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ON THE SIMULTANEOUS EXISTENCE OF FULL AND PARTIAL CAPITAL AGGREGATES

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1. Introduction

Some years ago Fisher (1965) examined the question of the existence of an aggregate capital stock in the context of a model in which technology was embodied in capital and capital was fixed (i.e., firm-specific) but labour was assigned to firms so as to maximize output. This was the beginning of a more general examination into the existence of aggregate production functions.[1] Not surprisingly, the conditions for such existence turned out to be quite restrictive. For example, the best known theorem in this area[2] states that under constant returns with one kind of capital per firm, a necessary and sufficient condition for the existence of an aggregate capital stock is that all technical differences be capital augmenting, with one unit of a different kind of capital being equivalent to a fixed number of units of a given type. Similar results hold for more general models.

Now, when each firm employs more than one type of capital, there are two related, but certainly not identical questions which arise. The first of these is that of the existence of a full capital aggregate; the second is that of the existence of an aggregate including some but not all of the capital types. To fix ideas, one may think of these as the question of the existence of a meaningful aggregate called "capital" on the one hand and of the existence of "equipment" or "plant" aggregates or both on the other. It is not the case that the conditions for one kind of aggregation are stronger than those for the other.[3]

Thus, to consider two particular examples, let the vth firm produce output y(v) according to the three-factor production function

(1.1) $$y(v) = f^v(K_1(v), K_2(v), L(v))$$

where $K_1(v)$ and $K_2(v)$ denote the amounts of two different types of capital (plant and equipment) which the vth firm has, and L(v) denotes the amount of labour assigned to the firm. Labour--assumed homogeneous--is allocated to firms to maximize the sum of outputs--also assumed homogeneous[4]--but the v superscript on the production function indicates the fact that technology is embodied in the capital stocks so that $K_j(v)$ and $K_j(v')$ can generally be physically quite different for $v \neq v'$, j = 1, 2 (physically different equipment used by different firms, for example). $f^v(\cdot,\cdot,\cdot)$ is assumed continuously twice differentiable with positive first derivatives and $f^v_{LL} < 0$, where subscripts indicate differentiation.

Now suppose that one of the firms, say the hth, has a production function of the form:

(1.2) $$f^h(K_1(h), K_2(h), L(h)) = A\, K_1(h)^{\alpha} K_2(h)^{\beta} L(h)^{\gamma} \quad .$$

Then no partial capital aggregate exists, no matter what other firms look like. A full capital aggregate will exist, however, so long as every other firm has a production function in the form:

(1.3) $$f^v(K_1(v), K_2(v), L(v)) = \{J^v(K_1(v), K_2(v))\}L(v)^{\gamma}$$ [5]

where $J^v(\cdot, \cdot)$ is monotonic in its arguments.

On the other hand, suppose instead that all firms (including the hth) have production functions of the form:

$$(1.4) \quad f^v(K_1(v), K_2(v), L(v)) = H_1^v(K_1(v)\{H_2^v(K_2(v) + c_v L(v)\}^{\gamma_v}$$

where $H_1^v(\cdot)$ and $H_2^v(\cdot)$ are monotonic functions and c_v and γ_v parameters. Then a partial aggregate exists for K_2. If, further, γ_v is the same for all firms, a partial aggregate exists for K_1 as well. Yet even in this case there exists no full aggregate of K_1 and K_2 together.[6]

In general, therefore, the cases permitting full and partial capital aggregation are different ones--both restrictive but not identical. A natural question, however, is that of what class of cases simultaneously permits the construction of <u>both</u> full and partial aggregates, fulfilling a macro-economist's production fantasies. That question was not examined in [1] and is the subject of the present paper.

Briefly, the results turn out as follows. First, there do exist such cases. Further, simultaneous existence of a total aggregate and one partial aggregate implies the existence of another partial aggregate consisting of all the kinds of capital left out of the first partial aggregate.[7] Similarly, the simultaneous existence of one partial aggregate and a subaggregate of that partial aggregate implies the existence of the complementary sub-partial aggregate. These tidings of apparent good news should not be overrated, however. They come about because the conditions for the existence of a total and even one partial aggregate are already extremely restrictive. Thus, for example, in the three-factor case under constant returns, a total aggregate and a K_1-aggregate will both exist if and only if all firms have production functions of the form:

