration, $k = 1$, then we add the arcs with uncertain measure 0.3. $d(2,5) = 1 \leq 1$, so $\eta_{25}^1 = 0.3$. All the directed edges have been added, so we stop the iteration. By the iteration process, distribution function of $d(2,5)$ is

$$\Phi_{(2,5)}(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.3, & \text{if } 1 \leq x < 2 \\ 0.4, & \text{if } 2 \leq x < 3 \\ 0.8, & \text{if } x \geq 3, \end{cases}$$

whose image is shown in Fig.2.

3.5 Effectiveness of MDA algorithm

In order to illustrate the effectiveness of this algorithm, we choose two classical algorithms for solving the shortest path problem: genetic algorithm (GA) and ant colony algorithm (ACO) to compare with the maximum directed graph algorithm (MDA) proposed in this paper. Each algorithm performs ten rounds of operation, and the final results are as follows. We select an uncertain digraph $\mathbb{D}$ with 20 vertices and 39 arcs, and the connection of its vertices is shown in Fig.3.

We study the shortest path distribution of $d(1,20)$ in uncertain digraph $\mathbb{D}$. The three algorithms run 10 times, respectively, and the final distributions are all the same. The final results of convergence iteration times and running time are shown in Fig. 4. We can find the following points:

(1) According to Fig. 4, the average running time of MDA algorithm is only 1.452s, which is less than 14.806s of GA algorithm and 22.565s of ACO algorithm. Compared with the two classical algorithms, the running time can be greatly shortened.

(2) The average convergence time of MDA algorithm is 0.854s, which is greatly improved compared with 9.577s of GA algorithm and 12.911s of ACO algorithm.

(3) The variance of average running time and average convergence time of MDA algorithm are 0.008996 and 0.003384, respectively, which are far less than GA algorithm and ACO algorithm. This algorithm has good stability.

Above all, since GA algorithm and ACO algorithm are heuristic algorithms, their running speed and results are greatly affected by initial parameters. MDA algorithm does not depend on the selection of initial parameters, but only related to the structure of uncertain digraph. It runs stably and efficiently. So, it is an efficient algorithm to solve the shortest path distribution problems in uncertain digraphs.

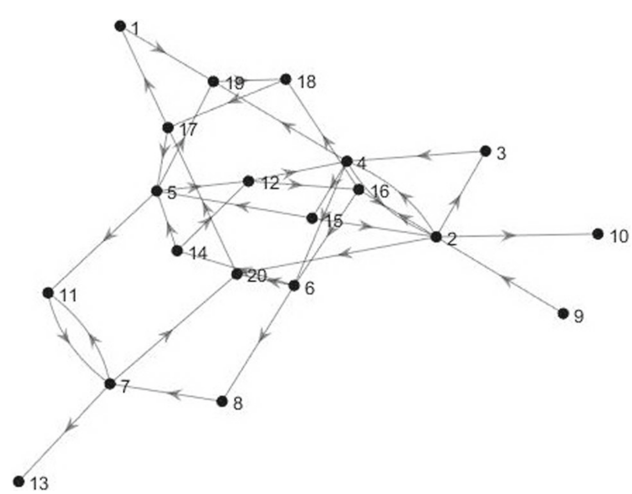


Fig. 3 Uncertain digraph $\mathbb{D}$

	Number of iteration to converge(Total)			Converge Time			Total Time		
	GA	ACO	UDA	GA	ACO	UDA	GA	ACO	UDA
1	20	28	-	5.8	12.2	0.84	8.66	21.41	1.45
2	38	33	-	9.99	16.21	0.86	14.9	27.94	1.46
3	42	21	-	11.38	10.65	0.86	16.99	19.72	1.46
4	36	18	-	9.76	12.64	0.88	14.57	21.8	1.49
5	40	24	-	12.36	10.18	0.87	20.27	17.26	1.47
6	30	30	-	7.56	15.46	0.89	11.29	28.1	1.48
7	33	17	-	8.95	8.22	0.86	13.35	15.23	1.43
8	36	28	-	9.34	14.02	0.94	14.59	24.17	1.62
9	36	27	-	9.16	13.88	0.7	13.67	23.93	1.21
10	43	31	-	11.47	15.65	0.84	19.77	26.09	1.45
Min	20	17	-	5.8	8.22	0.7	8.66	15.23	1.21
Max	43	33	-	12.36	16.21	0.94	20.27	28.1	1.62
Mean	35.4	25.7	-	9.577	12.911	0.854	14.806	22.565	1.452
Variance				3.374061	6.220669	0.003384	11.292364	16.835025	0.008996

