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On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems

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Abstract

This note develops a new method for the solution of EMP interaction problems. Basically it involves expanding the solution in terms of its singularities in the Laplace transform or complex frequency (or s) plane. In the time domain each term comes from an inverse transform of the corresponding term in the singularity expansion. Finite size objects with well behaved media have only poles in the finite s plane for their delta function response. These factor into terms involving the classical natural frequencies and modes but in addition bring out factors which we call coupling coefficients as well as the possibility of higher order poles besides simple poles, but still of finite order in the finite s plane. If the incident waveform has singularities in the finite s plane the response can be generally split into an object part (containing object poles) and a waveform part containing the waveform singularities. The object poles directly give amplitudes, frequencies, damping constants, and phases for the damped sinusoidal waveforms seen so commonly in EMP tests using pulsed waveforms. There is some latitude in the calculation of coupling coefficients and some difficulties are discussed.

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Foreword

Approximately this last summer I started looking at this singularity expansion method. Based on physical arguments the general form of the expansion and how the terms factored for finite objects to exhibit dependence on separate variables of the EMP interaction problem soon became apparent to me. In particular some physical observations generalized from many EMP tests in time domain led to the pole expansion concept in frequency domain. This forms the starting point for what follows in this note.

Next chronologically I started some discussion going on this subject, in particular with some of my colleagues at Northrop Corporate Labs in Pasadena. I certainly wish to thank them, in particular Dr. L. Marin, Dr. K. S. H. Lee, Dr. R. W. Latham, and Dr. F. Tesch (who is now with Dikewood) for many very stimulating conversations about this technique. They certainly helped me test and refine some of the concepts. They are also presently working on reports to further refine the method and calculate some example problems. In particular Drs. Marin and Latham have rather far advanced some analytic solutions of the magnetic-field integral equation for finite size perfectly conducting objects in terms of the singularity expansion.

In September there was a meeting in Pasadena with some significant attention given to this subject. I would like to thank everyone who came to that meeting for the stimulating discussion on this subject. On various occasions both at this meeting and on other occasions I have had occasion to discuss this matter with various people. In particular I would like to thank Prof. R. J. Garbacz of Ohio State U., Prof. C. T. Tai of U. of Michigan, Prof. S. W. Lee of U. of Illinois, Prof. C. Taylor of U. of Miss., and Dr. A. Poggio of Cornell Aeronautical Lab. Some of these people are already beginning studies on various aspects and specific problems concerned with the singularity expansion method.

...when suddenly a white rabbit with pink eyes ran close by her.

There was nothing so remarkable in that; nor did Alice think so very much out of the way to hear the Rabbit say to itself, "Oh, dear! Oh, dear! I shall be too late!" (when she thought it over afterward it occurred to her that she ought to have wondered at this, but at the time it all seemed quite natural); but when the Rabbit actually took a watch out of its waistcoat pocket, and looked at it, and then hurried on, Alice started to her feet, for it flashed across her mind that she had never before seen a rabbit with either a waistcoat pocket or a watch to take out of it, and, burning with curiosity, she ran across the field after it, and was just in time to see it pop down a large rabbit hole under the hedge.

In another moment down went Alice after it, never once considering how in the world she was to get out again.

(Lewis Carroll, Alice in Wonderland)

Contents

I. Introduction

This note takes off on a new course in the treatment of the interaction of electromagnetic fields with bodies located in free space or in other simple media, including the effects of the proximity of one body with respect to another (such as a body in an EMP simulator).

By way of introduction some physical observations are in order. Suppose one excites an object such as a missile, an aircraft, a building with various conductor geometries, etc. with a fast electromagnetic pulse. What are the general characteristics of the resulting waveforms for the various electromagnetic quantities (such as current, charge, etc.) associated with this object? Someone with much experience in EMP testing of such objects in EMP simulators could observe that an extremely common characteristic of such waveforms is the presence of one or more exponentially damped sinusoidal oscillations. This is the case not only in the excitation of the internal circuitry but also for the body geometry as well. This electromagnetic resonance phenomenon is particularly pronounced for long and comparatively slender conductors such as the main body resonance of a missile or the body and wing resonances of various aircraft. These damped sinusoids in the response are observed not only in EMP interaction studies (both experimental and theoretical) but also in fast pulse time-domain-type radar scattering studies.

Not all aspects of the electromagnetic response of objects look like damped sinusoids. Parts of the time domain response may look like the incident waveform, or perhaps its time derivative or time integral. It would also appear that in some cases even more complex types of responses occur.

Looking at the forms of the observed responses one might ask if there is some way that these observable features of the waveform can be found directly when one calculates the object electromagnetic response. Can the amplitude, frequency, damping constant, phase, etc. of each damped sinusoid be directly calculated? How do they depend on the incident wave? Can other kinds of response which give waveforms related to the incident waveform be directly calculated?

The purpose of this note is to begin to characterize electromagnetic interaction with objects in terms of quantities directly identifiable with various characteristics of resulting interaction waveforms. Some characteristics are associated with the object characteristics including the presence of neighboring objects. Other characteristics are associated with the waveform of the incident field. Yet others are associated with the spatial distribution of the incident fields, such as specified by direction of incidence and polarization. What is in effect accomplished here is a decomposition of the interaction problem into various quantities which depend on different variables of the problem. The dependence of the interaction on different variables can then be separately considered resulting in a considerable simplification in understanding how the resulting electromagnetic interaction can vary over all possible variations of the parameters of a particular problem being considered. This effectively extends the complexity of the object geometries one may be willing to consider for detailed calculations.

Having identified what appear to be exponentially decaying sinusoids in typical interaction experimental data one might use this as a clue toward finding a mathematical representation of the electromagnetic interaction which has these terms as part of the decomposition. Consider the Laplace transform of the various waveforms. (The two-sided Laplace transform is used throughout the note.) The Laplace transform of an exponentially damped sinusoid gives a pair of complex conjugate poles in the complex s plane of the form $1/(s - s_{\alpha_1})$ and $1/(s - s_{\alpha_2})$ where s is the Laplace transform variable, where $s_{\alpha_1} = \bar{s}_{\alpha_2}$ with the bar above a quantity indicates complex conjugate, and where α_1 and α_2 are sets of indices to label the poles being considered. If one could find these poles with their coefficients in the complex s plane from an integral equation or other form of the solution (such as from an eigenfunction expansion of the solution) then not only would he have a representation of part of the frequency or Laplace domain solution, but also of a part of the time domain waveform (damped sinusoids) as well. Such poles in the complex s plane are termed natural frequencies of the object since they are frequencies for which the object can have a response (in Laplace or complex frequency domain) with no excitation in the form of an incident wave. If the body is excited at a natural frequency then its response is infinite at that complex frequency.

Suppose one were to take a solution for some interaction problem expressed in the Laplace domain either explicitly or implicitly (such as in the form of an integral equation). Furthermore suppose one wishes to convert this into a time domain solution. This can be done using the inverse Laplace transform integral, a contour integral in the complex plane. This contour can be deformed in the complex s plane, passing it through regions where the response is an analytic function of s. On reaching singularities such as poles and branch points the contour can be deformed around the poles and branch cuts to obtain terms associated with each separate pole and branch cut and perhaps a contribution from a portion of the contour for $|s| \to \infty$. Our basic approach then in expressing the solution to electromagnetic interaction problems is to express it as a sum of such terms in both Laplace (frequency) domain and time domain. Different types of terms will have different properties and we wish to understand these properties in detail so as to take advantage of the decomposition of the interaction problem into its various parts.

Not only are the natural frequencies of the object of concern because there can be other singularities in the response. Associated with certain kinds of object geometries one may also need to consider branch cuts in the complex s plane. An example of such a body is an infinite length perfectly conducting circular cylinder for which the branch cuts can be associated with cylindrical Hankel functions. This note is mostly concerned with cases that the object response does not have branch cut contributions associated with the object characteristics. However the same approach as for the case of poles may be used by including terms for the branch cut integrals in the general expansion for the inverse Laplace transform integral. Back in Laplace domain the individual branch cut terms can be found by first subtracting all the pole contributions and then treating what is left. Since the only important contribution at the branch cut is the change in the function across it, then in calculating the branch cut integral one could use this change to define an appropriate term only associated with this change (along the entire branch cut). Such problems are not considered in this note but it is determined that for a class of objects of interest there are no branch cut contributions associated with the object geometry.

The incident waveform will also typically have singularities in its Laplace transform and there are terms in the object response which correspond to these. The object response can then be split to some extent into terms associated with the geometry and other electromagnetic characteristics of the object on the one hand, and the incident waveform characteristics (including how they couple to the object) on the other hand. In many cases the waveform contributions will be through rather simple singularities such as simple poles associated with exponential waveforms.

Having decomposed the object response into various terms in this manner, one can then see how accurately the first so many terms describe the complete object response in various of its characteristics in both frequency and time domains. It is not apparent that all interaction problems can be most conveniently characterized using the natural frequencies and modes and other singularities. However, for highly resonant structures it appears to considerably simplify the comprehension of the important features of electromagnetic interaction with the structure. Other techniques will continue to be valuable and waveform and frequency-response detailed calculations will still be needed. In some cases the waveforms, for example, will be useful in determining how many natural modes etc. are needed for the object and over what range of the object parameters the incomplete sum of modes is adequate.

This note first considers the general form of these solutions which separates out various aspects of the object response.

Then based on finite matrix formulation of integral equations in a form such as used for numerical solutions various properties of the solution can be found based on the analyticity properties of the matrix elements and vector components. For instance, the conclusion that only poles of finite order appear in the object response for finite size objects applies not only to perfectly conducting objects but ones of finite conductivity as well. Numerous topics are then briefly discussed to point out many areas for further investigation. Finally an appendix discusses the special "natural frequency" s = 0 and another appendix works out the object response for a perfectly conducting sphere as an illustration of natural frequency, mode, and coupling coefficient calculation and indexing.

In addition to what is discussed in this note many refinements of the singularity expansion method are possible and various extensions of the results would seem possible. In solving specific boundary value problems with this approach some other general results may be suggested by the data, thereby focusing attention on the proof or disproof of these conjectures and the consideration of other boundary value problems which better exhibit the same general results or test their validity.

While this note considers the solution of electromagnetic boundary value problems in terms of natural frequencies and other singularities the technique can be applied to experimental data as well. From frequency or time domain data one should be able to determine natural frequencies and modes and other singularity characteristics by extending the data to the remainder of the complex s plane using Laplace transform techniques or even analytic continuation. Of course there are numerical error problems as in other data reduction processes and this will also require quantitative understanding.

Since this is a new approach to the solution of electromagnetic boundary value problems there is clearly much work to be done to fully understand its many ramifications.

II. The Form and Some General Characteristics of the Singularity Expansion for Some of the Simpler Cases

Let us now look at some of the advantages associ ted with the singularity expansion method because of the form of the expansion in simple cases.

First we write the complex Laplace transform variable in terms of its real and imaginary parts for notational purposes as

$$s \equiv \Omega + i\omega \tag{2.1}$$

An arbitrary function of time f(t) which could be in general a scalar, vector, or tensor of arbitrary rank has a Laplace transform (bilateral or two sided) assuming the integral of $|f(t)|$ over any finite interval exists defined by[1]

$$L[f(t)] \equiv \tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)e^{-st}dt \tag{2.2}$$

where the integration is taken on the real t axis. This is the two sided Laplace transform (indicated by a tilde ~ over the function) where f(t) is required to have a behavior such that f(s) exists and is analytic in some strip $\Omega_- < \text{Re}[s] < \Omega_+$ in the s plane. The inverse transform (where f(t) is of bounded variation near t) is given by

$$L^{-1}[f(s)] = f(t) = \frac{1}{2\pi i}\int_{\Omega_o - i\infty}^{\Omega_o + i\infty} f(s)e^{st}ds \tag{2.3}$$

where the limits to $\pm i\infty$ can be interpreted in a Cauchy principal value sense and where $\Omega_- < \Omega_o < \Omega_+$ unless f(t) is discontinuous at t in which case the inversion gives $[f(t-) + f(t+)]/2$. In our cases of interest $f(t) = 0$ for $t < t_o$ in which case $\Omega_+ = \infty$ and the transform effectively reverts to a one sided Laplace transform with the lower limit as t_o. Typically also $\Omega_- = 0$ as long as f(t) does not grow as fast as an increasing exponential for $t \to +\infty$. Thus we normally have f(s) an analytic function of s for $\text{Re}[s] > 0$, the right half of the s plane, and the inversion integral is defined along $\text{Re}[s] = \Omega_o > 0$.

The essence of the singularity expansion method involves evaluating f(s) (which may be surface current density or various other electromagnetic quantities) by evaluating $\tilde{f}(s)$ in terms of the left half plane singularities ($\text{Re}[s] \le 0$). Express the time domain form f(t) in terms of these same singularities as would be done by deforming the contour for the inversion integral (equation 2.3) into the left half plane and splitting the integral into parts associated with each singularity. Note automatically that since we are only concerned with f(t) real for real t then the Laplace transformed f(t) has some symmetry which can be found by splitting the transform integral into real and imaginary parts. Denoting the complex conjugate by a bar - over the quantity we have

$$\tilde{f}(\bar{s}) = \overline{\tilde{f}(s)} \tag{2.4}$$

Singularities are then automatically symmetrically placed with respect to the Ω axis except that branch cuts can be moved around as long as the branch points stay symmetrically placed with respect to the Ω axis. For convenience we constrain the branch cuts to also be symmetrically placed with respect to the Ω axis so that equation 2.4 always holds except of course right at the singularities. Having found the term associated with one singularity we then automatically have the result for the conjugate singularity. Of course this does not help us for those singularities on the Ω axis. When we index the terms associated with each singularity with a set of integers we can also adopt the convention of a sign reversal on one of the integer indices corresponding to conjugate positioned singularities. For such an index positive integers can be associated by convention with $\omega > 0$ (upper half plane) a zero integer for $\omega = 0$ (the Ω axis) and a negative integer for $\omega < 0$ (the lower half plane). Alternately another symbol can be introduced to indicate which of a conjugate pair is meant.