$$(1.5) \quad f^{v}(K_1(v), K_2(v), L(v)) = F(b_v K_1(v) + c_v K_2(v), L(v))$$

where $F(\cdot,\cdot)$ is the same for all v and b_v and c_v are parameters which are allowed to vary. In this case, it is plain that $K_1(v)$ and $K_2(v)$ are perfect substitutes within firm v; a unit of one can be treated as equivalent to a fixed number of units of the other. Since $F(\cdot,\cdot)$ does not vary, $K_j(v)$ and $K_j(v')$ are also related in this way ($j = 1, 2; v \neq v'$) and we are looking at a generalization of the capital-augmenting theorem. In this context, it is hardly surprising that a K_2-aggregate exists as well.

This is the most restrictive case, however. I derive it below as a consequence of more general considerations, some of which lead to conditions at least somewhat less onerous.

2. The General Case: Preliminary Results

While the case of two kinds of capital within the firm is an interesting one, it is best to begin with the more general case of several kinds. The assumption of a single homogeneous labour and a single homogeneous output will be maintained for simplicity; this does not affect the results. Also for simplicity, I assume that all factors have positive marginal products in every firm and that it is always efficient to assign a positive amount of labour to every firm.[8]

Each firm, then, has m types of capital as well as labour. There are n firms. The vth firm's production function (twice continuously differentiable with strictly diminishing returns to labour) is given by:

$$(2.1) \qquad y(v) = f^{v}(K_1(v), \ldots, K_m(v), L(v)) \quad .$$

Labour is assigned to firms to maximize total output. This makes maximized output for the system as a whole a function of total labour and of the nm amounts of each of the m capital stocks possessed by each of the n firms. The questions being examined have to do with the simplification of that function by aggregation over capital types and over firms.

It is convenient to begin by stating the necessary and sufficient conditions for the existence of an aggregate consisting of the first p capital types.[9] They are:

Lemma 2.1: Necessary and sufficient conditions for the existence of an aggregate consisting of the first p capital types, $1 \leq p \leq m$, are given by (a), (b), and (c) below.

(a) Such an aggregate exists within each of the n firms taken separately. That is, for all v = 1, ..., n,

(2.2) $$f^v(K_1(v), \ldots, K_m(v), L(v)) = F^v(J(v), K_{p+1}(v), \ldots, K_m(v), L(v))$$

where

(2.3) $$J(v) = J^v(K_1(v), \ldots, K_p(v)).$$

(b) For all v = 1, ... n, all i = 1, ..., p, and all j = p + 1, ..., m,

(2.4) $$f^v_{K_iK_j} - \frac{f^v_{K_iL}\, f^v_{K_jL}}{f^v_{LL}} \equiv 0$$

(where the subscripts denote differentiation in the obvious way).

(c) There exists a function $g(\cdot)$, independent of v and of i, such that, for all v = 1, ..., n, i = 1, ..., p,

(2.5) $$\frac{f^v_{K_iL}}{f^v_{K_i}\, f^v_{LL}} = g(f^v_L) .$$

Proof: Immediate from [1, Theorem 6.1, p. 281]. The fact that $g(\cdot)$ is independant of i as well as of v follows from (a).

Lemma 2.2: In the presence of (a) of Lemma 2.1, (b) and (c) of Lemma 2.1 can be respectively replaced by (b') and (c') below.

(b') For all $v = 1, \ldots, n$ and all $j = p + 1, \ldots, m$,

$$(2.4') \qquad F^v_{JK_j} - \frac{F^v_{JL} F^v_{K_j L}}{F^v_{LL}} \equiv 0.$$

(c') There exists a function $g(\cdot)$, independent of v, such that, for all $v = 1, \ldots, m$,

$$(2.5') \qquad \frac{F^v_{JL}}{F^v_J F^v_{LL}} = g(F^v_L) \quad .$$

Proof: Obvious.

Lemmas 2.1 and 2.2 show that aggregation over the first p capitals for the system as a whole can be thought of in two stages. First there is aggregation at the level of the individual firm. Second, the individual firm aggregates, $J(v)$, are themselves aggregated over firms. This fact permits some notational simplification, since it means that--so far as the second stage of such aggregation (conditions (b) and (c) of Lemma 2.1) is concerned, we might as well take $p = 1$. I shall occasionally do so and later reinterpret the results.