Fig. 4 Model comparison

3.6 Numerical simulation of the shortest path model on COVID-19 transmission in social networks

The transmission of COVID-19 has severely threatened people's lives since 2019. Therefore, research about the transmission of COVID-19 has its theoretical and practical significance. According to the World Health Organization(WHO), COVID-19 has distinct community transmission characteristics. It can easily spread through the social network and the spreading rate of COVID-19 is much higher than the other viruses. In this part, we simulate the spread of viruses in social networks.

We use a vertex to represent a person. For any two person i and j, if COVID-19 is transmitted from i to j, there is an arc between the corresponding vertices, otherwise it does not exist. Due to the strong dependence of probability model on the amount of data. Since the virus transmission is affected by indeterminate factors, which are lack of supporting historical data, the distribution obtained by models of probability theory will be far from the actual situation. Then, we have to ask experts in relevant fields to give the experience distribution. As a result, we assume all the arcs exist with a certain measure. As the measures that COVID-19 transmit from i to j and from j to i are different, it can only be characterized by directed graphs instead of undirected graphs. At the same time, due to the lack of historical data of interpersonal communication, it is difficult for us to build a model by probability theory. As a result, we assume all the arcs exist with degree of uncertain measure.

When a small number of infected people appear in a social network, they have infected some people around because of the incubation period. At this time, the distance of uncertain digraph can be used to estimate the required isolation range. One reason why random graphs are not suitable here is that the probability measures of arcs in the graph are difficult to be obtained objectively. The other reason is that according to the independence of probability measure, the probability measure of a product event is equal to the product of all the probability measures. According to the little probability event principle, the transmission range is generally very small. However, this estimation is sometimes inconsistent with the reality, since the transmission range will be much larger due to the incubation period of the virus.

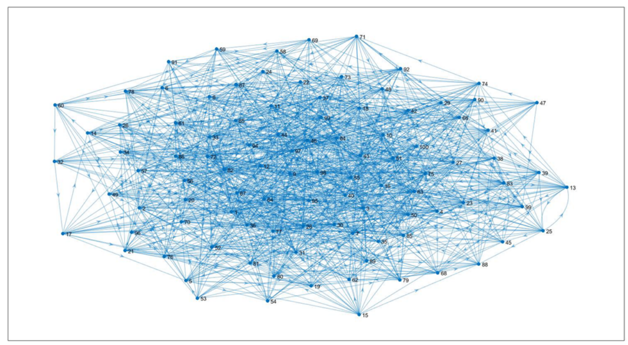


Fig. 5 Social network of 100 nodes

Then, we simulate how a person infects others around him. We assume the number of vertices in the transmission networks is 100. According to sociological research, we assume that the out-degrees and in-degrees of each vertex are around 25 originally. We consider the transmission range of infected person 1. Choose one of the remaining 99 healthy people, assuming the 100th one. We will explore how many people the virus needs to transmit from the first person to the 100th person. This is equivalent to calculating the distance between them in an uncertain digraph.

Let $\mathbb{D} = \{\mathcal{V}, \mathcal{A}\}$ be an uncertain digraph with 100 vertices. All the arcs exist with degree of uncertain measure that two people have close contact. We calculate the value of $\eta^k_{1,100}$ and the distribution function of uncertain variable $d(1, 100)$ using Algorithm 1 and 2. The original social network of 100 vertices is shown in Fig. 5. By algorithm 1, the value of $\eta^2_{1,100}$

is 0.1250. The same is true of the calculation of $\eta^k_{1,100}$. So we can get distribution function of $d(1, 100)$. Distribution function of $d(1, 100)$ is

$$\Phi_{(1,100)}(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.1250, & \text{if } 1 \le x < 2 \\ 0.3631, & \text{if } 2 \le x < 3 \\ 0.5781, & \text{if } 3 \le x < 4 \\ 0.8340, & \text{if } 4 \le x < 5 \\ 0.9006, & \text{if } 5 \le x < 6 \\ 0.9580, & \text{if } 6 \le x < 7 \\ 0.9681, & \text{if } x \ge 7, \end{cases}$$

whose image is shown in Fig. 6. Through calculation, we find that $d(1, 100)$ does not exceed 7 with an uncertain measure of 0.9580. In other words, patient 1 will infect the healthy person 100 through up to six times of transmission. In fact, if we choose other healthy people, the calculation result is almost the same, which means that an infected person will infect the whole community by seven rounds of transmission at most.