While there are various forms of incident waves that one might use we choose the commonly used plane wave for our examples. If one wishes, more complex field distribution can be found by superposition of many plane waves.[2] For our present purposes we consider an incident plane wave propagating in the $\vec{e}_1$ direction (independent of s) with electric field polarization in some combination of the $\vec{e}_2$ and $\vec{e}_3$ directions. Here $\vec{e}$ is used for a unit vector. The three unit vectors for our plane wave are all mutually orthogonal and form a right handed system of unit vectors as

$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \tag{2.5}$$

As shown in figure 2.1 this plane wave is incident on some object of finite linear dimensions. Let $\vec{r}$ denote the observer position and $\vec{r}'$ coordinates on the object. Then by an object of finite dimensions we can require $|\vec{r}'| < r_o$ for all $\vec{r}'$ where r_o is some finite radius. (Note that all dimensions are rationalized MKSA throughout the note.) Typically the coordinate origin ($\vec{r} = \vec{0}$) would be chosen near or even inside the object. If we

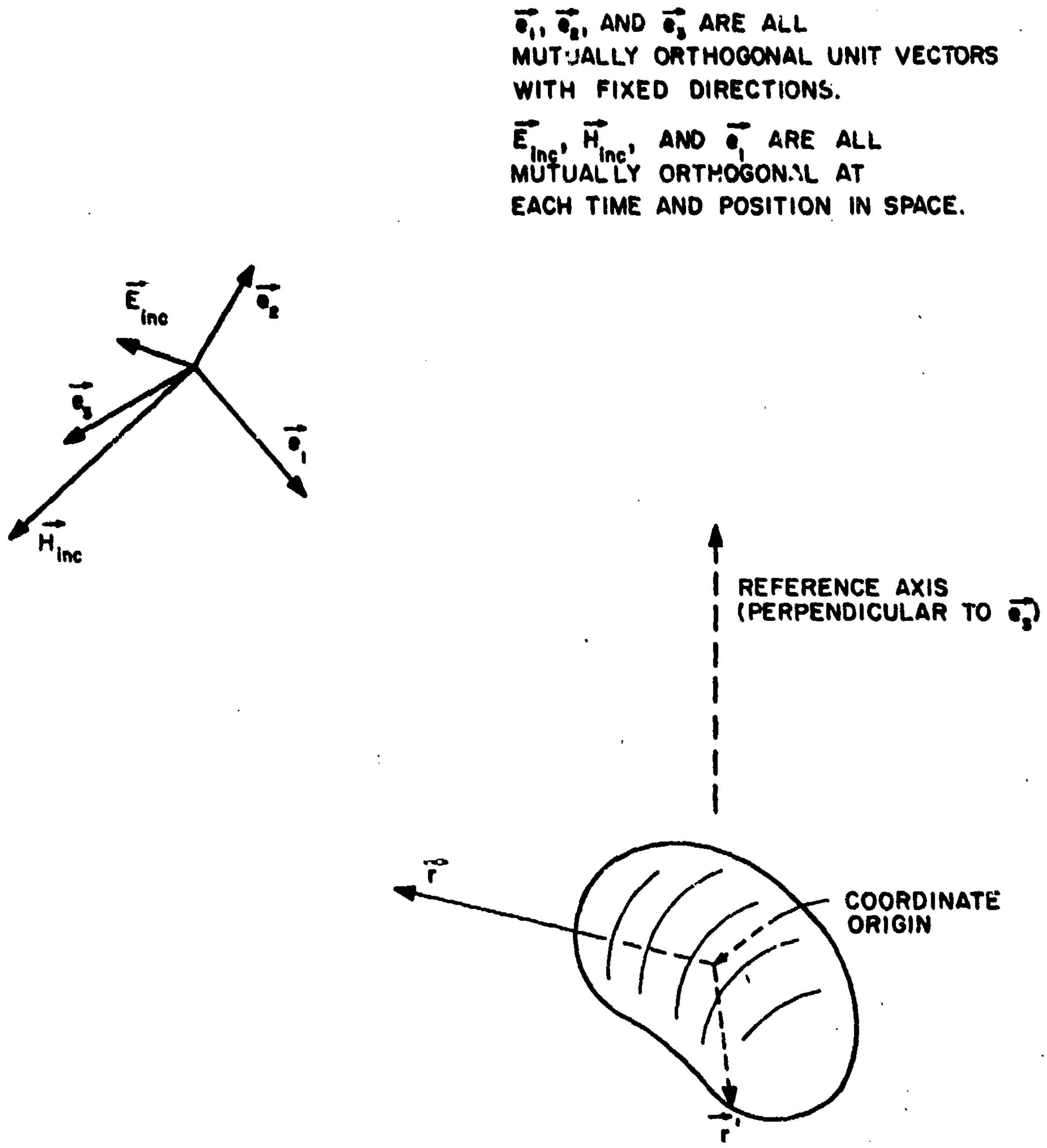

FIGURE 2.1 PLANE WAVE INCIDENT ON OBJECT

define some reference axis (as shown in figure 2.1) then given $\vec{e}_1$ we can choose $\vec{e}_2$ parallel to the plane which is parallel to both $\vec{e}_1$ and the reference axis; $\vec{e}_3$ will be perpendicular to this plane. If the object has an axis of symmetry this would normally be chosen as the reference axis. In spherical coordinates as in one the appendices the z axis (or $\theta = 0, \pi$) would be chosen as the reference axis for convenience.

Our incident plane wave is assumed to propagate in free space with a propagation constant

$$\gamma \equiv ik = \frac{s}{c} \tag{2.6}$$

where the speed of light in vacuum is

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} \tag{2.7}$$

The wave impedance of free space is

$$Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} \tag{2.8}$$

The permeability of free space is μ_o and the permittivity of free space is ε_o. It is not strictly necessary for the medium to be free space; μ_o and ε_o can be regarded as parameters of the large volume (ideally infinite in size) of the medium in which the object of interest is placed. However for some results it may be necessary that this medium have zero conductivity (be lossless) so zero conductivity is specified for the infinite medium for considerations in this note.

The general form of our incident plane wave can now be written as[3,4,5]

$$\begin{aligned} \vec{E}_{inc}(\vec{r},t) &= E_o\left[f_2\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right)\vec{e}_2 + f_3\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right)\vec{e}_3\right] \\ \vec{H}_{inc}(\vec{r},t) &= \frac{E_o}{Z_o}\left[f_2\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right)\vec{e}_3 - f_3\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right)\vec{e}_2\right] \end{aligned} \tag{2.9}$$

where E_o is a scale factor with dimensions volts/m. The two incident fields are related by

$$\vec{H}_{inc}(\vec{r},t) = \frac{1}{Z_o}\vec{e}_1 \times \vec{E}_{inc}(\vec{r},t)$$

$$\vec{E}_{inc}(\vec{r},t) = -Z_o\vec{e}_1 \times \vec{H}_{inc}(\vec{r},t) \tag{2.10}$$

In Laplace form the incident plane wave is

$$\tilde{\vec{E}}_{inc}(\vec{r},s) = E_o[\tilde{f}_2(s)\vec{e}_2 + \tilde{f}_3(s)\vec{e}_3]e^{-\gamma\vec{e}_1\cdot\vec{r}}$$

$$\tilde{\vec{H}}_{inc}(\vec{r},s) = \frac{E_o}{Z_o}[\tilde{f}_2(s)\vec{e}_3 - \tilde{f}_3(s)\vec{e}_2]e^{-\gamma\vec{e}_1\cdot\vec{r}} \tag{2.11}$$

Note that for each of the two independent polarizations we can have separate waveform functions $f_2(t^*)$ and $f_3(t^*)$ where the retarded time is defined by

$$t^* \equiv t - \frac{\vec{e}_1\cdot\vec{r}}{c} \tag{2.12}$$

The waveform functions have subscripts which relate them to the polarization vector for the electric field. For the magnetic field 2 and 3 are interchanged with a sign reversal in one case. No evanescent waves are allowed for our present purposes, so $\vec{e}_1$ (even when associated with the Laplace form) is taken as a real unit vector although for some purposes it need not be. This plane wave can be expanded in terms of the vector wave function for various other coordinate systems such as spherical coordinates[4] (which is used in appendix B) and cylindrical coordinates.[5]

For convenience one may introduce a unit dyadic plane wave for Laplace domain purposes as

$$\tilde{\vec{\vec{I}}}_1 \equiv (\delta_{b_1,b_2})e^{-\gamma\vec{e}_1\cdot\vec{r}} = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}e^{-\gamma\vec{e}_1\cdot\vec{r}} \tag{2.13}$$

(with b_1, b_2 indicating in this case a pair of indices each ranging from 1 to 3) which in the time domain is a dyadic delta function plane wave as

$$\vec{\vec{I}}_1 = (\delta_{b_1,b_2})\delta\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right) = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\delta\left(t - \frac{\vec{e}_1\cdot\vec{r}}{c}\right) \tag{2.14}$$

The subscript 1 indicates propagation in the $\vec{e}_1$ direction. Multiplying (in a dot product sense) this unit dyadic plane wave by a waveform and polarization gives our incident plane wave as

$$\overleftrightarrow{E}_{inc}(\vec{r},s) = E_o[\tilde{f}_2(s)\vec{e}_2 + \tilde{f}_3(s)\vec{e}_3] \cdot \overleftrightarrow{I}_1$$
$$\overleftrightarrow{H}_{inc}(\vec{r},s) = \frac{E_o}{Z_o}[\tilde{f}_2(s)\vec{e}_3 - \tilde{f}_3(s)\vec{e}_2] \cdot \overleftrightarrow{I}_1 \qquad (2.15)$$

In the time domain convolution is also needed which we might indicate by

$$\vec{E}_{inc}(\vec{r},t) = E_o[f_2(t)\vec{e}_2 + f_3(t)\vec{e}_3]* \cdot \vec{I}_1$$
$$\vec{H}_{inc}(\vec{r},t) = \frac{E_o}{Z_o}[f_2(t)\vec{e}_3 - f_3(t)\vec{e}_2]* \cdot \vec{I}_1 \qquad (2.16)$$

Note that the dyadic plane wave "contains" both longitudinal waves in the $\vec{e}_1$ part and transverse waves in the $\vec{e}_2$ and $\vec{e}_3$ parts. Only the transverse waves satisfy Maxwell's equations in source free media and so the 2 and 3 components give the most general uniform plane wave propagating parallel to $\vec{e}_1$.

The first problem in the solution of our electromagnetic interaction problem is then to find the object response to two incident waves

$$\overleftrightarrow{u}_2 \equiv \vec{e}_2 \cdot \overleftrightarrow{I}_1 , \quad \overleftrightarrow{u}_3 \equiv \vec{e}_3 \cdot \overleftrightarrow{I}_1 \qquad (2.17)$$

where these can be taken as either electric or magnetic fields because of the way they interchange with one another (with a sign reversal in one case). Knowing the response to these two waves (taken as both electric or both magnetic) then the response to a general plane wave incident in direction $\vec{e}_1$ can be formed in Laplace domain simply by reintroduction of the waveform functions $\tilde{f}_2$ and $\tilde{f}_3$ and other scale factors just as they appear in the incident plane wave as above. In time domain the response to $\vec{u}_2$ and $\vec{u}_3$ can be separately convoluted with f_2 and f_3 and scale factors reintroduced to obtain the general solution.

What then is the object response to $\overleftrightarrow{u}_1$ and to $\overleftrightarrow{u}_2$ taken for convenience as the electric field components to correspond directly to polarization? Here all the waveform characteristics are factored out leaving two problems, each of which has an "incident field function" which has no singularities in the entire s plane. These are termed entire functions; they are analytic for all finite s although in this case they each have an essential singularity at ∞. These are the simplest plane waves for our purposes.

There are many kinds of objects which one might consider. They can be of finite size or infinite size in various shapes. They can be composed of various media arranged in various distributions. In this note we concentrate our attention on a certain kind of such objects and work out some general results for this class of objects. Other kinds of objects can also be considered in this way and some comments are made regarding such other classes of objects. As shown in figure 2.1 we consider finite bodies described by $|\vec{r}'| < r_0$ for all points on the body. Furthermore for some of the considerations and the example in appendix B we take this object as perfectly conducting which reduces our considerations to the body surface which has surface charge and current densities as quantities of primary interest for describing the electromagnetic interaction. This body is not necessarily continuous but may be composed of several separate parts.

In a later section of this note it is demonstrated that current density (either surface or volume) for a finite object has its delta function response corresponding to any singularities in the finite s plane expressed in terms of poles of finite order. We identify these poles as s_α where α is some index set which indicates which pole is meant. This result is of fundamental importance for our general representation of the solution to the interaction problem. We have a series with $(s - s_\alpha)^{-n_\alpha}$ for $n_\alpha = 1, 2, 3, \ldots$ as a factor in each term. What we need now is the rest of the expression at each of these poles to complete the expansion of the two delta function plane wave responses in Laplace domain. If there is more than one order of pole at $s = s_\alpha$ the α index set can have a number to designate each term in the expansion corresponding to each order pole; clearly this number could be just n_α.

