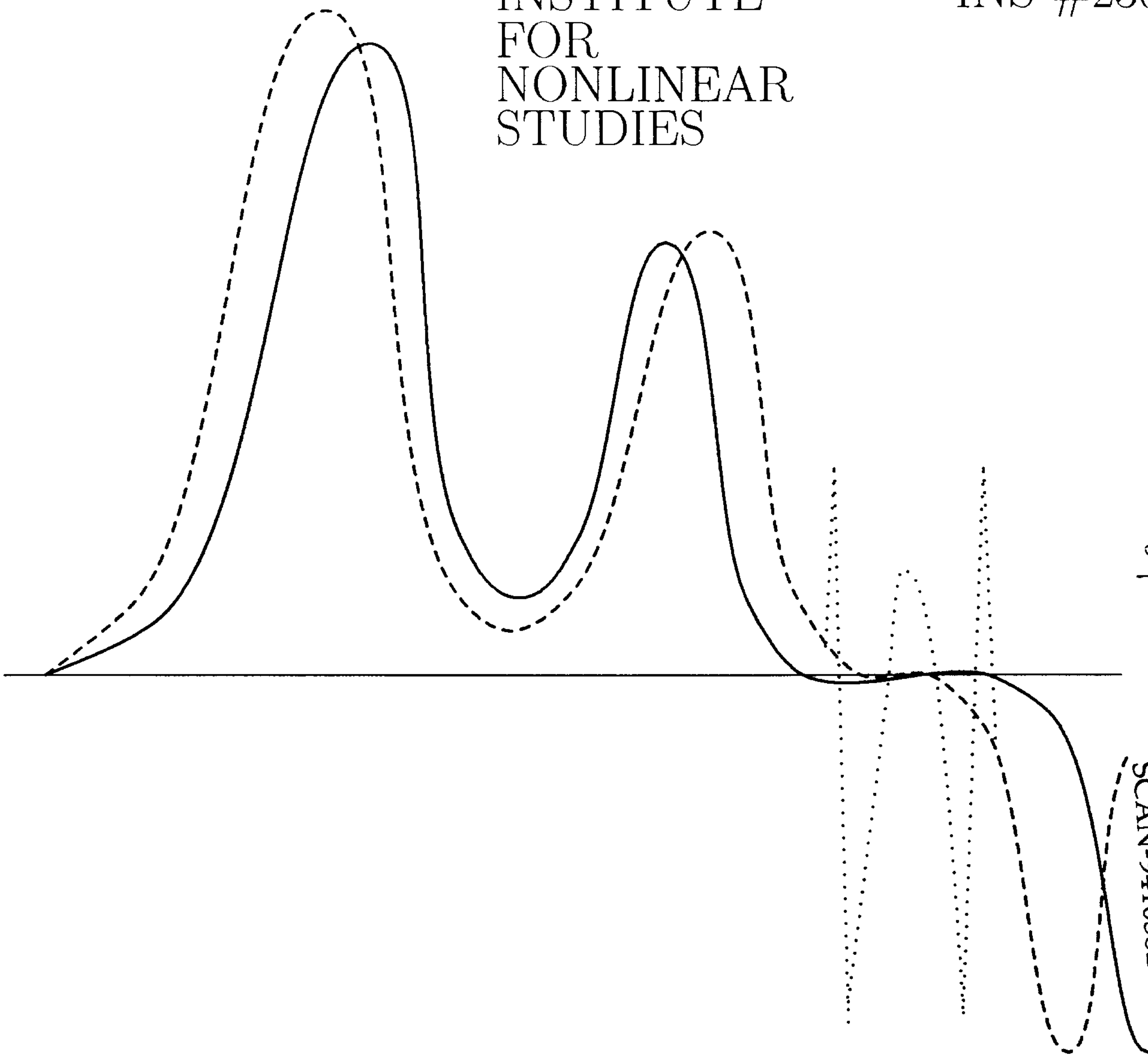
INSTITUTE FOR NONLINEAR STUDIES

INS #230



On the Simplest Integrable Equation in 2 + 1

by

A.S. Fokas

May 1993

CLARKSON UNIVERSITY
Potsdam, New York 13699, U.S.A.

On the Simplest Integrable Equation in 2 + 1

A.S. Fokas
Department of Mathematics and Computer Science
and Institute for Nonlinear Studies
Clarkson University
Potsdam, New York 13699-5815, U.S.A.