4 Shortest path problem in uncertain random digraph

In this section, we will propose the model of uncertain random digraph, and discuss the shortest path problem in uncertain random digraphs.

4.1 Problem description

In an uncertain random digraph with n vertices, all arcs are independent, and some arcs exist with degrees in probability measure while other arcs exist with degrees in uncertain measure.

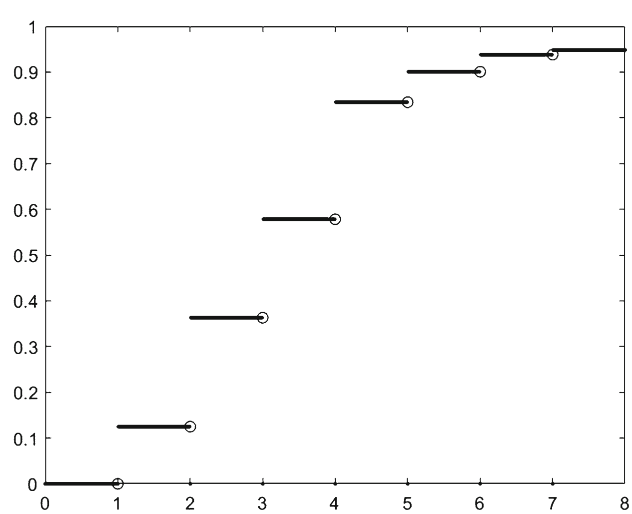


Fig. 6 Distribution function of $d(1, 100)$

We define two disjoint collections of arcs,

$$\mathcal{U} = \{(i, j) \mid 1 \le i, j \le n \text{ and}(i, j)\text{are uncertain arcs}\},$$
$$\mathcal{R} = \{(i, j) \mid 1 \le i, j \le n \text{ and}(i, j)\text{are random arcs}\},$$

with $\mathcal{U} \cup \mathcal{R} = \{(i, j) \mid 1 \le i, j \le n\}$. Note that deterministic arcs are regarded as special uncertain ones which exist with degrees in uncertain measure 1.

The *adjacency matrix* is an $n \times n$ matrix

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix},$$

where α_{ij} represent the truth values in uncertain measure or probability measure if the arc from i to j exist, $i, j = 1, 2, \ldots, n$. As digraphs considered in this paper are simple, $\alpha_{ii} = 0$, for $i = 1, 2, \ldots, n$, and $\mathcal{A}$ is normally asymmetric matrix, i.e., $\alpha_{ij} \ne \alpha_{ji}$, for $i \ne j$.

Definition 3 Assume $\mathcal{V}$ is the collection of vertices, $\mathcal{U}$ is the collection of uncertain arcs, $\mathcal{R}$ is the collection of random arcs, and $\mathcal{A}$ is the adjacency matrix. Then, the quartette $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ is said to be an uncertain random digraph.

For an uncertain random digraph $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$, write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

and

$$\mathbb{X} = \left\{ X \,\middle|\, \begin{array}{l} x_{ij} = 0 \text{ or } 1, \ \text{if } (i, j) \in \mathcal{R} \\ x_{ij} = 0, \ \text{if } (i, j) \in \mathcal{U} \\ x_{ii} = 0, \ i = 1, 2, \ldots, n \end{array} \right\}. \tag{8}$$

For any $X \in \mathbb{X}$, the *extension class* of X is defined by

$$X^* = \left\{ Y \,\middle|\, \begin{array}{l} y_{ij} = x_{ij}, \ \text{if } (i, j) \in \mathcal{R} \\ y_{ij} = 0 \text{ or } 1, \ \text{if } (i, j) \in \mathcal{U} \\ y_{ii} = 0, \ i = 1, 2, \ldots, n \end{array} \right\}. \tag{9}$$

As there are $n(n-1)$ possible arcs, there are $2^{n(n-1)}$ possible realization of arcs. Each one of them could be represented by a deterministic digraph, which is called a realization digraph. Since a digraph could be fully characterized by its adjacency matrix. For every $X \in \mathbb{X}$ and $Y \in X^*$, such that