These poles s_α are the natural frequencies of the object. By a natural frequency is meant a value of s for which the object has a response without an incident field exciting it. The natural frequencies are generally in the left half of the s plane ($\mathrm{Re}[s] < 0$) because of energy loss (to radiation in the case of a perfectly conducting body) except that natural frequencies can be on the $i\omega$ axis ($\Omega = 0$) with first order poles for lossless situations such as for interior cavity modes. However in the time domain such simple poles on the $i\omega$ axis correspond to undamped sinusoids which, if they can be excited by the incident wave, must continue to radiate power to infinity indefinitely by reciprocity. Since this would violate conservation of energy the residues of such poles must be zero and we drop them from consideration.

If the object responds at $s = s_\alpha$ with no incident field then such response must be independent of polarization ($\vec{e}_2$ and $\vec{e}_3$) and direction of incidence ($\vec{e}_1$) since they determine no incident fields for this ideal case. This leads to the concept of a natural mode. By a natural mode we mean a current, charge, field, etc. distribution associated with a body self oscillation at a complex natural frequency s_α. We designate natural modes by $\vec{\nu}_\alpha^{(\vec{F})}(\vec{r}\,')$ for a vector quantity $\vec{F}$ and $\nu_\alpha^{(F)}(\vec{r}\,')$ for a scalar quantity F. A scalar quantity F could be a surface charge density ρ_s and a vector quantity $\vec{F}$ could be a surface current density $\vec{J}_s$ parallel to the object surface.

There is of course the question of the uniqueness of the natural modes. Clearly the natural modes can be modified by a scale factor, but this is just a problem of an appropriate normalization of the natural modes. Depending on the problem of concern there may be various appropriate normalizations. Appendix B considers the case of a perfectly conducting sphere for which all the natural frequencies correspond to first order poles ($n_\alpha = 1$ only). For the perfectly conducting sphere the natural modes are types of spherical harmonics and here we use definitions which fit naturally with common usage.

The perfectly conducting sphere is an interesting example in that all the natural modes, coupling coefficients, and natural frequencies can be more readily calculated. Any general results for perfectly conducting finite sized objects must be true for the perfectly conducting sphere. Thus the sphere (and other shapes such as prolate and oblate spheriods and disks) can be used to test general results. Moreover they can be used to form a basis for conjecture for new general results. For example the perfectly conducting sphere has only simple poles ($n_\alpha = 1$ only) which completely describe its response to a delta function plane wave.

Aside from the scale factor there is another problem in defining the natural modes. This is the possible degeneracy of the modes. As can be seen for the sphere problem for example (appendix B) the modes can be degenerate in which case associated with each natural frequency there may be several modes. This problem can easily be handled by the α index set to designate separate independent modes. There are various different ways to define the different modes and what is needed is a convenient set of modes with the minimum number of modes necessary to span the space of all possible distributions of the quantity of interest associated with the particular natural frequency. Of course the sphere has a high degree of symmetry and one would expect degeneracy of the natural modes associated with symmetry. Bodies with a symmetry axis will also have a degeneracy of the form $\cos(n\phi)$ or $\sin(n\phi)$ as a factor in the natural modes (except for $n = 0$) giving at least two independent natural modes (for $n \geq 1$) for each natural frequency. Note that it is also possible (at least in the case of the sphere) to define the natural modes so that more than one natural frequency can have the same natural mode or even set of modes. In many cases this degeneracy problem may be unimportant, especially for irregularly shaped objects.

Further on in this note some calculational techniques for finding the natural frequencies and modes from general integral equation formulations of electromagnetic interaction problems are discussed. In performing such calculations one can observe for each natural frequency being considered whether other than first order poles are present and if more than one natural mode is needed for that natural frequency. Thus the form of the singularity representation can be checked in a problem being worked out and the results compared to more conventional numerical solutions at other frequencies in the s plane and/or in the time domain.

Having the natural modes $\vec{\nu}_\alpha(\vec{r}\,')$ which depend only on the object coordinates and the natural frequencies s_α which are fixed complex numbers we next need the coefficients which multiply $\vec{\nu}_\alpha(\vec{r}\,')(s - s_\alpha)^{-n_\alpha}$ to give the response to our incident plane waves. For the surface current density on the body we write

$$\tilde{\vec{J}}_s(\vec{r}\,',s) = \tilde{\vec{J}}_{s_2}(\vec{r}\,',s) + \tilde{\vec{J}}_{s_3}(\vec{r}\,',s) \tag{2.18}$$

where p = 2, 3 as a subscript designates the part associated with each polarization of the incident wave. Each part can now be written as

$$\tilde{\vec{J}}_{s_p}(\vec{r}\,',s) = \frac{E_o}{Z_o}\,\tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',s) \tag{2.19}$$

where $\tilde{\vec{U}}_p$ is the response of the surface current density to the $\vec{U}_p$ plane wave (Laplace transformed delta function wave) taken as the incident electric field. Note that $\vec{U}_p$ is dimensionless. For cases with volume current densities other normalizations would be appropriate. Our surface current density response functions (two of them, one for each incident electric polarization) are functions of the object coordinates $\vec{r}\,'$ and the complex frequency s (or of time t when inverse transformed) and of course depend on the direction of incidence $\vec{e}_1$.

Now we can write the surface current density response functions for finite sized perfectly conducting bodies as

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',s) = \left\{ \sum_{\alpha} \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')(s - s_\alpha)^{-n_\alpha} \right\} + \tilde{\vec{W}}_p^{(\vec{J}_s)}(\vec{e}_1,\vec{r}\,',s) \tag{2.20}$$

where $\tilde{\eta}_\alpha$ has dimensions of $(\text{time})^{-n_\alpha}$ and where the sum is taken over all the indices in the index set α except for p which is one index for $\tilde{\eta}$. Were α written as say two or three indices then a double or triple sum would be used in this expansion. Note that the coupling to the incident plane wave depends on frequency because there is at least a time delay or advance in when the mode is "turned on."

The coupling coefficients $\tilde{\eta}_\alpha(\vec{e}_1,s)$ are entire functions of s (no singularities in the finite s plane) with values at s_α which give the proper pole coefficients. Note that for $n_\alpha > 1$ the derivatives of $\tilde{\eta}_\alpha$ with respect to s at s_α take a pole of order n_α and give coefficients to terms of order $n_\alpha - 1$, $n_\alpha - 2$, etc. until a first order pole is reached. Then there is some flexibility in our definition of $\tilde{\eta}_\alpha$ as long as at each natural frequency $\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)$ gives the proper coefficient to the highest order pole there. The lower order poles at s_α then can be partially (if not entirely) included in the terms for the poles of higher order depending on the choice of the form of $\tilde{\eta}_\alpha$ for such higher order poles. This points out what might be termed the non uniqueness of the form of the singularity expansion. Certain features of the singularity expansion are fixed, but others have some flexibility. One might then ask what is the best definition of $\tilde{\eta}_\alpha$ consistent with the pole requirements? This might involve such criteria as simplicity of the resulting functions in frequency and/or time domain, asymptotic behavior for $|s| \to \infty$ so as to avoid poles at infinity as separate terms which complicate the form of the time domain expansion, etc.

Also in equation 2.20 we include $\tilde{\vec{W}}_p^{(\vec{J}_s)}(\vec{e}_1,\vec{r}\,',s)$ as an entire function of s containing none of the poles of the response in the finite s plane. This entire function is connected with the choice of the $\tilde{\eta}_\alpha$ and has similar flexibility in its choice. Only the resulting sum need be the unique solution for the current density etc. We have some flexibility in how we arrange the terms. In what follows in this section we consider $\tilde{\eta}_\alpha$ and a special form for the coupling coefficient as $c_\alpha e^{-st_0}$ in developing some of the consequences for the singularity expansion. The additional entire function $\tilde{\vec{W}}_p$ is only included in some of the expressions; it is usually dropped; it is not needed for the perfectly conducting sphere discussed in appendix B. This function is further considered in section 3. For perfectly conducting finite sized bodies the perfectly conducting sphere results suggest that a delay can be factored out so that a coupling coefficient c_α can be written as

$$\tilde{\eta}_\alpha(\vec{e}_1,s) = c_\alpha(\vec{e}_1)e^{-st_o(\vec{e}_1)} \qquad (2.21)$$

How general this result applies is not presently clear. My conjecture is that it applies to perfectly conducting finite objects if not more general objects such as lossy objects, perhaps with some nuances based on the order of the pole being considered. Here t_c is the time that a delta function plane wave first touches the object and is given by

$$t_o(\vec{e}_1) = \min_{\vec{r}'}\left[\frac{\vec{e}_1\cdot\vec{r}'}{c}\right] \qquad (2.22)$$

which is illustrated in figure 2.2. For the perfectly conducting sphere this result is immediately apparent as in appendix B.

Let us call $\tilde{\eta}_\alpha$ and c_α coupling coefficients and t_o the turn on time. In any event the object response for each and every mode is zero for $t < 0$. Note that if (as is often the case) the coordinate origin is inside the object of interest then t_o is a negative time (an advance). The allowable forms for $\tilde{\eta}_\alpha$, the resulting individual terms in the series (all forms giving the same sum), is a very important question in the singularity expansion method. Any entire function of s times $(s - s_\alpha)^{n_\alpha}$ for example can be added to $\tilde{\eta}_\alpha$ without introducing any new poles in the finite s plane. This question is considered further in the next section.

Now that the surface current density response functions for finite size objects are expressed in terms of natural modes, other quantities can be similarly expanded through their relationship to the surface current density. This includes scattered fields. However, for electromagnetic interaction questions we concentrate in this note on the surface current density and surface charge density. From the continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial\rho}{\partial t} = 0 , \quad \nabla \cdot \tilde{\vec{J}} + s\tilde{\rho} = 0 \qquad (2.23)$$

we can find the charge density from the current density. When we are dealing with surface current density $\vec{J}_s$ and surface charge density ρ_s the divergence has to be interpreted as a surface divergence with the spatial derivatives being taken with respect to two coordinates required to describe a position on the surface. Thus we can still write without ambiguity

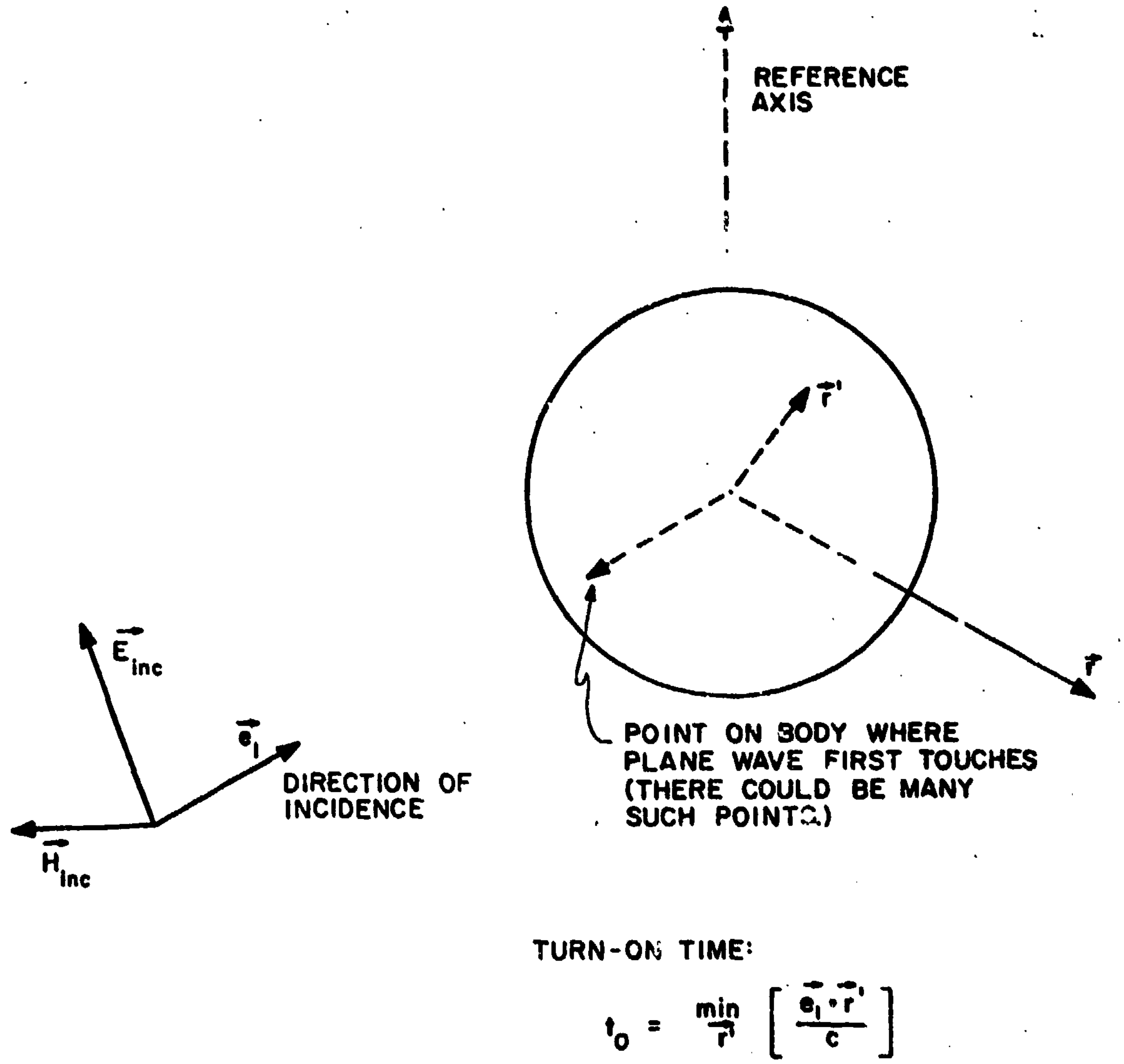

FIGURE 2.2 TURN-ON TIME FOR THE NATURAL MODES

$$\nabla' \cdot \vec{J}_s(\vec{r}',t) + \frac{\partial}{\partial t} \rho_s(\vec{r}',t) = 0 \ ,$$

$$\nabla' \cdot \tilde{\vec{J}}_s(\vec{r}',s) + s\tilde{\rho}_s(\vec{r}',s) = 0 \tag{2.24}$$

where the prime on the ∇ indicates derivatives with respect to the object coordinates $\vec{r}'$.