It is also convenient to state:

Lemma 2.3: In the presence of (a) of Lemma 2.1, if there exists a $g(\cdot)$ such that (2.5') holds, then (2.5) holds with that same $g(\cdot)$ for all $i = 1, \ldots, p$. Further, if there exists a $g(\cdot)$ such that (2.5) holds for any

$i = 1, \ldots, p$, then (2.5') holds with that same $g(\cdot)$ and hence (2.5) also holds with that same $g(\cdot)$ for <u>all</u> $i = 1, \ldots, p$.

<u>Proof</u>: Direct computation.

In view of the discussion following Lemma 2.2, this results applies even if the K_i in (2.5) are themselves subaggregates.

3. Complementary Subaggregates in the General Case

We can now state immediately:

<u>Theorem 3.1</u>: Suppose an aggregate consisting of the first <u>two</u> capital types exists. Suppose further that an aggregate consisting only of the second capital type exists. Then an aggregate consisting only of the first capital type also exists.

<u>Proof</u>: (a) of Lemma 2.1 is trivially satisfied here. Considering (b), it is evident that (2.4) holds for $i = 1$, $j = 3, \ldots, m$ by the existence of an aggregate containing the first two capital types. (2.4) is satisfied for $i = 1$, $j = 2$ because it is symmetric in i and j and an aggregate consisting of the second capital type only exists. So (b) is also satisfied. Finally, (2.5) is satisfied for $i = 1$, since an aggregate of the first two capital types exists, implying that (2.5) is satisfied for $i = 1, 2$.

<u>Corollary 3.1</u>: Suppose that an aggregate consisting of the first q capital types exists for some q, $1 < q \leq m$. Suppose further that an aggregate consisting of types $(p+1, \ldots, q)$ also exists for some p, $1 \leq p < q$. Finally, suppose that an aggregate consisting of the first p capital types exists for each firm taken separately. Then an aggregate consisting of the first p capital types exists for the system as a whole.

<u>Proof</u>: Follows from Lemmas 2.1-2.2 and Theorem 3.1.

Corollary 3.2: Suppose that an aggregate consisting of the first q capital types exists for some q, $1 < q \leq m$. Suppose further that an aggregate consisting of capital types (2, ..., q) also exists. Then an aggregate consisting of capital of type 1 exists as well.

Proof: Obvious.

Thus, in particular, if a full capital aggregate and also a partial capital aggregate exists for the system as a whole, then the complementary partial capital aggregate will also exist for the system as a whole provided that it exists at the level of each firm. This latter condition is guaranteed if only one capital type is left out of the first partial aggregate. Roughly speaking, provided that they all exist at the level of the individual firm, the existence of a full capital aggregate plus the existence of a plant aggregate implies the existence of an equipment aggregate (where all capital types are either plant or equipment). Similarly, the existence of a full aggregate and of an equipment aggregate implies the existence of a plant aggregate.

This seems like relatively good news, in contrast to the usual results as to the restrictive conditions required for aggregation, and in a way that is true. Indeed, the news will appear to get better, for it will turn out that the condition in Corollary 3.1 that an aggregate of the first p capital types exist at the firm level is redundant. Alas, reconsideration of what is going on reveals that this occurs only because the simultaneous existence of two aggregates, one of which involves a subset of the capital types involved in the other, is already extremely restrictive.

As might be expected in view of Lemma 2.3, such further results involve (a) and (b) of Lemma 2.1 which restrict the form of individual production functions, in contrast to (c) which, while also so restrictive, involves a condition across production functions. As a result, it is convenient to drop the v superscript and argument in what follows. We prove:

Lemma 3.1: Let $f(\cdot)$ be one of the production functions. Suppose that for some q, $1 < q \leq m$, $f(\cdot)$ can be written as:

$$f(K_1, \ldots K_m, L) = F(J(K_1 \ldots K_q), K_{q+1}, \ldots K_m, L) \tag{3.1}$$

and also, for every $i = 1, \ldots, q-1$,

$$f_{K_q K_i} - \frac{f_{K_q L} f_{K_i L}}{f_{LL}} \equiv 0. \tag{3.2}$$

Then $J(K_1 \ldots K_q)$ can be written as

$$J(K_1 \ldots K_q) = R(K_q) + S(K_1, \ldots, K_{q-1}). \tag{3.3}$$

Proof: Using (3.1), (3.2) becomes

$$0 = F_J J_{K_q K_i} + F_{JJ} J_{K_q} J_{K_i} - \frac{F_{JL}^2 J_{K_q} J_{K_i}}{F_{LL}} \tag{3.4}$$