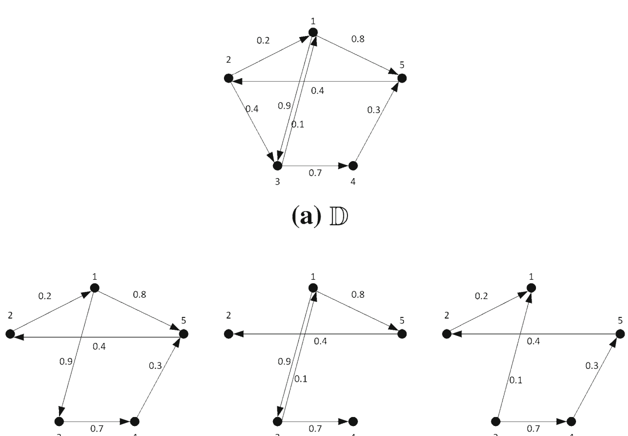
Y is the adjacency matrix of a realization digraph H. The chance measure of the event that the realization digraph H appears, is

$$\left(\prod_{(i,j)\in\mathcal{R}} w_{ij}(Y)\right)\left(\min_{(i,j)\in\mathcal{U}} w_{ij}(Y)\right), \tag{10}$$

where

$$w_{ij}(Y) = \begin{cases} \alpha_{ij}, & \text{if } y_{ij} = 1 \\ 1-\alpha_{ij}, & \text{if } y_{ij} = 0. \end{cases}$$

Example 2. Let $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ be an uncertain random digraph (shown in Fig.7), $A = \{(1, 2), (1, 3), (1, 5), (2, 3), (3, 1), (3, 4), (4, 5), (5, 2)\}$. where $\mathcal{V} = \{1, 2, 3, 4, 5\}$, $\mathcal{R} = \{(1, 2), (1, 3), (3, 4)\}$, $\mathcal{U} = \{(1, 5), (2, 3), (3, 1), (4, 5), (5, 2)\}$, and



(a) $\mathbb{D}$ **(b)** H_1 **(c)** H_2 **(d)** H_3

Fig. 7 Uncertain random digraph $\mathbb{D}$ and its realization digraphs

$$\mathcal{A} = \begin{pmatrix} 0 & 0.2 & 0.9 & 0 & 0.8 \\ 0 & 0 & 0.4 & 0 & 0 \\ 0.1 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.3 \\ 0 & 0.4 & 0 & 0 & 0 \end{pmatrix}.$$

As $\mathbb{D}$ has 8 edges, it has 2^8 realizations graphs, three of which are shown in Fig.7. The chance measure of the event that H_1 appears is

$$0.2 \times 0.9 \times 0.7 \times \min\{0.6, 0.9, 0.4, 0.8, 0.3\} = 0.0378.$$

Similarly, the chance measure of the events that H_2 and H_3 appear are 0.0504 and 0.0014, respectively.

Remark 3 If $\mathcal{U} = \emptyset$, an uncertain random digraph $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ becomes a *random digraph*. Then,

$$\mathbb{X} = \left\{ X \left| \begin{array}{l} x_{ij} = 0 \text{ or } 1,\ i, j = 1, 2, \ldots, n \\ x_{ii} = 0,\ i = 1, 2, \ldots, n \end{array} \right. \right\}. \tag{11}$$

For any $X \in \mathbb{X}$, X is the adjacency matrix of a realization digraph, which appears with probability

$$\prod_{1\le i,j\le n} w_{ij}(X).$$

If $\mathcal{R} = \emptyset$, an uncertain random digraph $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ becomes an *uncertain digraph* in Definition 1.

The shortest path length from i to j, denoted by $d(i, j)$, is a function of all x_{ij}. Since x_{ij} is a uncertain random digraph, $d(i, j)$ is an uncertain random variable. Therefore, we will naturally study the chance distribution and the properties of uncertain random variable $d(i, j)$.

4.2 Distribution of shortest path in uncertain random digraph

In this section, we will give the formula of the shortest path distribution function in uncertain random digraphs. According to Theorem 1, we have the following theorem.