As before split the surface charge density as

$$\tilde{\rho}_s(\vec{r}',s) = \tilde{\rho}_{s_2}(\vec{r}',s) + \tilde{\rho}_{s_3}(\vec{r}',s) \tag{2.25}$$

to correspond to the two polarizations. Write each part as

$$\tilde{\rho}_{s_p}(\vec{r}',s) = \epsilon_o E_o \tilde{f}_p(s) \tilde{U}_p^{(\rho_s)}(\vec{r}',s) \tag{2.26}$$

where the normalization using $\epsilon_o E_o$ is again chosen to make $\tilde{U}_p$ dimensionless. From the surface current density natural modes construct a set of surface charge density natural modes as

$$\nu_\alpha^{(\rho_s)}(\vec{r}') = -a_\alpha \nabla' \cdot \vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}') \tag{2.27}$$

where a_α is a scale factor with dimension of length which we can choose for convenience, such as to allow some desired normalization condition on the ν_α. If desired the a_α could be all the same and perhaps chosen as some characteristic dimension of the object. Note that some ν_α for the surface charge density might be identically zero for some index sets α. This is possible because current density can be split into two parts, one with zero divergence but nonzero curl and one with zero curl but nonzero divergence if the current density is confined to a volume of finite dimensions.[6]

All the above definitions for surface current and charge density modes can be directly extended to volume modes or combined modes for volume and surface densities.

Taking out common factors the continuity equation allows us to write the surface charge density response functions (using equation 2.20) as

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = \left\{\sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\,\nu_\alpha^{(\rho_s)}(\vec{r}')\left(\frac{sa_\alpha}{c}\right)^{-1}(s-s_\alpha)^{-n_\alpha}\right\} + \tilde{W}_p^{(\rho_s)}(\vec{e}_1,\vec{r}',s) \tag{2.28}$$

where the splitting of η_α can be done as before. All our previous results for the surface current density then carry over to the surface charge density. The same set of natural frequencies and coupling coefficients apply to both (and even to scattered fields if you like). Note that $\tilde{W}_p^{(\rho s)}$ comes from $\tilde{W}_p^{(\vec{J}s)}$ via equations 2.23.

However, there is what looks like a new pole at $s = 0$ introduced into the surface charge density response function. Using a relation for separating poles at separate frequencies we have

$$s^{-1}(s-s_\alpha)^{-1} = s_\alpha^{-1}(s-s_\alpha)^{-1} - s_\alpha^{-1}s^{-1}$$

$$s^{-1}(s-s_\alpha)^{-n_\alpha} = s_\alpha^{-1}(s-s_\alpha)^{-n_\alpha} - s_\alpha^{-2}(s-s_\alpha)^{-n_\alpha+1}$$

$$+ \cdots + (-1)^{n_\alpha-1}s_\alpha^{-n_\alpha}(s-s_\alpha)^{-1} + (-s_\alpha)^{-n_\alpha}s^{-1} \tag{2.29}$$

By this expansion we can write the surface charge density response functions as

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = \sum_\alpha \sum_{m=1}^{n_\alpha} (-1)^{m-1}\frac{c}{a_\alpha}s_\alpha^{-m}\tilde{\eta}_\alpha(\vec{e}_1,s)\,\nu_\alpha^{(\rho_s)}(\vec{r}')(s-s_\alpha)^{-n_\alpha+m-1}$$

$$+ \frac{1}{s}\sum_\alpha (-s_\alpha)^{-n_\alpha}\frac{c}{a_\alpha}\tilde{\eta}_\alpha(\vec{e}_1,s)\,\nu_\alpha^{(\rho_s)}(\vec{r}') + \tilde{W}_p^{(\rho_s)}(\vec{e}_1,\vec{r}',s) \tag{2.30}$$

where the second summation could also be included in α.

Consider for a moment the static response characteristics ($s \to 0$) of a body of finite dimensions. For a plane wave of unit amplitude as $s \to 0$ the object response goes to the static limit in which both current density and charge density are proportional to the field strength. For small $|s|$ the response is negligibly changed from the $s = 0$ case. Thus there is no singularity at s = 0 in the s plane for either current density or charge density. As discussed before for the current density there are no poles with nonzero residue on the $i\omega$ axis for $|\omega| > 0$ because of the restriction of conservation of energy together with reciprocity when considering the time domain response. The static response characteristics rule out poles in the response at s = 0 so that all poles s_α lie in the left half plane $Re[s] < 0$. Thus without loss of generality in our expansions we can require

$$Re[s_\alpha] < 0 \tag{2.31}$$

Cavity modes or any other modes which do not couple to the incident field are excluded from our consideration.

Note that there are static current density and/or charge density distributions which can exist on finite sized objects in the absence of any incident wave. However, such static distributions do not couple to the incident wave and can be included as an additive term in the surface current density and surface charge density response, but with no dependence on the incident waveform. As the response to the incident wave does not depend on these modes we do not include them in our expansion. We can call this case the natural frequency at s = 0 and this is briefly considered in appendix A. This case is like the internal cavity modes which have natural frequencies on the axis $s = i\omega$; these also do not couple to the incident wave and have no dependence on the incident waveform; they can be added into the results at the end if desired.

Referring back to equation 2.30 for the surface charge density response functions note that a pole at s = 0 is not allowed. Thus we have letting $s \to 0$ in the limit

$$\sum_\alpha (-s_\alpha)^{-n_\alpha} \frac{c}{a_\alpha} \tilde{\eta}_\alpha(\vec{e}_1, 0)\, \nu_\alpha^{(\rho_s)}(\vec{r}\,') + \tilde{W}_p^{(\rho_s)}(\vec{e}_1, \vec{r}\,', 0) = 0 \tag{2.32}$$

For cases that η_α can be made to factor (such as for the sphere) as in equation 2.21 then we can write (also dropping $\tilde{W}_p$)

$$\sum_\alpha (-s_\alpha)^{-n_\alpha} \frac{c}{a_\alpha} c_\alpha(\vec{e}_1)\, \nu_\alpha^{(\rho_s)}(\vec{r}\,') = 0 \tag{2.33}$$

This constrains a relation among the natural modes and their coupling coefficients.

For the case that $n_\alpha = 1$ for all α (which is the case for the perfectly conducting sphere, and perhaps for all finite size, perfectly conducting objects) and that the coupling factors as in equation 2.21, the response functions for surface current density and surface charge density can be written as (dropping the W functions).

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',s) = e^{-st_o}\sum_\alpha c_\alpha(\vec{e}_1)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')(s - s_\alpha)^{-1}$$

$$\tilde{U}_p^{(\rho_s)}(\vec{r}\,',s) = e^{-st_o}\sum_\alpha \frac{c}{s_\alpha a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,') \qquad (2.34)$$

This is a rather simple looking result with each term quite factored. This also points out the importance of understanding under what circumstances (as for the sphere) the delay factoring of the coupling coefficient can be used.

Now the waveform functions $\tilde{f}_p(s)$ can be reintroduced and multiplied on both sides of equations 2.34 or equations 2.20 and 2.30. Recombining the polarizations as in equations 2.18, 2.19, 2.25, and 2.26 gives complete representations of the solutions for surface current density and surface charge density. This would give individual terms with frequency dependence in the form $\tilde{f}_p(s)\tilde{\eta}_{p\alpha}(\vec{e}_1,s)(s - s_\alpha)^{-n_\alpha+m-1}$ which, for the case as before that $\tilde{\eta}$ factors, has the frequency dependence in the form $e^{-st_o}\tilde{f}_p(s)(s - s_\alpha)^{-n_\alpha+m-1}$ which for $n_\alpha = 1$ reduces to $e^{-st_o}\tilde{f}_p(s)(s - s_\alpha)^{-1}$.

However, why stop here? The behavior of $\tilde{f}_p(s)$ may allow us to conveniently express it in terms of waveform singularities, just as we have been considering the object delta function response in terms of object singularities. The general idea then is to expand $\tilde{f}_p(s)$ in terms of its singularities and separate the waveform and object singularities into separate terms. We might call the resulting separate terms as the waveform part and object part of the response for convenience. Consider an example by letting

$$\tilde{f}_p(s) = \frac{1}{s - s_w}, \quad f_p(t) = e^{s_w t}u(t) \qquad (2.35)$$

so that the waveform is a decaying exponential with a simple pole at s_w (a waveform pole). Note that the commonly used double exponential waveform for EMP environments is nothing more than the sum of two terms such as this.

Taking the surface current density first we have (for $s_w \neq s_\alpha$ for any α)

$$\tilde{\vec{V}}_p^{(\vec{J}_s)}(\vec{r}',s) \equiv \tilde{f}_p(s)\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}',s) = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}')(s-s_w)^{-1}(s-s_\alpha)^{-n_\alpha}$$

$$\equiv \tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}',s) + \tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}',s) \tag{2.36}$$

where the object part is

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}',s) = \sum_\alpha \sum_{m=1}^{n_\alpha} (-1)^{m-1}(s_\alpha-s_w)^{-m}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}^{(\vec{J}_s)}(\vec{r}')(s-s_\alpha)^{-n_\alpha+m-1} \tag{2.37}$$

where the second sum could also be included in α and where the waveform part is

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}',s) = \frac{1}{s-s_w}\sum_\alpha (s_w - s_\alpha)^{-n_\alpha}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}') \tag{2.38}$$

Note that here and for most of what follows the W functions are not included but can be included in a convenient way in the object-waveform split. For the special case of $n_\alpha = 1$ this reduces to

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}',s) = \sum_\alpha (s_\alpha-s_w)^{-1}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}')(s-s_\alpha)^{-1}$$

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}',s) = \frac{1}{s-s_w}\sum_\alpha (s_w-s_\alpha)^{-1}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}') \tag{2.39}$$

For more general waveforms (but still $n_\alpha = 1$) this suggests a definition of the splitting as

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}',s) = \sum_\alpha \tilde{f}_p(s_\alpha)\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}')(s-s_\alpha)^{-1} \tag{2.40}$$

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) = \sum_{\alpha} \tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')\frac{\tilde{f}_p(s)-\tilde{f}_p(s_\alpha)}{s-s_\alpha}$$

As long as the waveform singularities are separate from all s_α then no object poles appear in the waveform part of the response. If the s_α do lie on waveform singularities then special treatment is needed but the general idea of equations 2.40 would still seem appropriate. For n_α more than just 1 the object part can be defined through a Taylor expansion of $\tilde{f}_p$ around each s_α giving

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}\,',s) \equiv \sum_{\alpha}\sum_{m=1}^{n_\alpha}\frac{f_p^{(m-1)}(s_\alpha)}{(m-1)!}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')(s-s_\alpha)^{-n_\alpha+m-1}$$

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) \equiv \tilde{\vec{V}}_{p}^{(\vec{J}_s)}(\vec{r}\,',s) - \tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}\,',s) \tag{2.41}$$

As a special case consider the unit step waveform by letting $s_w = 0$. Then the waveform part of the response we write as

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) = \frac{1}{s}\sum_{\alpha}\tilde{\eta}_\alpha(\vec{e}_1,s)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')(-s_\alpha)^{-1} \tag{2.42}$$

With $\tilde{\eta}_\alpha$ factored as before and assuming $n_\alpha = 1$ the results reduce to

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) = \frac{e^{-st_o}}{s}\tilde{\vec{U}}_{p}^{(\vec{J}_s)}(\vec{r}\,',0)$$

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}\,',s) = e^{-st_o}\sum_{\alpha}s_\alpha^{-1}c_\alpha(\vec{e}_1)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,')(s-s_\alpha)^{-1} \tag{2.43}$$

where the static surface current density response is

$$\tilde{\vec{U}}_{p}^{(\vec{J}_s)}(\vec{r}\,',0) = \sum_{\alpha}(-s_\alpha)^{-1}c_\alpha(\vec{e}_1)\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,') \tag{2.44}$$

The step function response of the object is readily constructed from the delta function response. A factor of $(-s_\alpha)^{-1}$ is included with each object pole, and then a new term is added which is nothing more than the static response time a step function with turn-on time of t_0. Thus beside tabulating the natural frequencies, modes, and coupling coefficients of finite size objects we can tabulate the static response so as to readily construct the step response of the object in both Laplace and time domains.