whence

$$\frac{J_{K_q K_i}}{J_{K_q} J_{K_i}} = \frac{F_{JL}^2}{F_J F_{LL}} - \frac{F_{JJ}}{F_J} \tag{3.5}$$

and this holds for all $i = 1, \ldots, q-1$. However, the right-hand side of (3.5) depends only on J, $K_{q+1}, \ldots, K_m$, and L and not (given the value of J) on $K_1, \ldots, K_q$. Hence (3.5) can only hold if both sides are the same constant. This implies that the left-hand side must be the same constant for all $i = 1, \ldots, q-1$. There are two cases to consider.

(a) The constant involved is zero. Then $J_{K_q K_i} \equiv 0$ for all $i = 1, \ldots, q-1$ and the desired result is immediate.

(b) The constant involved is $C \neq 0$. Then

$$\frac{\partial \log J_{K_q}}{\partial K_i} = C \frac{\partial J}{\partial K_i} \tag{3.6}$$

for $i = 1, \ldots, q-1$. Integrating with respect to K_i,

$$\log J_{K_q} = CJ + \log G^i(K^i) \tag{3.7}$$

where $K^i \equiv (K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_q)$ and $G^i(\cdot)$ is some function. However, the fact that C is the same for all $i = 1, \ldots, q-1$ implies with (3.7) that $G^i(K^i)$ is the same for all such i. Hence we can write

$$G^i(K^i) = r(K_q) \tag{3.8}$$

for all $i = 1, \ldots q-1$. Substituting in (3.7), we obtain:

$$J_{K_q} = r(K_q)e^{CJ} , \tag{3.9}$$

$$e^{-CJ} \frac{\partial J}{\partial K_q} = r(K_q) \quad . \tag{3.10}$$

Integrating with respect to K_q, this implies

$$-\frac{1}{C} e^{-CJ} = -R(K_q) - S(K^q) \tag{3.11}$$

where $-R(K_q)$ is an integral of $r(K_q)$ and $S(K^q)$ is some function of $K_1, \ldots, K_{q-1}$. Then

$$(3.12) \qquad J = -\frac{1}{C}\log C - \frac{1}{C}\log\ \{R(K_q) + S(K^q)\}.$$

However, the log transformation in (3.12) and the constants can be absorbed into the definition of J by suitable redefinition of $F(\cdot,\ldots,\cdot)$ so this is equivalent to the statement of the lemma, completing the proof.

Remark 3.1: Lemma 3.1 itself shows that the constant, C, involved in the proof can always be taken as zero by writing J as in (3.3) and redefining F appropriately. Note that, if $q = m$ and F exhibits constant returns in its arguments, then that constant is already zero and $J(\cdot,\ldots,\cdot)$ additive in that further redefinition. This follows from (3.5) and Euler's Theorem applied to F_J and F_L which are both homogeneous of degree zero.

This result enables us to state the most general result of this section.

Theorem 3.2: Suppose that an aggregate consisting of the first q capital types exists for some q, $1 < q \leq m$. Suppose further that an aggregate consisting of capital types $(p+1, \ldots, q)$ also exists for some p, $1 \leq p < q$. Then an aggregate consisting of the first p capital types also exists.

Proof: Lemmas 2.1 and 2.2 and the discussion following them show that an aggregate of capital types $(p+1, \ldots, q)$ exists at the level of each firm. Further, we might as well start with such an aggregate and take $p = q-1$. Now the existence of an aggregate over the first q capital types together with Lemma 2.1 implies that (3.1) is applicable to $f^v(\cdot)$ for all $v=1, \ldots, n$. Further, the existence of an aggregate consisting of capital of type q implies that (3.2) also holds. This makes Lemma 3.1 applicable whence an aggregate consisting of the first p capital types exists for each firm taken separately. The desired result now follows from Corollary 3.1.

We can also state a partial converse.

Theorem 3.3: Suppose that an aggregate consisting of the first p capital types exists for the system as a whole, for some p, $1 \leq p < m$. Suppose further that an aggregate consisting of capital types (p+1, ..., q) exists for the system as a whole for some q, $p < q \leq m$. Finally, suppose an aggregate consisting of the first q capital types exists for each firm separately. Then an aggregate consisting of the first q capital types exists for the system as a whole.