Theorem 5 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ be an uncertain random digraph. For any vertices $i, j \in \mathcal{V}$, $\mathbb{X}$ is the class of matrices satisfying (8), X^* is the extension class of X satisfying (9). The chance measure that the distance from i to j is at most k is denoted by $\eta^k_{ij}(\mathbb{D})$. And we have*

$$\eta^k_{ij}(\mathbb{D}) = \sum_{X\in\mathbb{X}} \Big(\prod_{(i,j)\in\mathcal{R}} w_{ij}(X)\Big) f^*_k(X),$$

where

$$f_k^*(X) = \begin{cases} \sup\limits_{Y \in X^*, f_k(Y)=1} \min\limits_{(i,j)\in\mathcal{U}} w_{ij}(Y), \\ \quad \text{if} \sup\limits_{Y \in X^*, f_k(Y)=1} \min\limits_{(i,j)\in\mathcal{U}} w_{ij}(Y) < 0.5 \\ 1 - \sup\limits_{Y \in X^*, f_k(Y)=0} \min\limits_{(i,j)\in\mathcal{U}} w_{ij}(Y), \\ \quad \text{if} \sup\limits_{Y \in X^*, f_k(Y)=1} \min\limits_{(i,j)\in\mathcal{U}} w_{ij}(Y) \geq 0.5. \end{cases}$$

and

$$w_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0. \end{cases} \tag{12}$$

Remark 4 According to the duality axiom, the chance measure that i and j are in two different components is denoted by η_{ij}^{∞}. And we have $\eta_{ij}^{\infty} = 1 - \eta_{ij}^{k}(\mathbb{D})$.

Corollary 1 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{A})$ be a random digraph. $\mathbb{X}$ is the class of matrices satisfying (8), X^* is the extension class of X satisfying (9). w_{ij} satisfies (12). For any vertices $i, j \in \mathcal{V}$, we have*

1. the probability measure that the distance from i to j is at most k is

$$P\{d(i,j) \leq k\} = \sum_{X \in \mathbb{X}, g_k(Y)=1} \left(\prod_{1 \leq i,j \leq n} w_{ij}(X) \right).$$

2. the probability measure that i and j are in two different components is

$$P\{d(i,j) = \infty\} = 1 - P\{d(i,j) \leq n-1\},$$

According to Theorem 5 and the definition of distribution function, it is easy to calculate the distribution function of $d(i,j)$ by the following theorem.

Theorem 6 *Let $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ be an uncertain random digraph. For any two distinct vertices i and j, $\Phi_{(i,j)}(x)$ is the chance distribution function of uncertain random variable $d(i,j)$, and we have*

$$\Phi_{(i,j)}(x) = \begin{cases} 0, & \text{if } x < 1 \\ \eta_{ij}^{[x]}(\mathbb{D}), & \text{if } 1 \leq x < n-1 \\ \eta_{ij}^{n-1}(\mathbb{D}), & \text{if } x \geq n-1. \end{cases}$$

Although the method of formulating chance distribution function has been given, it seems to be theoretical and quite complicated because we have to concern all the realization digraphs. So we need a simplified formula to calculate final result. According to Theorem 5, after random arcs are fixed, the uncertain random digraph will become an uncertain digraph. As a result, we can calculated f^* by Algorithm 1. In next part, we will propose an efficient method to calculate $\eta_{ij}^{k}(\mathbb{D})$.

4.3 Algorithm and example

According to Algorithm 1 and Theorem 5, we give an efficient Algorithm to calculate $\eta_{ij}^{k}(\mathbb{D})$ of an uncertain random digraph.

Algorithm 3 Algorithm for calculating the $\eta_{ij}^{k}(\mathbb{D})$of an uncertain random digraph

Step 1. *Divide the realization digraph of uncertain random digraph $\mathbb{D} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A})$ into l parts by the arcs with degree of random measure, where $l = 2^{|\mathcal{R}|}$. For each part of the $\mathbb{D}$, we assume there exist only arcs with degree of random measure from the beginning. Let $j = 1$.*

Step 2. *Calculate f_k^* by Algorithm 1. Let $j = j + 1$.*

Step 3. *If $j = l$, stop and calculate the $\eta_{ij}^{k}(\mathbb{D})$ by Theorem 5; if $j < l$, set $j = j + 1$ and go to Step 2.*