Now the neat thing about the static response is that we need not consider it in terms of direction of incidence and polarization. For the static surface current density response we can solve for the magnetic field using the Laplace equation with a uniform magnetic field incident on the object. The exponential factor in the incident plane wave (equations 2.11) becomes irrelevant, going to 1. Thus we need only consider three separate cases of the incident magnetic field, corresponding to three orthogonal direction such as the cartesian axes (x, y, z). For a unit incident static H field we have $\vec{J}_s$ response functions $\vec{U}_{s_x}$ etc. for each of the three axes giving a dyadic surface current density response function as

$$\vec{\vec{U}}_s^{(\vec{J}_s)}(\vec{r}\,') \equiv \vec{U}_{s_x}^{(\vec{J}_s)}(\vec{r}\,')\vec{e}_x + \vec{U}_{s_y}^{(\vec{J}_s)}(\vec{r}\,')\vec{e}_y + \vec{U}_{s_z}^{(\vec{J}_s)}(\vec{r}\,')\vec{e}_z \qquad (2.45)$$

Since the H field is in the direction $\vec{e}_1 \times \vec{e}_p$ for p = 2, 3 we can write

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',0) = \vec{\vec{U}}_s^{(\vec{J}_s)}(\vec{r}\,')\cdot[\vec{e}_1\times\vec{e}_p]$$

$$= \vec{U}_{s_x}^{(\vec{J}_s)}(\vec{r}\,')[\vec{e}_x\cdot(\vec{e}_1\times\vec{e}_p)] + \vec{U}_{s_y}^{(\vec{J}_s)}(\vec{r}\,')[\vec{e}_y\cdot(\vec{e}_1\times\vec{e}_p)] + \vec{U}_{s_z}^{(\vec{J}_s)}(\vec{r}\,')[\vec{e}_z\cdot(\vec{e}_1\times\vec{e}_p)]$$

(2.46)

showing a set of direction cosines for three static solutions. The static response dyadic for the surface current density is then a useful tool for directly extending the delta function response to the step function response. One could go to other types of "static" terms for s^{-2} (ramp) and higher order such waveforms. The step function waveform is a useful tool because of its fast rise time to a finite amplitude making the static response correspondingly important.

The surface charge density has similar properties when the waveform is reintroduced as

$$\tilde{V}_p^{(\rho_s)}(\vec{r}\,',s) \equiv \tilde{f}_p(s)\tilde{U}_p^{(\rho_s)}(\vec{r}\,',s) = \tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s) + \tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) \tag{2.47}$$

As before the idea is to associate the object poles with the object part of the response $\tilde{V}_{p_o}$ and the rest with the waveform part $\tilde{V}_{p_w}$. If n_α takes on more values than one the double sum as in equation 2.30 extends into a triple sum including coefficients $\tilde{f}_p(s)$ and the first $n_\alpha - 1$ derivatives all evaluated at s_α.

For simplicity just consider the case of $n_\alpha = 1$ with $\tilde{n}_\alpha$ factorable as above so that we have the simple surface charge density response function as in equations 2.34. The object and waveform parts of the response may then be written as

$$\begin{aligned}
\tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s) &= e^{-st_o}\sum_\alpha \tilde{f}_p(s_\alpha)\frac{c}{s_\alpha a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,')(s-s_\alpha)^{-1} \\
\tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) &= e^{-st_o}\sum_\alpha \frac{c}{s_\alpha a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,')\frac{\tilde{f}_p(s)-\tilde{f}_p(s_\alpha)}{s-s_\alpha}
\end{aligned} \tag{2.48}$$

The response to an exponential waveform as in equations 2.35 takes the forms

$$\begin{aligned}
\tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s) &= e^{-st_o}\sum_\alpha (s_\alpha-s_w)^{-1}\frac{c}{s_\alpha a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,')(s-s_\alpha)^{-1} \\
\tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) &= \frac{e^{-st_o}}{s-s_w}\sum_\alpha (s_w-s_\alpha)^{-1}\frac{c}{s_\alpha a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,')
\end{aligned} \tag{2.49}$$

For $s_w = 0$ we have the step response as

$$\begin{aligned}
\tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s) &= e^{-st_o}\sum_\alpha \frac{c}{s_\alpha^2 a_\alpha}\, c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,')(s-s_\alpha)^{-1} \\
\tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) &= \frac{e^{-st_o}}{s}\tilde{U}_p^{(\rho_s)}(\vec{r}\,',0)
\end{aligned} \tag{2.50}$$

where the static surface charge density is given by

$$\tilde{U}_p^{(\rho_s)}(\vec{r}\,',0) = \sum_\alpha (-s_\alpha)^{-1} \frac{c}{s_\alpha a_\alpha} c_\alpha(\vec{e}_1)\nu_\alpha^{(\rho_s)}(\vec{r}\,') \tag{2.51}$$

The step function surface charge density response is then also constructed from the delta function response in the same way as the surface charge density with the same form of the results, at least for $n_\alpha = 1$ and a common delay factorable from $\tilde{n}_\alpha$.

For the static surface charge density response we need consider only the response to three orthogonal incident static electric fields, directly analogous to the relationship of the static incident magnetic field and surface current density. For a unit incident static E field we have ρ_s response functions (scalars) as U_{s_x} etc. for each of the three axes giving a vector surface charge density response function as

$$\vec{U}_s^{(\rho_s)}(\vec{r}\,') \equiv U_{s_x}^{(\rho_s)}\vec{e}_x + U_{s_y}^{(\rho_s)}\vec{e}_y + U_{s_z}^{(\rho_s)}\vec{e}_z \tag{2.52}$$

Since the E field is in the direction $\vec{e}_p$ for p = 2, 3 we can write

$$\begin{aligned}\tilde{U}_p^{(\rho_s)}(\vec{r}\,',0) &= \vec{U}_s^{(\rho_s)}(\vec{r}\,')\cdot\vec{e}_p \\ &= U_{s_x}^{(\rho_s)}[\vec{e}_x\cdot\vec{e}_p]+U_{s_y}^{(\rho_s)}[\vec{e}_y\cdot\vec{e}_p]+U_{s_z}^{(\rho_s)}[\vec{e}_z\cdot\vec{e}_p]\end{aligned} \tag{2.53}$$

showing a set of direction cosines to weight the three static solutions. Thus for step response purposes it is useful to tabulate the vector surface charge density response and the dyadic surface current density response which can be multiplied in a dot or inner product sense with an appropriate scale factor times the static field of interest to obtain the static response. This applied in both frequency domain for small $|s|$ and in time domain for an important term in the step function response.

As discussed near the beginning of this section the fact that we are dealing with real valued time functions makes the Laplace transformed functions have certain symmetry properties with respect to the Ω axis as expressed by equation 2.4. Basically $\tilde{f}(\Omega + i\omega)$ has its real part symmetric with respect to a sign reversal of ω while its imaginary part is antisymmetric with respect to a sign reversal of ω. Also as discussed before all object poles with nonzero coupling to the incident wave lie in the left half plane $\Omega < 0$. These two results tell something about the object pole pattern in the complex s plane. For convenience then let us split the α index set into three parts. For $\omega > 0$ (the upper half plane) use α_+, for $\omega = 0$ (the Ω axis) use α_o, and for $\omega < 0$ (the lower half plane) use α_-. Since poles come in conjugate pairs for those not on the Ω axis then we can relate α_- to α_+ as

$$s_{\alpha_-} = \bar{s}_{\alpha_+} \tag{2.54}$$

which specifies which α_- goes with which α_+ except for the case of multiple poles at s_α in which case we make the identification of the α_- set to the α_+ set with the value of n_α in each case the same so that we require

$$n_{\alpha_-} = n_{\alpha_+} \tag{2.55}$$

Equations 2.54 and 2.55 define a one to one correspondence between α_- and α_+ index sets, unless we have mode degeneracy in which case we also require α_- and α_+ correspond to conjugate modes with conjugate coupling coefficients as well.

From the conjugate symmetry requirement the natural modes and coupling coefficients can be made to have the same conjugate relations. Thus we set

$$\begin{aligned}
\vec{\nu}_{\alpha_-}^{(\vec{J}_s)}(\vec{r}\,') &= \bar{\vec{\nu}}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}\,') \;, & \mathrm{Im}\left[\vec{\nu}_{\alpha_o}^{(\vec{J}_s)}(\vec{r}\,')\right] &= \vec{0} \\
\nu_{\alpha_-}^{(\rho_s)}(\vec{r}\,') &= \bar{\nu}_{\alpha_+}^{(\rho_s)}(\vec{r}\,') \;, & \mathrm{Im}\left[\nu_{\alpha_o}^{(\rho_s)}(\vec{r}\,')\right] &= 0 \\
a_{\alpha_-} &= \bar{a}_{\alpha_+} \;, & \mathrm{Im}[a_{\alpha_o}] &= 0 \\
\tilde{\eta}_{\alpha_-}(\vec{e}_1,s) &= \bar{\tilde{\eta}}_{\alpha_+}(\vec{e}_1,\bar{s}) \;, & \tilde{\eta}_{\alpha_o}(\vec{e}_1,s) &= \bar{\tilde{\eta}}_{\alpha_o}(\vec{e}_1,\bar{s}) \\
c_{\alpha_-}(\vec{e}_1) &= \bar{c}_{\alpha_+}(\vec{e}_1) \;, & \mathrm{Im}[c_{\alpha_o}(\vec{e}_1)] &= 0
\end{aligned} \tag{2.56}$$

With these relations we can now write the surface current density response functions as

$$\tilde{U}_p^{(\vec{J}_s)}(\vec{r}',s) = \sum_{\alpha_o} \tilde{n}_{\alpha_o}(\vec{e}_1,s)\,\vec{v}_{\alpha_o}^{(\vec{J}_s)}(\vec{r}')(s-s_{\alpha_o})^{-n_\alpha}$$

$$+ \sum_{\alpha_+}\Big[\tilde{n}_{\alpha_+}(\vec{e}_1,s)\,\vec{v}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}')(s-s_{\alpha_+})^{-n_{\alpha_+}}$$

$$+ \bar{\tilde{n}}_{\alpha_+}(\vec{e}_1,\bar{s})\,\bar{\vec{v}}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}')(s-\bar{s}_{\alpha_+})^{-n_{\alpha_+}}\Big]$$

$$= e^{-st_o}\sum_{\alpha_o} c_{\alpha_o}(\vec{e}_1)\,\vec{v}_{\alpha_o}^{(\vec{J}_s)}(\vec{r}')(s-s_{\alpha_o})^{-n_\alpha}$$

$$+ e^{-st_o}\sum_{\alpha_+}\Big[c_{\alpha_+}(\vec{e}_1)\,\vec{v}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}')(s-s_{\alpha_+})^{-n_{\alpha_+}}$$

$$+ \bar{c}_{\alpha_+}(\vec{e}_1)\,\bar{\vec{v}}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}')(s-\bar{s}_{\alpha_+})^{-n_{\alpha_+}}\Big] \qquad (2.57)$$

where the case where $\tilde{n}_\alpha$ factors out a simple delay is also included. Note now that each term in the sum has the conjugate symmetry of equation 2.4, is real on the Ω axis, and corresponds to real valued time function. Similar properties apply to the surface charge density response functions. Taking the simpler form with $\tilde{n}_\alpha$ factored as in equations 2.34 with only $n_\alpha = 1$ we can write

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = e^{-st_o}\sum_{\alpha_o} \frac{c}{s_{\alpha_o}a_{\alpha_o}}\, c_{\alpha_o}(\vec{e}_1)\,\nu_{\alpha_o}^{(\rho_s)}(\vec{r}')(s-s_\alpha)^{-1}$$

$$+ e^{-st_o}\sum_{\alpha_+}\Big[\frac{c}{s_{\alpha_+}a_{\alpha_+}}\, c_{\alpha_+}(\vec{e}_1)\,\nu_{\alpha_+}^{(\rho_s)}(\vec{r}')(s-s_{\alpha_+})^{-1}$$

$$+ \frac{c}{\bar{s}_{\alpha_+}\bar{a}_{\alpha_+}}\, c_{\alpha_+}(\vec{e}_1)\,\bar{\nu}_{\alpha_+}^{(\rho_s)}(\vec{r}')(s-\bar{s}_{\alpha_+})^{-1}\Big] \qquad (2.58)$$

The same idea is readily applied to the more complex forms.

If $\tilde{\eta}_\alpha$ factors as in equation 2.21 then the s dependence of each term is a delay with a term of the form $(s - s_\alpha)^{-n'}$ where n' might be n_α or other more complicated exponents. Converting this term to the time domain through the well known Laplace transform pair gives

$$L^{-1}\left[e^{-st_o}(s-s_\alpha)^{-n'}\right] = \frac{(t-t_o)^{n'-1}}{(n'-1)!} e^{s_\alpha(t-t_o)} u(t-t_o) \qquad (2.59)$$

For the extremely interesting case of n' = 1 this is

$$L^{-1}\left[e^{-st_o}(s-s_\alpha)^{-1}\right] = e^{s_\alpha(t-t_o)} u(t-t_o) \qquad (2.60)$$

This is a damped sinusoidal waveform like that which suggested looking at the natural frequencies in the first place. The damping constant is just $\Omega_\alpha = \text{Re}[s_\alpha]$ (plus or minus as one wishes) and the radian oscillation frequency is just $\omega_\alpha = \text{Im}[s_\alpha]$ where α would be taken as α_+ or α_o for this frequency to make $\omega_\alpha \geq 0$. Since $\Omega_\alpha < 0$ for all poles of interest then each term in the response function goes to zero for large time as we would expect. Writing out the real and imaginary parts gives

$$L^{-1}\left[e^{-st_o}(s-s_\alpha)^{-n'}\right] = \frac{(t-t_o)^{n'-1}}{(n'-1)!} e^{\Omega_\alpha(t-t_o)}$$

$$[\cos(\omega_\alpha(t-t_o)) + i\sin(\omega_\alpha(t-t_o))]u(t-t_o) \qquad (2.61)$$

If the $\tilde{\eta}_\alpha$ terms do not factor as above then we need their inverse transforms to convolute with terms similar to these.