May 1993

INS #230

Abstract An integrable two dimensional generalization of the nonlinear Schrödinger equation is discussed. This equation is the simplest scalar evolution equation in two dimensions, which can be integrated by the inverse spectral method.

The nonlinear Schrödinger equation (NLS)

$$iq_t + q_{xx} - 2\lambda |q|^2 q = 0, \quad \lambda = \pm 1, \tag{1}$$

is considered as the simplest scalar evolution equation in one spatial dimension, which can be solved by the inverse spectral method [1]. I claim that the simplest scalar evolution equation in two spatial dimensions, which can be solved by the inverse spectral method is the equation

$$iq_t + q_{xx} - 2\lambda q \int_{-\infty}^{y} dy' |q(x,y',t)|_x^2 = 0, \quad \lambda = \pm 1. \tag{2}$$

Under the reduction $y = x$, equation (2) becomes (1).

Equation (2) is a particular case of the following integrable equation for the scalar function $q(\xi,\eta,t)$,

$$iq_t - (\alpha-\beta)q_{\xi\xi} + (\alpha+\beta)q_{\eta\eta} - 2\lambda q \left[(\alpha+\beta)\int_{-\infty}^{\xi} d\xi' |q|_\eta^2 - (\alpha-\beta)\int_{-\infty}^{\eta} d\eta' |q|_\xi^2 + u_1(\eta,t) + u_2(\xi,t)\right] = 0. \tag{3}$$

In equation (3), $\lambda = \pm 1$, α, β are arbitrary real constants, and $u_1(\eta,t)$, $u_2(\xi,t)$ are arbitrary real-valued functions of the arguments indicated.

Equation (3) contains three interesting particular cases: (i) $\alpha = \beta = \frac{1}{2}$, $u_1 = u_2 = 0$, yields equation (2). (ii) $\alpha = 0$, $\beta = 1$, yields the celebrated Davey-Stewartson (DS) I equation [2]. (iii) $\alpha = 1$, $\beta = 0$ yields an integrable equation recently obtained in [3] by using ideas [4] of the direct linearizing method [5] and called, by the authors of [3], DSIII. It is claimed in [3] that this equation is new. However, it was actually discovered much earlier by Santini and the author using the symmetry approach [6] (DSIII is precisely equation (3.39d) of [6a]).

It is clear from the symmetry approach [6] that the linear combination of DSI and DSIII, i.e. equation (3), is also integrable. Here, I shall rederive equation (3), as well as its Lax pair and the associated nonlocal Riemann-Hilbert (RH) problem, using the dressing method.

The starting point of the dressing method is to postulate the nonlocal Riemann-Hilbert (RH) [7]-[9] problem

$$\mu^+(x,y,t,k) = \int_{\mathbb{R}} dl \mu^-(x,y,t,l) F(x,y,t,k,l), \quad k\epsilon\mathbb{R}, \tag{4.a}$$

$$\mu(x,y,t,k) \sim I + \frac{\mu^{(1)}(x,y,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty. \tag{4.b}$$

In equation (4.a), μ^+, μ^-, F are matrix-valued functions, $+(-)$ denotes holomorphicity in the upper(lower) half of the complex k-plane, and $\int_{\mathbb{R}}$ denotes integration over the real axis. I assume that for given F, equation (4.a) can be solved uniquely for μ^+ and μ^-. I shall denote by $P_F\mu^-$ the rhs of equation (4.a).

Let the operators D_x, D_y, D_t be defined by

$$D_x\mu = \mu_x + ik\mu\sigma_3, \quad D_y\mu = \mu_y - ik\mu, \quad D_t\mu = \mu_t - ik^2\mu(\alpha I + \beta\sigma_3), \tag{5}$$

where α, β are real scalar constants, $I = diag(1,1)$, $\sigma_3 = diag(1,-1)$. The function F is specified by the requirement that the operators D_x, D_y, D_t commute with the integral operator P_F. This implies

$$F_x = i(l\sigma_3 F - kF\sigma_3), \quad F_y = i(k-l)F, \quad F_t = ik^2 F(\alpha I + \beta\sigma_3) - il^2(\alpha I + \beta\sigma_3)F. \tag{6}$$