Proof: Condition (a) of Lemma 2.1 holds by assumption. So does condition (c). Condition (b) follows from Lemma 2.3.

In the case of $q = m$ where a full aggregate is involved, Theorem 3.2 can be strengthened to:

Theorem 3.4: Suppose that a full capital aggregate exists for each firm taken separately. Suppose further that an aggregate consisting of capital types (p+1, ..., m) exists for the system as a whole for some p, $1 \leq p < m$. Then a full capital aggregate and also an aggregate consisting of the first p capital types also exist for the system as a whole.[10]

Proof: Consider first the existence of a full capital aggregate. By assumption, (a) of Lemma 2.1 is satisfied (with $p = m$). Further, (b) is vacuous since all capital types are to be included in the aggregate. Finally, (c) holds by Lemma 2.3 and the existence of an aggregate consisting of capital types (p+1, ..., m). Hence a full capital aggregate exists. The simultaneous existence of an aggregate consisting of the first p capital type now follows from Theorem 3.2.

The fly in the ointment occurs because Lemma 3.1, on which these results depend, does not merely ensure the existence of an aggregate of the first q-1 capital types for each firm separately given that an aggregate of

K_q exists. This would involve only

(3.13) $$J(K_1,\ldots,K_q) = H(R(K_q), S(K_1,\ldots,K_{q-1}))$$

whereas (3.3) shows that J is actually additively separable. As we shall see most clearly when we come to closed form results, this is so restrictive as to take any surprise out of Theorem 3.4. We state the restrictive consequence as:

Theorem 3.5: Suppose that an aggregate consisting of the first q capital types exists. Suppose further that an aggregate consisting of capital types (p+1, ..., q) exists for some p, $1 < p \leq q$. Then every firm's production function can be written in the form:

(3.14) $$f^v(K_1(v),\ldots, K_m(v), L(v))$$

$$= F^v(\{S^v(K_1(v),\ldots,K_p(v)) + R^v(K_{p+1}(v),\ldots,K_q(v)\},K_{q+1}(v),\ldots,K_m(v),L(v)).$$

Proof: Follows from Lemmas 2.1, 2.2, and 3.1.

Corollary 3.3: Suppose that a full capital aggregate exists (either for the system as a whole or for each firm separately) and that an aggregate consisting of capital types (p+1, ..., m) also exists for the system as a whole for some p, $1 \leq p < m$. Then each firm's production function can be written in the form:

(3.15) $$f^v(K_1(v), \ldots, K_m(v), L(v))$$

$$= F^v(\{S^v(K_1(v), \ldots, K_p(v)) + R^v(K_{p+1}(v),\ldots,K_m(v)\}, L(v))$$

Proof: Follows from Theorems 3.3 and 3.4.

The simplest interpretation of these conditions occurs when we consider constant returns and related technologies for which necessary and sufficient conditions for full capital aggregation are known in closed form.[11]

4. Constant Returns and Related Technologies

Definition 4.1: Suppose that the vth firm's production function can be written in the form

$$(4.1)\quad f^v(K_1(v),\ldots, K_m(v), L(v)) = G^v(H^1(K_1(v)),\ldots,H^m(K_m(v)), L(v))$$

where the $H^i(\cdot)$ are monotonic and $G^v(\cdot, \ldots,\cdot,\cdot)$ is homogeneous of degree one in its arguments. Then $f^v(\cdot, \ldots,\cdot,\cdot)$ will be called capital-generalized, constant-returns (CGCR) (See Fisher (1965), pp. 270-71.)

A CGCR function is one which can be made constant returns by (possibly non-linear) stretching of the capital axes. Equations (1.2) and (1.3) above provide examples.[12] We prove:

Theorem 4.1: Let every firm have a CGCR production function.

(a) A necessary and sufficient condition for the simultaneous existence of a full capital aggregate and an aggregate consisting of capital types (p+1, ..., m) for some p, $1 \leq p < m$ is that, for all v=1, ...,n,

$$(4.2)\quad f^v(K_1(v),\ldots,K_m(v)) = F(\{S^v(K_1(v),\ldots,K_p(v)) + R^v(K_{p+1}(v),\ldots,K_m(v)\},L(v))$$

where $F(\cdot,\cdot)$ is homogeneous of degree one and is independent of v.

(b) Further, in this case, an aggregate consisting of the first p capital types also exists.