Consider then the surface current density delta function response (for η factored). From equation 2.57 we have

$$\vec{U}_p^{(\vec{J}_s)}(\vec{r}',t) = \sum_{\alpha_o} c_{\alpha_o}(\vec{e}_1)\vec{\nu}_{\alpha_o}^{(\vec{J}_s)}(\vec{r}')\frac{(t-t_o)^{n_\alpha-1}}{(n_{\alpha_o}-1)!} e^{\Omega_{\alpha_o}(t-t_o)} u(t-t_o)$$

$$+ \sum_{\alpha_+} 2\text{Re}\left[c_{\alpha_+}(\vec{e}_1)\vec{\nu}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}')e^{i\omega_{\alpha_+}(t-t_o)}\right]\frac{(t-t_o)^{n_{\alpha_+}-1}}{(n_{\alpha_+}-1)!} e^{\Omega_{\alpha_+}(t-t_o)} u(t-t_o)$$

(2.62)

For the surface charge density for $n_\alpha = 1$ we have the delta function response as

$$U_p^{(\rho_s)}(\vec{r}',t) = \sum_{\alpha_o} \frac{c}{s_{\alpha_o} a_{\alpha_o}} c_{\alpha_o}(\vec{e}_1) \nu_{\alpha_o}^{(\rho_s)}(\vec{r}') e^{\Omega_{\alpha_o}(t-t_o)} u(t-t_o)$$

$$+ \sum_{\alpha_+} 2\mathrm{Re}\left[\frac{c}{s_{\alpha_+} a_{\alpha_+}} c_{\alpha_+}(\vec{e}_1) \nu_{\alpha_+}^{(\rho_s)}(\vec{r}') e^{i\omega_{\alpha_+}(t-t_o)}\right] e^{\Omega_{\alpha_+}(t-t_o)} u(t-t_o) \quad (2.63)$$

From the way that the terms are split up we have separated the damped exponential parts of the response from the damped oscillatory parts. Depending on the real and imaginary parts of the natural modes, natural frequencies, and coupling coefficients the initial "phase angle" of the sinusoidal oscillation may vary with direction of incidence $\vec{e}_1$, polarization p, and/or position $\vec{r}'$ on the object. If one wished he might consider the real and imaginary parts of a natural mode as separate modes and consider the coupling to each part with different distributions perhaps over the object.

With the incident waveforms reintroduced we can also directly write down response waveforms for cases that $\tilde{\eta}_\alpha$ factors as before. Consider the case that no waveform singularities are at any of the s_α. In particular consider the step function response for all $n_\alpha = 1$. For the surface current density the object and waveform parts from equations 2.43 and 2.46 are

$$\vec{V}_{p_o}^{(\vec{J}_s)}(\vec{r}',t) = \sum_{\alpha_o} s_{\alpha_o}^{-1} c_{\alpha_o}(\vec{e}_1) \vec{\nu}_{\alpha_o}^{(\vec{J}_s)}(\vec{r}') e^{\Omega_{\alpha_o}(t-t_o)} u(t-t_o)$$

$$+ \sum_{\alpha_+} 2\mathrm{Re}\left[s_{\alpha_+}^{-1} c_{\alpha_+}(\vec{e}_1) \vec{\nu}_{\alpha_+}^{(\vec{J}_s)}(\vec{r}') e^{i\omega_{\alpha_+}(t-t_o)}\right] e^{\Omega_{\alpha_+}(t-t_o)} u(t-t_o) \quad (2.64)$$

$$\vec{V}_{p_w}^{(\vec{J}_s)}(\vec{r}',t) = \vec{U}_s^{(\vec{J}_s)}(\vec{r}') \cdot [\vec{e}_1 \times \vec{e}_p] u(t-t_o)$$

For the surface charge density the object and waveform parts of the step response functions from equations 2.50 and 2.53 are

$$v_{p_o}^{(\rho_s)}(\vec{r}',t) = \sum_{\alpha_o} \frac{c}{s_{\alpha_o}^2 a_{\alpha_o}} c_{\alpha_o}(\vec{e}_1)\nu_{\alpha_o}^{(\rho_s)}(\vec{r}')e^{\Omega_{\alpha_o}(t-t_o)} u(t-t_o)$$

$$+ \sum_{\alpha_+} 2\mathrm{Re}\left[\frac{c}{s_{\alpha_+}^2 a_{\alpha_+}} c_{\alpha_+}(\vec{e}_1)\nu_{\alpha_+}^{(\rho_s)}(\vec{r}')e^{i\omega_{\alpha_+}(t-t_o)}\right] e^{\Omega_{\alpha_+}(t-t_o)} u(t-t_o) \tag{2.65}$$

$$v_{p_w}^{(\rho_s)}(\vec{r}',t) = \vec{U}_s^{(\rho_s)}(\vec{r}')\cdot\vec{e}_p u(t-t_o)$$

In the time domain only one term is added to the delta function response, namely a step function. The coefficients of the damped sinusoidal (and simple exponential) terms are rather simply altered.

The response functions for other kinds of incident waveforms can be readily found for the object part by introducing coefficients $\tilde{f}_p(s_\alpha)$ in the time domain waveforms in the same forms as introduced in the Laplace versions such as in equations 2.41 and 2.48. The waveform part can be more difficult if only because of the many possibilities one might choose for incident waveforms. Different incident waveforms give different types of time domain waveforms when combined with the terms arising from the object poles. Note that the response to an incident waveform cannot always be simply split into object and waveform parts. As a trivial example suppose $\tilde{f}_p(s)$ itself has a pole at some s_α. In such a case the contribution from s_α to the response has a higher order pole than the delta function response. However the response associated with s_α is then easily treated separately and the same type of time domain function as in equation 2.59 results.

In this section we have tried to give some insight into the power of the singularity expansion method for representing solutions in both frequency and time domains, at least for finite sized objects. While many variations of this problem have been considered, complicating the notation somewhat, the solution of specific problems may be expressed somewhat simpler with the notation adapted to the results of the problem at hand. Specific convenient choices of the α index set are needed for each problem and the possibilities for n_α can be limited. The natural modes and/or coefficients may be expressible as purely real quantities in some cases, even for natural frequencies off the Ω axis. A basic question concerns the coupling coefficients $\tilde{\eta}_\alpha$ and the other entire functions $\tilde{W}_p$. These need to be optimally chosen for the different problems at hand depending on early times, late times, etc.

There are still types of problems which have singularity expansions with terms such as branches which have not been considered here. Infinite sized objects have object responses with such terms. However the reader should have a general idea by now of what the singularity expansion method is all about. What has been done for expanding in terms of natural frequencies for object response plus another term for waveform response can be carried over to natural frequencies and branches for the object response with perhaps increased complexity associated with the branches.

III. Properties of the Singularity Expansion of the Object Response for Finite Size Objects Viewed from the Finite Matrix Formulation of Integral Equations

There are two somewhat complementary ways to view some of the questions regarding the form of the singularity expansion for objects of finite size. One approach involves consideration of the properties of an integral equation formulation for the continuous object geometry. Various types of integral equations such as those classed as electric field formulation, magnetic field formulation, extended boundary condition formulation, etc. can be investigated to obtain mathematical theorems concerning the properties of the singularity expansion for various kinds of object classes. This might be termed the continuous integral operator approach. Drs. Marin and Latham are presently using the magnetic field integral equation to consider the question of there being only poles in the finite s plane in the object response for finite size perfectly conducting object from this viewpoint. Clearly this kind of approach is needed for considering many such questions so as to establish general theorems applying for all frequencies and time and stated exactly in an analytical form. This could be considered a viewpoint which is based on the continuous nature of the object geometries. Eigenfunction expansions can also be used for such considerations but the tabulated cases of such expansions are limited. Eigenfunction expansions are possible for general kinds of object geometries but they must be numerically calculated.[7] However the analytic properties of such eigenfunction expansions can still be used to investigate the singularity expansion. Viewed another way such eigenfunction expansions are representations of the integral operators defined over the object characteristics.

The second and complementary approach might be termed the discrete approach. This refers to zoning the object into many discrete zones and treating each zone as a position with a particular current density etc. associated with it. The integral equation then takes the form of a vector-matrix equation which has the general form

$$(g_{n,m}(s)) \cdot (\tilde{J}_m(s)) = (\tilde{I}_n(s)) \qquad n,m = 1, 2, \cdots, N \qquad (3.1)$$

where N is some typically large integer. Here the vectors $(\tilde{J}_m)$ and $(\tilde{I}_n)$ each have N components and are not the same as the three component space vectors. The index n refers to the $\vec{r}$ coordinates for the "incident" quantities shown here in general as $(\tilde{I}_n)$ which might come from an electric or magnetic field that is incident and therefore specified. Note that in forming the linear equations summarized as equation 3.1 each spatial zone can have one, two, or three nonzero current density components and this influences how the indices n and m are set up. The $(\tilde{J}_m)$ refers to the unknown current ($\vec{r}\,'$ coordinates) with one, two, or three components in each zone. The matrix $(g_{n,m})$ has elements which couple the n zone-component with the m zone-component; it is basically a spatially discrete form of the integral operator. While equation 3.1 is shown in a form suggesting that it is intended for solving for the current density this is not necessarily the case; it could be solving for another quantity from which the current density would be obtained by a subsequent calculation. For our present discussion, however, consider it the current density.

Considering the current density components in each zone as our unknowns is not the only way to obtain a set of linear equations from a given integral equation. The current density can be expanded in a more general set of expansion functions and a matrix-vector equation formed to obtain the coefficients in this expansion. Of course, only a finite set is used to obtain a finite $N \times N$ matrix and the sets of functions involved should be in some sense complete in the limit of large N. This general approach is often termed the method of moments.[8] Both the zoning approach and more general function expansions are valuable from the viewpoint of numerical calculations. In terms of the singularity expansion method there is clearly much work to be done in refining the numerical techniques to find natural frequencies, modes, etc. most efficiently and most accurately. Much that has been done for other numerical problems can likely be applied here.

Besides the practical aspect of numerical computations the matrix-vector formulation can be used as a theoretical technique for establishing some of the general characteristics of singularity expansions. If in the limit of large N the solution for the current density

$$(\tilde{J}_m(s)) = (g_{n,m})^{-1} \cdot (\tilde{I}_n) \tag{3.2}$$

converges to the exact solution of the integral equation then by understanding general properties of $(\tilde{J}_m)$ we can find general properties for the continuous case, i.e. $\tilde{\vec{J}}(\vec{r}', s)$. Only where the matrix elements are not uniquely defined or the inverse matrix

$$F \equiv (f_{m,n}) \equiv (g_{n,m})^{-1} \tag{3.3}$$

does not exist is $(\tilde{J}_m)$ not uniquely defined or non existent.

Let us then look at the singularity expansion characteristics of the approximate numerical solution $(\tilde{J}_m)$. In the limit of large N for convergent matrix-vector formulations the singularity expansion of $(\tilde{J}_m)$ will be that of $\vec{\tilde{J}}(\vec{r}', s)$. Consider the case that some zoning technique is used to convert the integral equation to a matrix-vector equation by dividing the body surface or volume as appropriate into discrete zones of finite linear dimensions. The number of zones is then less than or equal to N. Clearly then we are only considering finite sized objects here because an infinite surface area or infinite volume cannot be divided up into a finite number of areas and/or volumes with finite linear dimensions unless some very strange extensions to infinity with finite volume and/or surface area are included. Let us consider only finite sized objects here so that all distances $|\vec{r}_n - \vec{r}_m|$ between zones are finite. The coordinates $\vec{r}$ and $\vec{r}'$ go over the $\vec{r}_n$ and $\vec{r}_m$ and are centered on each zone in some sense. The incident quantities and resulting current density are evaluated in an appropriate average sense at each zone center.

Let $\vec{\tilde{I}}(\vec{r}, s)$ be based on an incident delta function plane wave. It might come from the electric and/or magnetic field or some spatial projection of these on the object. Since this is an entire function of s (analytic in the entire finite s plane for all $\vec{r}$ on the finite size object) then the discrete formulation $(\tilde{I}_n)$ has each element $(\tilde{I}_n(s))$ as an entire function of s. In this formulation let the current density be the sum of <u>all</u> terms representing charge motion, including $\sigma\vec{\tilde{E}}$ (conduction current density), $s(\varepsilon - \varepsilon_0)\vec{\tilde{E}}$ (displacement current density), and $\nabla \times [(\mu - \mu_0)\vec{\tilde{H}}]$ (magnetization current density). Split electric and magnetic fields each into the sum of incident plus scattered parts. The incident parts are given in the problem definition. The scattered parts can then be represented as integrals over the current density with kernels involving the free space Green's function which uses $\gamma = s/c$ instead of some propagation constant involving local medium parameters which may vary with position.

Thus we have a pair of volume integral equations (which may reduce to surface integral equations say for perfectly conducting bodies) which equate scattered electric and magnetic fields (which could be thought of as a six component vector or even a four tensor in relativistic formulation) to integrals involving incident plus scattered fields. Move the terms involving incident fields to one side of the equations and use these to form the N component vector $(\tilde{I}_n)$. Note that σ, ε, and μ are assumed to be single valued analytic functions of s except possibly for poles which can be removed by multiplying through the equations by an appropriate zero to make each $\tilde{I}_n(s)$ an entire function. The scattered parts are used to form $(g_{n,m})$ and $(\tilde{J}_m)$ which ~~an~~ is an appropriate scattered field vector or the current density vector less that part directly proportional to the incident fields. Note that σ, ε, and μ enter the coefficients on the scattered side and are analytic single valued here then as well while the exponential terms use $\gamma = s/c$ which is an entire function.