Next, we will discuss the complexity of Algorithm 3. We assume $|\mathcal{R}(\mathbb{D})| = r$, m be the number of different uncertain values with which arcs exist. In Step 2, we need to calculate f_k^* by Algorithm 1, whose the complexity is $O(mn^2)$. Since there are 2^r iterations, the complexity of Step 2 is $O(2^r mn^2)$. In Step 3, there are 2^r multiplications and each multiplication has complexity of $O(r)$. Then, we have to calculate 2^r additions. Thus, the complexity of Algorithm 3 is

$$O(2^r mn^2) \times (O(r2^r) + 2^r) = 4^r rmn^2.$$

Example 2 $\mathbb{D} = \{\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{A}\}$ be an uncertain random digraph. $\mathcal{R} = \{(3,1), (2,1)\}$ and other arcs exist with degree of uncertain measure. We calculate the value of $\eta_{25}^{3}(\mathbb{D})$ using Algorithm 3 (Figs. 8, 9).

Since there are 2 arcs exist with degree of probability measure, we divide the realization of digraph into $2^2 = 4$

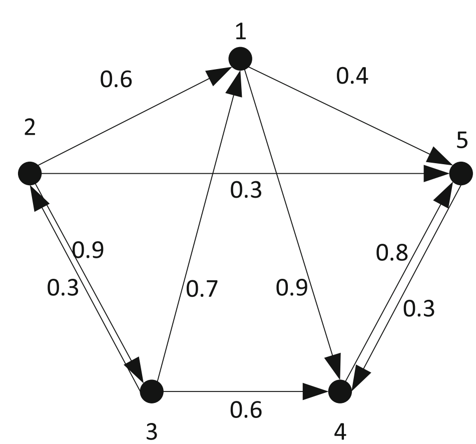


Fig. 8 Uncertain random digraph $\mathbb{D}$

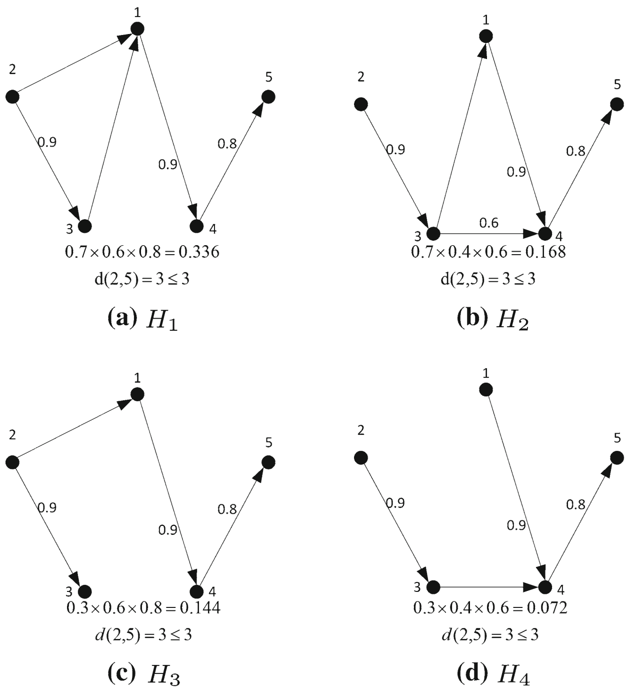


Fig. 9 The result of the four cases

cases. We calculate the f_k^*, respectively. We give the solution procedure of the case 1, another 3 cases can be solved by the same method. According to Algorithm 1, four maximum d_{ij}^k digraph are found and f_k^* equals to 0.8, 0.6, 0.8, 0.6, respectively. By Theorem 5, $\eta_{25}^3(\mathbb{D})$ is given by the following equation