(c) When the aggregates exist, the aggregate production function can be written as

$$(4.3)\quad y^* = F(\{\sum_{v=1}^{n} S^v(K_2(v), \ldots, K_p(v) + \sum_{v=1}^{n} R^v(K_{p+1}(v), \ldots, K_m(v)\}, L)$$

where y^* is maximized output and L is total labour.[13]

<u>Proof:</u> (a)1. Necessity. From Lemma 2.1 the existence of a full aggregate implies that for every v=1, ..., n,

$$(4.4)\quad f^v(K_1(v), \ldots, K_m(v), L(v)) = F^v(J^v(K_1(v), \ldots, K_m(v)), L(v)).$$

Since $f^v(\cdot, \ldots, \cdot, \cdot)$ is CGCR, $F^v(\cdot, \cdot)$ can be taken to be homogeneous of degree one in its arguments by suitable redefinition of $J^v(\cdot, \ldots, \cdot)$. The fact that the $F^v(\cdot, \cdot)$ must then all be the same is the standard capital-augmenting result for the existence of a full capital aggregate given in [1, Theorem 3.2, p. 268 and Theorem 4.2, p. 272].[14] That $J^v(K_1(v), \ldots, K_m(v)$ can be written in the additive form given in (4.2) follows from Corollary 3.3 above. This does not quite complete the proof of necessity, however, since one must show that the two arguments just given hold simultaneously--that is, that $F^v(\cdot, \cdot)$ can be taken as homogeneous of degree one when $J^v(\cdot, \ldots, \cdot, \cdot)$ is written in its additive form. This follows from Remark 3.1.

2. Sufficiency. This can be proved in (at least) two ways. One such way is by construction as in the proof of (c), below. The other way is as follows. From (4.2) and the fact that $F(\cdot, \cdot)$ is homogeneous of degree one [1, Theorem 4.2, p. 272] implies that a full aggregate exists. Further, (4.2) itself shows the existence at the firm level of two aggregates consisting respectively

of capital types (p+1, ..., q) and the first p capital types, so that (a) of Lemma 2.1 is satisfied for such partial aggregates. Lemma 2.3 then shows that (c) of Lemma 2.1 is also so satisfied. Since $F(\cdot,\cdot)$ is constant returns, F_J and F_L are homogeneous of degree zero (where $J(v)$ is the first argument of $F(\cdot,\cdot)$). As in Remark 3.1, Euler's Theorem applied to F_J and and F_L shows that the right-hand side of (3.5) above is zero and, since the left-hand side is also zero when K_q and K_i are replaced by S^v and R^v, respectively, (3.5) and thus (3.4) and (3.2) hold so that (b) of Lemmas 2.1 and 2.2 are also satisfied. Lemma 2.1 now shows the existence of the partial aggregates.

(b) The proof of sufficiency just given proves the simultaneous existence of both partial aggregates. This would also follow from Theorem 3.2 or Theorem 3.3.

(c) The nature of the aggregate follows from an alternative proof of sufficiency along lines following Solow [5, pp. 104-105] (see also [4, pp. 559-60].) Begin with (4.2). Optimal allocation of labour requires the same marginal product of labour in every use. Since $F(\cdot,\cdot)$ is constant returns and $F_{LL} < 0$, this means the ratio of the two arguments of $F(\cdot,\cdot)$ must be the same for every v. Letting $J(v)$ denote the first argument, there then exist scalars, λ_v, v=1, ...,n, such that $\lambda_v > 0$, $\sum_{v=1}^{n} \lambda_v = 1$, and

$$L(v) = \lambda_v L \; ; \quad J(v) = \lambda_v J \tag{4.5}$$

where

$$J \equiv \sum_{v=1}^{n} J(v) \equiv \sum_{v=1}^{n} S^v(K_1(v),\ldots,K_p(v)) + \sum_{v=1}^{n} R^v(K_{p+1}(v), \ldots, K_m(v)) \tag{4.6}$$

when labour is optimally allocated. Then, at such points,

$$y^* = \sum_{v=1}^{n} F(J(v), L(v)) = \sum_{v=1}^{n} F(\lambda_v J, \lambda_v L) = \sum_{v=1}^{n} \lambda_v F(J, L) = F(J, L) \tag{4.7}$$

using the fact that $F(\cdot,\cdot)$ is homogeneous of degree one. This simultaneously shows the existence of a full capital aggregate, J, and of partial capital aggregates,

$$(4.8) \quad S \equiv \sum_{v=1}^{n} S^v(K_1(v), \ldots, K_p(v)) \text{ and } R \equiv \sum_{v=1}^{n} R^v(K_{p+1}(v), \ldots, K_n(v))$$

as well as proving (c).