Through this procedure we can construct an integral equation relating current density and/or surface current density as appropriate to the incident fields involving only analytic single valued functions as long as the media are suitably well behaved. By various manipulations such a pair of integral equations can be converted into various more desirable forms. Then the integral equations can be converted into a single matrix-vector equation which can have various forms.

One advantage of a zoning formulation for these considerations is the somewhat physical picture one can associate with the discrete zones. Infinitesimal differential line, surface, and volume elements become discrete ones of small size. As long as complex radian wavelengths have magnitude large compared to zone size then the interaction between adjacent zones and one zone on itself are described by statics. We can start thinking of the zoned object as a big circuit with simple elements connecting adjacent zones but more complex ones connecting distant zones. In each zone there are a few equivalent sources associated with the discrete source elements $\tilde{I}_n$.

The kernels of the various integral equations are based on the scalar Green's function (for free space) as

$$G(s,|\vec{r}-\vec{r}\,'|) = \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|} \tag{3.4}$$

with

$$\gamma = \frac{s}{c} \tag{3.5}$$

This scalar Green's function is an analytic function of s for finite $|\vec{r}-\vec{r}\,'|$ with $|\vec{r}-\vec{r}\,'| \neq 0$. This function forms the basis for the various kernels used in the various types of integral equation formulations of general interaction or scattering problems. The dyadic Green's function can be written in the form[9,10]

$$\vec{\vec{G}}(s,|\vec{r}-\vec{r}\,'|) = \left[1 - \frac{1}{\gamma^2}\nabla\nabla\cdot\right][G(s,|\vec{r}-\vec{r}\,'|)\vec{\vec{1}}] = \left[\vec{\vec{1}} - \frac{1}{\gamma^2}\nabla\nabla\right]G(s,|\vec{r}-\vec{r}\,'|) \tag{3.6}$$

with the identity dyadic

$$\vec{\vec{1}} = (\delta_{b_1,b_2}) = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{for } b_1,b_2 = 1,\ 2,\ 3 \tag{3.7}$$

Note that the dyadic Green's function is formed from the scalar Green's function by multiplication by γ^{-2} (which contains s^{-2}) and spatial derivatives as in $\nabla\nabla$. Thus for finite $|\vec{r}-\vec{r}'|$ (but not zero) the dyadic Green's function is also an analytic function of s, except at $s = 0$. However in going to static limits the s^{-2} term combines with other terms in s to avoid an object pole at $s = 0$ as discussed before. Similarly terms like ∇G etc. appear in the kernels and the same analyticity properties with respect to s apply.

Now in forming the elements of the matrix $(g_{n,m})$ for $\vec{r}_n \neq \vec{r}_m$ the coordinates $\vec{r}_n$ and $\vec{r}_m$ are used in the various Green's function type terms. In such cases then the matrix element must be an analytic function of s except possibly for poles which don't bother us. The spatial derivatives can be replaced by finite difference operators based on the spacing between nearby zones in the zoning system defined to segment the body. For $\vec{r}_g = \vec{r}_h$ (including the case $g = h$) the problem is basically a static one. The scalar Green's function can be written as

$$G(s,|\vec{r}-\vec{r}'|) = \frac{1}{4\pi|\vec{r}-\vec{r}'|} - \frac{\gamma}{4\pi} + \frac{\gamma^2}{8\pi}|\vec{r}-\vec{r}'| + O(s^3) \tag{3.8}$$

Taking the first few terms as needed then the matrix elements $g_{n,m}$ for $\vec{r}_n = \vec{r}_m$ (including $g_{n,n}$) can be defined and they too are analytic functions of s. Thus the matrix $(g_{n,m})$ can be defined as an analytic function of s except for possible poles of finite order. Note that finite zone size is important here because zone size contributes to the matrix elements and vector components and we want to avoid infinite values for these.

One should be careful of various approximate formulations so that the resulting matrix elements are analytic in s without branch cuts. For example it is common in various thin wire type formulations to obtain terms containing functions like $\ln(\gamma w)$ where here w is the wire radius. Such terms result from integrating the Green's function to remove a coordinate such as the azimuthal angle around the wire, perhaps approximating the result and then integrating say along the wire in the process of finding an asymptotic form for small wire radius. Such thin wire formulations are important and perhaps even useful in developing singularity expansions but for finite size bodies they do have this limitation which needs to be recognized. For our present purposes we do not use such types of formulations.

The cofactor matrix D for $(g_{n,m})$ is

$$D \equiv (d_{n,m}) = \left((-1)^{n+m}\det G_{n,m}\right) \tag{3.9}$$

where $G_{n,m}$ is the N - 1 by N - 1 matrix formed by deleting the nth row and mth column from $(g_{n,m})$. The determinant of $(g_{n,m})$ we indicate by

$$\Delta \equiv \det(g_{n,m}) \tag{3.10}$$

Note that the transpose is

$$D^T \equiv (d_{m,n}) \tag{3.11}$$

The solution of our matrix equation can then be written

$$(\tilde{J}_m(s)) = (f_{m,n}(s)) \cdot (\tilde{I}_n(s)) \tag{3.12}$$

where

$$(f_{m,n}) = \frac{1}{\Delta}(d_{m,n})$$

$$d_{m,n} = (-1)^{n+m} \det G_{n,m} \tag{3.13}$$

Now since the $g_{n,m}$ are analytic functions of s except possibly for poles, then the same is true of both the $d_{m,n}$ and Δ since the determinant is a polynomial function of the matrix elements. Therefore since the $\tilde{I}_n(s)$ are entire functions of s and the $f_{m,n}(s)$ are analytic except for poles, then the $\tilde{J}_m(s)$ must also be analytic functions of s except for poles. Furthermore any poles of the $g_{n,m}$ elements are associated with poles in the medium parameters or with the Green's function with various operators on it. Poles associated with the Green's function are at s = 0 and do not concern us because they give no resulting poles with non zero coefficients based on physical grounds if the media are passive. Poles in the medium parameters σ, ε, μ can be troublesome so for our present discussion we restrict ourselves to the case of no such medium poles. Such poles in passive medium parameters could, however, be included in the expansion technique if desired. With this restriction then the poles in $(\tilde{J}_m(s))$ are the zeros of Δ and the order of each pole is less than or equal to the order of the corresponding zero of Δ.

The order of the zero at s_α must be finite since if Δ and all its derivatives are zero at s_α then as an analytic function Δ must be identically zero implying no unique solution for $(\tilde{J}_m)$ violating the uniqueness theorem for the solution of Maxwell's equations. Furthermore the zeros of Δ in the finite s plane must be isolated, i.e. only a finite number are allowed in a finite region of the s plane. This again is a property of analytic functions. As an example suppose that there were a line of zeros for Δ. Then by analytic continuation Δ would be zero in the entire s plane, again not allowed.

This result is of course of fundamental importance to the singularity expansion for finite objects. It relies on restrictions on the form of the medium parameters and the convergence of the particular matrix-vector formulation of various possible integral equations. The response has only pole singularities of finite order and isolated in the finite s plane. Since we also restrict the object to be passive then these poles cannot be in the right half plane. Furthermore as discussed previously (and in appendix A) any poles on the $i\omega$ axis must have zero coupling coefficients and so we can state for our purposes for passive objects that all poles in the finite s plane have negative real part ($\Omega_\alpha < 0$). Stated another way how can the $(\tilde{J}_m)$ have branches (removing multiple values) when the elements $g_{n,m}$ are chosen to have no branches because they are single valued?

Another way to view this question of only poles for the singularities in the finite s plane concerns the numerical results per se. Matrix-vector formulation of various integral equations has been common for a long time and has been used to obtain many accurate solutions for finite size objects. For formulations which have used analytic matrix elements and incident vector components (except for poles in the finite s plane) the numerical results obtained (usually on the $i\omega$ axis and sometimes converted to time domain) must be representable by a singularity expansion which has only poles in the finite s plane by our previous discussion. Thus in cases where accurate results have been obtained such results are accurately representable by such a singularity expansion. The accuracy of such a singularity expansion for finite size objects is thus related to the accuracy of the corresponding matrix-vector representation of the integral equation from which it is derived.

Knowing that the inverse matrix $(f_{m,n})$ has only poles in the finite s plane this still leaves open the question of singularities at infinity associated with an entire function such as e^{-st_0}. One might argue that this matrix is like an admittance matrix for a large circuit composed of inductors, resistors, and capacitors and like such cases might have only ratios of polynomials in s for matrix elements. However for large N the degree of such polynomials could get arbitrarily large. Thus for completeness at the moment let us write

$$(f_{m,n}(s)) = \left\{\sum_{\alpha} (s-s_\alpha)^{-n_\alpha}(f_{m,n})_\alpha\right\} + (f_{m,n}(s))_e \qquad (3.14)$$

where the $(f_{m,n})_\alpha$ are constant matrices which are the coefficients of each pole of order $n_\alpha \geq 1$ and $(f_{m,n}(s))_e$ is an entire matrix function of s. An interesting question for further investigation concerns the properties of $(f_{m,n}(s))_e$ for some matrix size N by N and in the limit of large N; perhaps it is zero for many cases of interest.

Written in matrix-vector form the current density expansion as in equation 2.20 for the delta function response with polarization p is then

$$(\tilde{J}_m(s))_p = \sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)(\nu_m)_\alpha (s-s_\alpha)^{-n_\alpha} + (\tilde{W}_m(\vec{e}_1,s))_p \tag{3.15}$$

At each s_α we can expand each $\tilde{\eta}_\alpha$ in a power series since each $\tilde{\eta}_\alpha$ is an entire function. The successive terms in such an expansion produce poles of order $n_\alpha - 1$, $n_\alpha - 2$, etc. associated with each $n_\alpha \geq 2$. Note that for each s_α there may be several n_α while each n_α can have several $\vec{\nu}_\alpha$ (degeneracy) each of which has an $\tilde{\eta}_\alpha$. Expanding the inverse matrix $(f_{m,n}(s))$ as in equation 3.14, substituting this in equation 3.12, and multiplying by $(s-s_\alpha)^{N_\alpha}$ we then obtain for each polarization

$$\sum_{\substack{n_\alpha=N_\alpha \\ \text{(all modes)}}} \tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)(\nu_m)_\alpha = (f_{m,n})_\alpha \Big|_{n_\alpha=N_\alpha} \cdot (\tilde{I}_n(\vec{e}_1,s_\alpha))_p \tag{3.16}$$

where N_α is the maximum pole order n_α at the s_α of interest. Note that the number of independent $\vec{\nu}_\alpha$ for this case need not be infinite because the coefficient matrices $(f_{m,n})_\alpha$ come from the inversion of $(g_{n,m}(s))$ near the s_α where $\Delta = 0$. Similarly we can expand $(\tilde{I}_n)$ near s_α to obtain the coefficients of the next higher order pole contribution (order $N_\alpha - 1$) as

$$\sum_{n_\alpha=N_\alpha-1} \tilde{\eta}_\alpha(\vec{e}_1,s_\alpha)(\nu_m)_\alpha + \sum_{n_\alpha=N_\alpha} \left[\frac{d}{ds}\tilde{\eta}_\alpha(e_1,s)\right]_{s=s_\alpha} (\nu_m)_\alpha$$

$$= (f_{m,n})_\alpha \Big|_{n_\alpha=N_\alpha-1} \cdot (\tilde{I}_n(\vec{e}_1,s_\alpha))_p + (f_{m,n})_\alpha \Big|_{n_\alpha=N_\alpha}$$

$$\cdot \frac{d}{ds}\left[(\tilde{I}_n(e_1,s))_p\right]_{s=s_\alpha} \tag{3.17}$$

Knowing the $\vec{\nu}_\alpha$ and the choice for $\tilde{\eta}_\alpha(\vec{e}_1,s)$ then the terms for $n_\alpha = N_\alpha$ can be calculated to leave this last equation as one for the $n_\alpha = N_\alpha - 1$ terms. This can be repeated to obtain all the terms for the poles of all orders at s_α. If the $\vec{\nu}_\alpha$ are degenerate for a given n_α and are constructed to give an orthogonal set for this case in the sense

$$(\tilde{\nu}_m)_{\alpha'} \cdot (\nu_m)_\alpha = 0 \quad \text{for } \alpha \neq \alpha' \tag{3.18}$$

then the coupling coefficients have the relation for $n_\alpha = N_\alpha$ as

$$\tilde{\eta}_\alpha(\vec{e}_1,s_\alpha) = \left[(\tilde{\nu}_m)_\alpha \cdot (\nu_m)_\alpha\right]^{-1}(\tilde{\nu}_m)_\alpha \cdot (f_{m,n})_\alpha \cdot (\tilde{I}_n(\vec{e}_1,s_\alpha)) \tag{3.19}$$

For purposes of calculation with the inverse matrix suppose Δ has a zero at s_α of ord N_α, the maximum pole order of $(f_{m,n}(s))$ at s_α. Say that near s_α

$$\begin{aligned} \Delta(s) &= (s-s_\alpha)^{N_\alpha}\Delta_{N_\alpha} + (s-s_\alpha)^{N_\alpha+1}\Delta_{N_\alpha+1} + \cdots \\ \frac{1}{\Delta(s)} &= (s-s_\alpha)^{-N_\alpha}\delta_{-N_\alpha} + (s-s_\alpha)^{-N_\alpha+1}\delta_{-N_\alpha+1} + \cdots \end{aligned} \tag{3.20}$$

with the coefficients related as

$$\delta_{-N_\alpha} = \frac{1}{\Delta_{N_\alpha}}, \quad \delta_{-N_\alpha+1} = -\frac{\Delta_{N_\alpha+1}}{\Delta^2_{N_\alpha}}, \tag{3.21}$$

and so on by inverting the series for $\Delta(s)$ to one for $\Delta^{-1}(s)$. Then the $(f_{m,n})_\alpha$ can be calculated from

$$(f_{m,n}(s)) = \frac{1}{\Delta(s)}(d_{m,n}(s)) \tag{3.22}$$

$$(f_{m,n})_\alpha \Big|_{n_\alpha} = \sum_{\ell=0}^{N_\alpha - n_\alpha} \delta_{-N_\alpha+\ell} (d_{m,n})_\ell$$

$$(d_{m,n})_\ell = \frac{1}{\ell!}\left[\frac{d^\ell}{ds^\ell}(d_{m,n}(s))\right]_{s=s_\alpha} \tag{3.23}$$

and so on by collecting terms in expanding the series for $\Delta(s)$ around s_α into a series for $1/\Delta(s)$. Note that there is a different set of $(d_{m,n})_\ell$ coefficient matrices at each s_α so they can also use the general α subscript set.