I look for linear combinations involving the operators D_x, D_y, D_t, which vanish as $k \to \infty$. Then, since I assume that the homogeneous RH problem has only the trivial solution, these combinations must be identically zero. The simplest such combination is $D_x\mu + \sigma_3 D_y\mu + Q\mu$; its $0(k)$ term cancels as $k \to \infty$, also if $Q = i[\sigma_3, \mu^{(1)}]$, its $0(1)$ term cancels as $k \to \infty$, and hence this combination is zero,

$$\mu_x + \sigma_3\mu_y + ik[\mu, \sigma_3] + Q\mu = 0, \tag{7}$$

$$Q = i[\sigma_3, \mu^{(1)}]. \tag{8}$$

Equation (7) is the time-independent part of the Lax pair [10]. I note for future use that the $0(1/k)$ term of equation (7) yields

$$\mu_x^{(1)} + \sigma_3\mu_y^{(1)} + i[\mu^{(2)}, \sigma_3] + Q\mu^{(1)} = 0. \tag{9}$$

The time dependent part of the Lax pair is given by

$$D_t\mu - i(\alpha I + \beta\sigma_3)D_y^2\mu - BD_y\mu - A\mu = 0,$$

where the matrix-valued functions A and B are chosen in such a way that the $0(k)$ and $0(1)$ terms of this equation vanish as $k \to \infty$. Writing explicitly the operators $D_t\mu$ and $D_y\mu$ and then using equation (7) to eliminate the k^2 term, the above equations becomes

$$\mu_t = i(\alpha I + \beta\sigma_3)\mu_{yy} + k(2\alpha\mu_y - \beta\mu_x + \beta\sigma_3\mu_y - \beta Q\mu - iB\mu) + B\mu_y + A\mu.$$

Eliminating the $0(k)$ terms as $k \to \infty$, implies $B = i\beta Q$. Thus

$$\mu_t = i(\alpha I + \beta\sigma_3)\mu_{yy} + k(2\alpha\mu_y - \beta\mu_x + \beta\sigma_3\mu_y) + i\beta Q\mu_y + A\mu. \tag{9}$$

Eliminating the $0(1)$ terms as $k \to \infty$, implies

$$A = -2\alpha\mu_y^{(1)} + \beta\mu_x^{(1)} - \beta\sigma_3\mu_y^{(1)}. \tag{10}$$

Equation (9), where A is defined by equation (10), is the t-part of the Lax pair. The compatibility condition of equations (7) and (9) yields a system of two equations for the two entries of Q. An alternative way to obtain this system of equations is to look at the $0(1/k)$ term of equation (9) as $k \to \infty$,

$$\mu_t^{(1)} = i(\alpha I + \beta\sigma_3)\mu_{yy}^{(1)} + 2\alpha\mu_y^{(2)} - \beta\mu_x^{(2)} + \beta\sigma_3\mu_y^{(2)} + i\beta Q\mu_y^{(1)} + A\mu^{(1)}. \tag{11}$$

Let

$$Q = \begin{pmatrix} 0 & q_1 \\ q_2 & 0 \end{pmatrix}, \quad \mu^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mu^{(2)} = \begin{pmatrix} a^{(2)} & b^{(2)} \\ c^{(2)} & d^{(2)} \end{pmatrix}. \tag{12}$$

Equations (8) and (9) imply

$$b = \frac{q_1}{2i}, \quad c = -\frac{q_2}{2i}, \quad a = \frac{1}{4i}\int_{-\infty}^{\xi} d\xi' q_1q_2, \quad d = -\frac{1}{4i}\int_{-\infty}^{\eta} d\eta' q_1q_2,$$

$$b^{(2)} = -\frac{q_{1\xi}}{2} + \frac{q_1}{8}\int_{-\infty}^{\eta} d\eta' q_1q_2, \quad c^{(2)} = -\frac{q_{2\eta}}{2} + \frac{q_1}{8}\int_{-\infty}^{\xi} d\xi' q_1q_2, \tag{13}$$

where ξ and η are characteristic coordinates, i.e.

$$\xi = x + y, \quad \eta = x - y. \tag{14}$$

Using equations (13), the off-diagonal elements of equation (11) yield

$$iq_{1_t}-(\alpha-\beta)q_{1_{\xi\xi}}+(\alpha+\beta)q_{1_{\eta\eta}}-\frac{q_1}{2}\left[(\alpha+\beta)\int_{-\infty}^{\xi}d\xi'(q_1q_2)_\eta-(\alpha-\beta)\int_{-\infty}^{\eta}d\eta'(q_1q_2)\xi\right]=0,$$

$$iq_{2_t}+(\alpha-\beta)q_{2_{\xi\xi}}-(\alpha+\beta)q_{2_{\eta\eta}}+\frac{q_2}{2}\left[(\alpha+\beta)\int_{-\infty}^{\xi}d\xi'(q_1q_2)_\eta-(\alpha-\beta)\int_{-\infty}^{\eta}d\eta'(q_1q_2)\xi\right]=0. \quad (15)$$

The reduction $q_2 = 4\lambda\bar{q}_1$, yields equation (3) with $u_1 = u_2 = 0$. A simple modification of the dressing method discussed here, yields an equation similar to equation (15) but with the extra terms $u_1(\eta,t)+u_2(\xi,t)$. This extended dressing method for the case of DSI is discussed in [11].