$$\eta_{25}^3(\mathbb{D}) = 0.336 + 0.168 + 0.144 + 0.072 = 0.72.$$

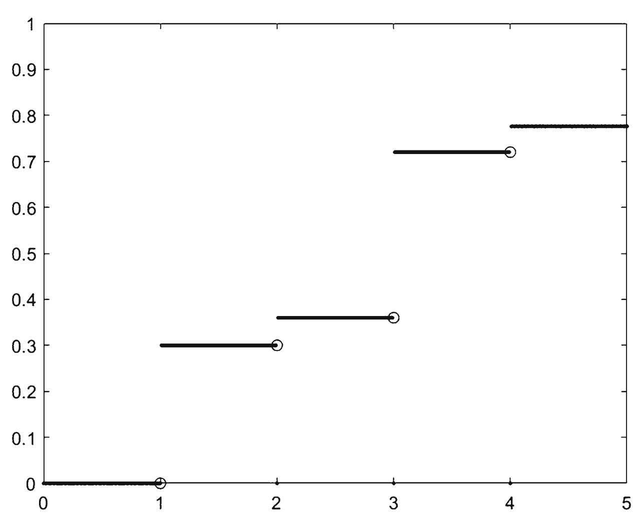


Fig. 10 Distribution function of $d(2,5)$

As a result, the distribution function $\Phi_{(2,5)}(x)$ is

$$\Phi_{(2,5)}(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.3, & \text{if } 1 \le x < 2 \\ 0.36, & \text{if } 2 \le x < 3 \\ 0.72, & \text{if } 3 \le x < 4 \\ 0.776, & \text{if } x \ge 4, \end{cases}$$

whose image is shown in Fig. 10.

5 Conclusions

In this paper, we discussed the shortest path problem in uncertain digraphs and uncertain random digraphs. Key properties of $d(i,j) \le k$ have been discussed. We gave the formulas for calculating $\eta_{ij}^k(\mathbb{D})$ and distribution function of $d(i,j)$. And an efficient polynomial algorithm was proposed to calculate $\eta_{ij}^k(\mathbb{D})$. When the number of vertices is small, the algorithm performs well and can greatly reduce the amount of computation. We also used this model to study the transmission of covid-19 in social networks.

Due to the strong dependence of probability model on the amount of data. If there is no or lack of historical data, the distribution obtained by using the model of probability theory will be far from the actual situation. In this case, the theory of uncertainty will have great theoretical value and practical significance. Of course, if there is enough historical data, the classical probability model may have better results.

Further research will focus on the following aspects. First, MDA algorithm is a more efficient algorithm. With the help of the property that the shortest path increases with the increase in edges in directed graphs, this method can be considered when studying the index properties of edge increase in other uncertain directed graphs in the future. Second, important indices, such as diameter, radius could be considered in the frame of uncertain random digraphs. Finally, similar models could be introduced, such as uncertain fuzzy graph.

Funding The authors have not disclosed any funding.

Data availibility Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

References

Abbaszadeh Sori A, Ebrahimnejad A, Motameni H (2020) The fuzzy inference approach to solve multi-objective constrained shortest path problem. J Intell Fuzzy Syst 38(4):4711–4720

Bollobás B (2011) Random graphs, 2nd edn. Cambridge University Press, Cambridge

Bondy A, Murty USR (2008) Graph theory. Springer, Berlin

Chen L, Peng J, Rao C, Rosyida I (2018) Cycle index of uncertain random graph. J Intell Fuzzy Syst 34:4249–4259

Chuang TN, Kung JY (2005) The fuzzy shortest path length and the corresponding shortest path in a network. Comput Oper Res 32(6):1409–1428

Deng Y, Chen Y, Zhang Y, Mahadevan S (2012) Fuzzy Dijkstra algorithm for shortest path problem under uncertain environment. Appl Soft Comput 12(3):1231–1237

Dubois D, Prade H (1980) Fuzzy sets and systems: theory and applications. Academic Press, New York

Ebrahimnejad A (2021) An acceptability index based approach for solving shortest path problem on a network with interval weights. RAIRO Oper Res 55(1):1767–1787

Ebrahimnejad A, Karimnejad Z, Alrezaamiri H (2015) Particle swarm optimisation algorithm for solving shortest path problems with mixed fuzzy arc weights. Int J Appl Decis Sci 8(2):203–222

Ebrahimnejad A, Madjid Tavava, Hamnidreza Alrezaamiri (2016) A novel artificial bee colony algorithm for shortest path problems with fuzzy arc weights. Measurement 93:48–56

Ebrahimnejad A, Tabatabaei S, Santos-Arteaga FJ (2020) A novel lexicographic optimization method for solving shortest path problems with interval-valued triangular fuzzy arc weights. J Intell Fuzzy Syst 39(1):1277–1287

Erdos P, Renyi A (1959) On random graphs. Publicationes Mathematicae 6:290–297

Esfahanian H (2008) On the evolution of graph connectivity algorithms

Gao Y (2011) Shortest path problem with uncertain arc lengths. Comput Math Appl 62(6):2591–2600

Gao X (2013) Cycle index of uncertain graph. Information 15(12):270–277

Gao X (2014) Regularity index of uncertain graph. J Intell Fuzzy Syst 27:1671–1678