Theorem 4.1 shows fairly plainly what is going on. In the CGCR case, the requisite full and partial aggregates will exist in the system as a whole if and only if: (1) the corresponding partial aggregates exist at the firm level <u>and</u> are perfect substitutes for each other within the firm; and (2) firms differ <u>only</u> in the way in which the partial aggregates are constructed but not in the way in which their capital aggregates are combined with labour to produce final output. This leads to a situation in which, once aggregation has been performed at the firm level, the firm's two partial capital aggregates are perfect substitutes for those of any other firm as well as for each other.

This set of conditions is even plainer in its implication when we consider constant returns and partial capital aggregates consisting of a single capital type.

<u>Corollary 4.1</u>: Suppose all firms have constant returns production functions. Then the results of Theorem 4.1 apply. Further, $R^v(\cdot, \ldots, \cdot)$ and $S^v(\cdot, \ldots, \cdot)$ can be taken as homogeneous of degree one in their arguments.

<u>Proof</u>: Constant returns is a special case of CGCR, so only the final statement requires a separate proof which is left to the reader.[15]

<u>Corollary 4.2</u>: Suppose all firms have constant returns production functions. A necessary and sufficient condition for the simultaneous existence of a

full capital aggregate and an aggregate consisting of capital type m only is as in Theorem 4.1 with $R^v(K_m(v)) = b_m(v)\ K_m(v)$ for some scalar $b_m(v)$, so that aggregate capital for the system as a whole is given by

$$(4.9) \qquad J \equiv \sum_{v=1}^{n} b_m(v)\ K_m(v) + \sum_{v=1}^{n} S^v(K_1(v),\ldots,K_{m-1}(v)).$$

The other results of Theorem 4.1 continue to apply.

Proof: Obvious from Corollary 4.1.

Corollary 4.3: Suppose all firms have constant returns production functions. A necessary and sufficient condition for the simultaneous existence of a full capital aggregate and of m-1 partial capital aggregates each consisting of a single capital type is that, for every v=1, ..., n, there exist scalars, $b_1(v),\ldots,b_m(v)$, such that:

$$(4.10) \qquad f^v(K_1(v),\ldots,K_m(v),L(v)) = F(\sum_{i=1}^{m} b_i(v)K_i(v),L(v))$$

where $F(\cdot,\cdot)$ is the same for all firms. In this case, aggregate capital stock is given by

$$(4.11) \qquad J \equiv \sum_{v=1}^{n} \sum_{i=1}^{m} b_i(v)K_i(v)$$

and the aggregate production function by

$$(4.12) \qquad y^* = F(J,\ L)\ .$$

In this case, the mth possible partial aggregate consisting of a single capital aggregate and, indeed, every possible partial capital aggregate also exists.

Proof: Let the m-1 partial aggregates whose existence is assumed be types 2, ...,m. Let the full aggregate for firm v be $J^v(K_1(v),\ldots,K_m(v))$

which can be taken to be homogenous of degree one. As in the proof of Theorem 4.1, consideration of Remark 3.1 and (3.5), (3.4), and (3.2) shows that

$$J^v_{K_i K_j} = 0 \qquad i, j = 1, \ldots, m \quad i \neq j \tag{4.13}$$

whence, by homogeneity, there exist $b_1(v), \ldots, b_m(v)$ such that

$$J^v(K_1(v), \ldots, K_m(v)) = \sum_{i=1}^{m} b_i(v) K_i(v) . \tag{4.14}$$

The corollary now follows from Theorem 4.1.

Corollary 4.3 exhibits the case in which all capital aggregates exist. In it, any capital of any type within any firm is equivalent to a fixed amount of capital of any other type in or out of the same firm, where the equivalences depend both on type and firm. All capital assets are perfect substitutes both within and across firms and there is a natural capital aggregate consisting of capital measured in efficiency units. Such an aggregate only exists, however, because firms differ only as regards the size of their capital stock measured in efficiency units. This is the capital-augmentation theorem applied both across and within firms, as it were.