Remarks (i) It can be verified directly [12] that equation (2) and the equation obtained from (2) by exchanging x and y commute.

(ii) Equation (4) can be used to derive large classes of solutions of equation (15). Indeed, let F satisfy the linear equations (6), and let μ be a solution of the nonlocal RH problem (4). Then equation (8) yields a solution of equation (15), where $\mu^{(1)} = \lim_{k\to\infty} k(\mu - I)$.

(iii) DSI and DSIII support certain localized solutions [13], [14], called dromions in [15]. Such solutions also exist for equation (3).

(iv) The Cauchy problem for equation (3), where $q(\xi,\eta,0)$, $u_1(\eta,t)$, and $u_2(\xi,t)$ are given, can be solved by following the method developed in [15].

(v) Equations in 2 + 1 admit two hierarchies of symmetries. The starting members of these hierarchies are q_x and q_y (these symmetries are the consequence of invariance under x and y translations). DSI and DSIII are the next members of these hierarchies. In this sense, the existence of DSIII is a novelty of multidimensions (since DSIII comes from q_y). In terms of dressing, DSIII comes from replacing k^2 by $k^2 I$ in the D_t operator. This modification yields non trivial results in multidimensions only, which again shows that the existence of DSIII is a multidimensional effect. The fact that the above simple modification of the dressing method gives no new equations in $1+1$, is perhaps the reason why it was not considered earlier.

(vi) The discussion of (v) is valid for any $2+1$ integrable equation possessing a matrix Lax pair. For scalar equations such as the Kadomtsev-Petviashvili (KP) equation, the situation is similar. The symmetry approach again yields two hierarchies associated with q_x and q_y. From the dressing point of view, these hierarchies correspond to different powers of k in the D_t operator (q_x and q_y correspond to k and k^2, while KP and the next member of the q_y hierarchy correspond to k^3 and k^4).

Acknowledgements

This work was partially supported by the National Science Foundation under Grant Number DMS-9204075, and by the Air Force Office of Scientific Research under Grant Number F49620-93-1-0088-DEP.

References

[1] V. Zakharov and A. Shabat, Soviet Phys. JETP, **34**, 62 (1972).

[2] A. Davey and K. Stewartson, Proc. R. Soc. London Ser. A **338**, 101 (1974).

[3] M. Boiti, F. Pempinelli, P.C. Sabatier, Nonlinear Evolution Equations from an Inverse Spectral Problem, preprint (1993).

[4] P.C. Sabatier, Inverse Problems, **8**, 263 (1992); Phys. Lett. A, 345 (1992).

[5] A.S. Fokas and M.J. Ablowitz, Phys. Rev. Lett. **47** 1096 (1981).

[6] P.M. Santini and A.S. Fokas, Commun. Math. Phys. **115**, 375-419 (1988); A.S. Fokas and P.M. Santini, Commun. Math. Phys. **116**, 449-474 (1988).

[7] S.V. Manakov, Physica D **3**, 420 (1981).

[8] A.S. Fokas and M.J. Ablowitz, Stud. Appl. Math. **69**, 211 (1983).

[9] V.E. Zakharov and S.V. Manakov, Funct. Anal. App. **19**, No. 2, 11 (1985).

[10] P. Lax, Comm. Pure Appl. Math. **21**, 467 (1968).

[11] A.S. Fokas and V.E. Zakharov, J. Nonlinear Sci., **2**, 109-134 (1992).

[12] I. Dorfman, private communication.

[13] M. Boiti, J. Leon, L. Martina and F. Pempinelli, Phys. Lett. A **132**, 432-439 (1988).

[14] F. Pempinelli, Localized Soliton Solutions for DSI and DSIII Equations, preprint (1943).

[15] A.S. Fokas and P.M. Santini, Physica D **44**, 99-130 (1990).