Gao X (2016) Tree index of uncertain graphs. Soft Comput 20(4):1449–1458

Gao X, Gao Y (2013) Connectedness index of uncertain graph. Int J Uncertain Fuzz Knowl Based Syst 21(1):127–137

Gao Y, Qin Z (2016) On computing the edge-connectivity of an uncertain graph. IEEE Trans Fuzzy Syst 24(4):981–991

Gao R, Yao K (2016) Importance index of components in uncertain random systems. Knowl Based Syst 109(1):208–217

Gao Y, Yang L, Li S, Kar S (2015) On distribution function of the diameter in uncertain graph. Inf Sci 296:61–74

Gao R, Sun Y, Ralescu DA (2016) Order statistics of uncertain random variables with application to k-out-of-n system. Fuzzy Optim Decis Mak 16(2):1–23

Gao X, Guo C, Yin X, Yu X (2019) The computation on α-connectedness index of uncertain graph. Clust Comput 22(3):5691–5701

Gilbert EN (1959) Random graphs. Ann Math Stat 30(4):1141–1144

Hernandes F, Lamata MT, Verdegay JL, Yamakami A (2007) The shortest path problem on networks with fuzzy parameters. Fuzzy Sets Syst 158(14):1561–1570

Hou Y, Peng W (2014) Distance between uncertain random variables. Math Model Eng Probl 1:17–24

Ke H, Su T, Ni Y (2015) Uncertain random multilevel programming with application to production control problem. Soft Comput 19(6):1739–1746

Kolmogoroff A (1933) Grundbegriffe der Wahrscheinlichkeitsrechnung. Julius Springer, Berlin

Li H, Gao X (2020) On the vertex-connectivity of an uncertain random graph. IEEE Access 8:85504–85514

Li H, Zhang H (2020) On the edge-connectivity of an uncertain random graph. IEEE Access 99:1

Li H, Zhang B, Peng J (2018) On the matching number of an uncertain graph. Iran J Fuzzy Syst 15(2):89–108

Liu B (2007) Uncertainty theory, 2nd edn. Springer, Berlin

Liu B (2009) Some research problems in uncertainy theory. J Uncertain Syst 3:3–10

Liu Y (2013a) Uncertain random programming with applications. Fuzzy Optim Decis Mak 12(2):153–169

Liu Y (2013b) Uncertain random variables: a mixture of uncertainty and randomness. Soft Comput 17(4):625–634

Liu B (2014) Uncertain random graph and uncertain random network. J Uncertain Syst 8(1):3–12

Liu Y, Ralescu DA (2014) Risk index in uncertain random risk analysis. Int J Uncertain Fuzz Knowl Based Syst 22(4):491–504

Matula DW (1987) Determining edge connectivity in 0 (nm). In: Symposium on Foundations of Computer Science

Okada S, Soper T (2000) A shortest path problem on a network with fuzzy arc lengths. Fuzzy Sets Syst 109(1):129–140

Qin Z (2018) Uncertain random goal programming. Fuzzy Optim Decis Mak 17(4):375–386

Rosyida I, Peng J, Chen L, Widodo W, Indrati ChR, Sugeng KA (2018) An uncertain chromatic number of an uncertain graph based on α-cut coloring. Fuzzy Optim Decis Mak 17(1):103–123

Sheng Y, Gao Y (2016) Shortest path problem of uncertain random network. Comput Ind Eng 99:66

Sheng Y, Mei X (2020) Uncertain random shortest path problem. Soft Comput 24(4):66

Soifer A (2009) The mathematical coloring book. Springer, New York

Wen M, Kang R (2016) Reliability analysis in uncertain random system. Fuzzy Optim Decis Mak 15(4):491–506

Yao K, Gao J (2016) Law of large numbers for uncertain random variables. IEEE Trans Fuzzy Syst 24(3):615–621

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353

Zhang B, Peng J (2012) Euler index in uncertain graph. Appl Math Comput 218(20):10279–10288

Zhang B, Peng J, Li S (2017) Euler index of uncertain random graph: concepts and properties. Int J Comput Math 94:217–229

Zhou J, Yang F, Wang K (2014a) An inverse shortest path problem on an uncertain graph. J Netw 9(9):2353–2359

Zhou J, Yang F, Wang K (2014b) Multi-objective optimization in uncertain random environments. Fuzzy Optim Decis Mak 13(4):397–413