This is, of course, the most restrictive case. Theorem 4.1 or Corollaries 4.1 and 4.2 exhibit cases where perfect substitution only occurs between groups of capital types without requiring it among individual capital types. Still the restrictive implications are clear. Suppose for simplicity that all capital is considered either plant or equipment. Then the simultaneous existence of a full capital aggregate and of either a plant or an equipment aggregate (the existence of either of which implies the existence of the other given a full aggregate) implies that plant and equipment--as aggreagates--are perfect substitutes. Under constant returns or CGCR (essentially the only important cases where aggregates are likely to exist at all), firms can differ only in how the plant and equipment aggregates are constructed. They cannot differ as to how they are used or as to the fact that they are perfect substitutes.

Notes

1. See Fisher (1969) for a relatively non-technical summary and for a more general bibliography.

2. See Fisher (1965), p. 268. A number of authors have provided alternate proofs. See the works cited in Fisher (1965), p. 268, n. 1 and Fisher (1969), p. 558, n. 17.

3. See the discussion in Fisher (1969), p. 561.

4. These assumptions as to labour and output homogeneity make no difference in the present context. The mutually isomorphic questions of labour and output aggregation were analyzed in Fisher (1968a). See also Fisher (1968b).

5. See Fisher (1965), pp. 277, 286 and Fisher (1969), pp. 562-63.

6. The existence of the partial aggregates can be verified by direct--although tedious--computation from Fisher (1965), Theorem 5.1, p. 276. (In the statement of that theorem, (5.10) should be replaced by (5.8).) See also Lemmas 2.1 and 2.1' (Ibid., p. 267). The non-existence of the full aggregate follows from the fact that the marginal rate of substitution between $K_1(v)$ and $K_2(v)$, even within the vth firm, is not independent of L(v). See Fisher (1965), p. 280.

7. Somewhat similar results hold for labour or output aggregation. See Fisher (1968a) and (1968b). The results are not the same, however. In the case of labour aggregation, there is a natural partial aggregate consisting of any single labour so the existence of such a partial aggregate is not restrictive, as it is in the capital case (see especially Lemma 3.1 below). On the other hand, for the result in question on complementary subaggregates to apply, it is necessary to assume the existence of the complementary subaggregate at the firm level which is not additionally necessary for capital.

8. For discussion of these matters see Fisher (1965), (1968a), (1968b), and (1969).

9. Throughout this paper I use unmodified the phrase "existence of an aggregate" to mean the existence of an aggregate for the system as a whole rather than the existence of an aggregate within particular firms' production functions. When the latter is meant, I shall say so. Of

course, the two are closely related; indeed, existence within each firm's production function is necessary but not sufficient for existence for the system as a whole.

10. Note that the result is not symmetric as regards the assumptions made on full and partial aggregates. The existence of a full aggregate for the system as a whole plus the existence of a partial aggregate for each firm separately does not suffice for the existence of any partial aggregate for the system as a whole. Otherwise a full aggregate for the whole system would immediately imply an aggregate for the whole system of each separate capital type and this is false. (Equations (1.2) and (1.3) above provide a counter example.) This is because of the crucial role of (b) of Lemma 2.1 which is vacuous for a full aggregate.

11. These are not the only such cases, however. (See Fisher (1965), p. 273.) A full aggregate exists if all production functions are in the form

$$(3.16)\quad f^v(K_1(v),\ldots,K_m(v),\ L(v)) = F^v(J^v(K_1(v),\ldots,K_m(v)) + c_v L(v)).$$

The full set of non-constant-returns-related cases permitting capital aggregation is not known. It is plain, however, that such cases are very special in that, for most non-constant-returns-related cases, capital aggregates will not exist for the system as a whole even if they exist for each firm separately and all firms are exactly alike.

12. A more general class of technologies would be ones which are homogeneous of degree one in labour and monotonic functions of the first q capital types but not necessarily in monotonic functions of the remaining capital types. The necessity results of this section concerning the nature of the aggregate apply to such cases with the existence of a full aggregate replaced by the existence of an aggregate consisting of the first q capital types and partial aggregates being understood to involve capital types lower-numbered than q. Explicit discussion of this case does not seem useful because closed-form necessary and sufficient conditions for the existence of such aggregates are not known for $q < m$.

13. Generalization to more than two partial aggregates is left to the reader.

14. See also Fisher (1965), p. 267, n.1.

15. Cf. Fisher (1968a), p. 398.

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