Instead of using the inverse matrix approach we can set up an eigenvector equation, factoring out the coupling coefficient, by first expanding $(g_{n,m}(s))$ in a Taylor series around s_α as

$$(g_{n,m}(s)) = \sum_{\ell=0}^{\infty} (s-s_\alpha)^\ell (g_{n,m})_\ell$$

$$(g_{n,m})_\ell = \frac{1}{\ell!}\left[\frac{d^\ell}{ds^\ell}(g_{n,m}(s))\right]_{s=s_\alpha} \tag{3.24}$$

Note again that there is a separate set of coefficient matrices for each s_α so the $(g_{n,m})_\ell$ can also use the α set. Substitute this series together with equation 3.15 into equation 3.1 to obtain

$$\left\{\sum_{\ell=0}^{\infty} (s-s_\alpha)^\ell (g_{n,m})_\ell\right\} \cdot \left\{\sum_\alpha \tilde{\eta}_\alpha(\vec{e}_1,s)(\nu_m)_\alpha (s-s_\alpha)^{-n_\alpha} + (\tilde{W}_m(\vec{e}_1,s))\right\}$$

$$= (\tilde{I}_n(s)) \tag{3.25}$$

Since the right side has no poles at s_α then collect the coefficients of each pole on the left side and equate them to zero beginning with the highest order pole at s_α for $n_\alpha = N_\alpha$ as

$$(g_{n,m})_o \cdot (\nu_m)_\alpha \Big|_{n_\alpha = N_\alpha} = (0) = (g_{n,m}(s_\alpha)) \cdot (\nu_m)_\alpha \Big|_{n_\alpha = N_\alpha} \tag{3.26}$$

where the coupling coefficient is assumed non zero and factored out and (0) is an N component vector with all zero elements. There may be several natural modes for $n_\alpha = N_\alpha$ perhaps associated with symmetry. If there is mode degeneracy for $n_\alpha = N_\alpha$ then construct an orthogonal set which spans the space of such modes so that the property in equation 3.18 applies. Going to the $n_\alpha = N_\alpha - 1$ term we have (for $N_\alpha \geq 2$)

$$\sum_{n_\alpha = N_\alpha} \tilde{\eta}_\alpha(\vec{e}_1, s_\alpha)(g_{n,m})_1 \cdot (\nu_m)_\alpha + \sum_{n_\alpha = N_\alpha - 1} \left[\frac{d}{ds} \tilde{\eta}_\alpha(\vec{e}_1, s)\right]_{s=s_\alpha} (g_{n,m})_o$$

$$\cdot (\nu_m)_\alpha = (0) \tag{3.27}$$

At this point let us consider the form of $\tilde{\eta}_\alpha$ to simplify some of these terms.

Suppose that we restrict $\tilde{\eta}_\alpha$ to have the form

$$\tilde{\eta}_\alpha(\vec{e}_1, s) \equiv \frac{1}{\tilde{T}(s)} c_\alpha(\vec{e}_1) \tag{3.28}$$

where we have factored out the complex frequency dependence as a common factor for all α. This common factor $\tilde{T}(s)$ is specified to be an entire function in the complex s plane; $\tilde{T}(s)$ may even have zeros in the finite s plane if we wish to remove some poles before making a singularity expansion. The new or modified set of coupling coefficients $c_\alpha(e_1)$ are specified to be constants and are the same type of constant coupling coefficients as used in section 2 as for example in equation 2.21. One choice of $\tilde{T}(s)$ is clearly e^{st_o} as used in section 2.

Now multiply the matrix-vector equation 3.1 by $\tilde{T}(s)$ to obtain

$$(g_{n,m}(s)) \cdot [\tilde{T}(s)(\tilde{J}_m(s))] = \tilde{T}(s)(\tilde{I}_n(s)) \tag{3.29}$$

The right hand side is still an entire function. Then restrict for our present analysis the coupling coefficients to have the form in equation 3.28. Solve for the c_α and divide the results by $\tilde{T}(s)$.

Equation 3.25 can now be rewritten as

$$\left\{\sum_{\ell=0}^{\infty}(s-s_\alpha)^{\ell}\,(g_{n,m})_\ell\right\}\cdot\left\{\sum_\alpha c_\alpha(\vec{e}_1)\,(\nu_m)_\alpha\,(s-s_\alpha)^{-n_\alpha}+\tilde{T}(s)\,(\tilde{W}_m(\vec{e}_1,s))\right\}$$

$$= \tilde{T}(s)\,(\tilde{I}_n(s)) \tag{3.30}$$

The s derivatives of the coupling coefficients as in equation 3.27 now do not appear. Equation 3.26 can then be generalized for $n_\alpha = 1, 2, \cdots, N_\alpha$ as

$$(g_{n,m})_\ell \cdot (\nu_m)_\alpha \bigg|_{n_\alpha = N_\alpha - \ell} = (0) \tag{3.31}$$

from which natural modes can be calculated in the form of orthogonal natural modes constructed for each n_α for which there is modal degeneracy.

Similarly using the expansion of the inverse matrix we can write

$$\tilde{T}(s)\,(\tilde{J}_m(s)) = \left\{\sum_\alpha (s-s_\alpha)^{-n_\alpha}(f_{m,n})_\alpha + (f_{m,n}(s))_e\right\}\cdot[\tilde{T}(s)\,(\tilde{I}_n(s))]$$

$$= \sum_\alpha c_\alpha(\vec{e}_1)\,(\nu_m)_\alpha\,(s-s_\alpha)^{-n_\alpha} + \tilde{T}(s)\,(\tilde{W}_m(\vec{e}_1,s)) \tag{3.32}$$

Around s_α make the Taylor expansion

$$\tilde{T}(s)\,(\tilde{I}_n(s)) = \sum_{\ell=0}^{\infty}(s-s_\alpha)^{\ell}\,(T_n)_\ell$$

$$(T_n)_\ell = \frac{1}{\ell!}\left[\frac{d^\ell}{ds^\ell}[\tilde{T}(s)\,(\tilde{I}_n(s))]\right]_{s=s_\alpha} \tag{3.33}$$

where the coefficient vectors $(T_n)_\ell$ can also use the α index set. Then for the pole at s_α of order M we have

$$\sum_{n_\alpha=M} c_\alpha(\vec{e}_1)(\nu_m)_\alpha = \sum_{\ell=0}^{N_\alpha-M} (f_{m,n})_\alpha \Big|_{n_\alpha=M+\ell} \cdot (T_n)_\ell \tag{3.34}$$

from which we can obtain both the natural modes (constructed as orthogonal set) and the coupling coefficients. Note that the dependence on $\vec{e}_1$ is contained in the vectors $(T_n)_\ell$. If we let $M \geq N_\alpha$ then we can relate the entire functions as well. Alternately having found all the c_α and $\vec{\nu}_\alpha$ these can be subtracted from equation 3.32 to leave the entire function $\tilde{T}(s)(\tilde{W}_m(\vec{e}_1,s))$ expressed in terms of $\tilde{T}(s)(\tilde{I}_n(s))$ together with the inverse matrix coefficients.

With the natural modes known equation 3.34 gives an expression for the coupling coefficients for $\alpha = \alpha'$ as

$$c_{\alpha'}(\vec{e}_1) = \left[(\bar{\nu}_m)_{\alpha'} \cdot (\nu_m)_{\alpha'}\right]^{-1} \sum_{\ell=0}^{N_\alpha-M} (\bar{\nu}_n)_{\alpha'} \cdot (f_{m,n})_\alpha \Big|_{n_\alpha=M+\ell} \cdot (T_n)_\ell \tag{3.35}$$

The coefficient matrices for the inverse matrix can be related to the Taylor expansion of $(g_{n,m}(s))$ through

$$(g_{n,m'}(s)) \cdot (f_{m',m}(s)) = (\delta_{n,m}) = (f_{n,m'}(s)) \cdot (g_{m',m}(s))$$

$$(\delta_{n,m}) = \left\{\sum_{\ell=0}^{\infty} (s-s_\alpha)^\ell (g_{n,m'})_\ell\right\} \cdot \left\{\sum_{q=1}^{N_\alpha} (s-s_\alpha)^{-q}(f_{m',m})_q + (f_{m',n}(s))_e\right\} \tag{3.36}$$

where now the coefficient matrices $(f_{m',n})_q$ apply for the particular s_α of interest; the remaining terms at other s_α can be included with the remainder function for present purposes. Equating the resulting singular terms to zero gives for the highest order pole

$$(g_{n,m'})_o \cdot (f_{m',m})_{N_\alpha} = (0_{n,m}) = (f_{n,m'})_{N_\alpha} \cdot (g_{m',m})_o \tag{3.37}$$

where $(0_{n,m})$ is an $N \times N$ matrix with zero elements. From this result the columns (fixed m) of $(f_{m',m})_{N_\alpha}$ must be eigenvectors of $(g_{n,m'})_o$ and the rows (fixed n) of $(g_{n,m'})_o$ must be eigenvectors of the transpose of $(g_{n,m'})_o$. Also the rows of $(f_{n,m'})_{N_\alpha}$

are eigenvectors of the transpose of $(g_{m',m})_o$ and the columns of $(g_{m',m})_o$ are eigenvectors of $(f_{n,m'})_{N_\alpha}$. For poles of order n' in equation 3.36 we have the general result

$$\sum_{q-\ell=n'} (g_{n,m'})_\ell \cdot (f_{m',n})_{n'-\ell} = (0_{n,m}) = \sum_{q-\ell=n'} (f_{n,m'})_{n'-\ell} \cdot (g_{m',m})_\ell \tag{3.38}$$

where $1 \le n' \le N_\alpha$.

Consider the special but practically important case that the pole at s_α is simple ($N_\alpha = 1$). Then the natural modes are the eigenvectors from

$$(g_{n,m}(s_\alpha)) \cdot (\nu_m)_\alpha = (0) \tag{3.39}$$

The coupling coefficients are

$$c_\alpha(\vec{e}_1) = \left[(\bar{\nu}_m)_\alpha \cdot (\nu_m)_\alpha\right]^{-1} (\bar{\nu}_m)_\alpha \cdot (f_{m,n})_1 \cdot (T_n)_o$$

$$= \left[(\bar{\nu}_m)_\alpha \cdot (\nu_m)_\alpha\right]^{-1} (\bar{\nu}_m)_\alpha \cdot (f_{m,n})_1 \cdot [\tilde{T}(s_\alpha)(\tilde{I}_n(s_\alpha))] \tag{3.40}$$

where

$$(f_{m,n})_1 = \lim_{s \to s_\alpha} (s-s_\alpha)(f_{m,n}(s))$$

$$= \left[\lim_{s \to s_\alpha} \frac{s-s_\alpha}{\Delta(s)}\right](d_{m,n}(s_\alpha)) = \frac{1}{\Delta_1}(d_{m,n}(s_\alpha)) \tag{3.41}$$

with the relation

$$(g_{n,m'}(s_\alpha)) \cdot (f_{m',m})_1 = (0_{n,m}) = (f_{n,m'})_1 \cdot (g_{m',m}(s_\alpha)) \tag{3.42}$$

In another form the natural modes and coupling coefficients are found from

$$\sum_{n_\alpha=1} c_\alpha(\vec{e}_1)(\nu_m)_\alpha = (f_{m,n})_1 \cdot (T_n)_o$$

$$= (f_{m,n})_1 \cdot [\tilde{T}(s_\alpha)(\tilde{I}_n(s_\alpha))] \qquad (3.43)$$

which for the case of no modal degeneracy reduces to

$$c_\alpha(\vec{e}_1)(\nu_m)_\alpha = (f_{m,n})_1 \cdot [\tilde{T}(s_\alpha)(\tilde{I}_n(s_\alpha))] \qquad (3.44)$$

There is an alternate way of calculating the coupling coefficients. Again let $N_\alpha = 1$ at the s_α pole. From equation 3.42 the columns of $(f_{m,n})_1$ are eigenvectors of $(g_{n,m}(s_\alpha))$. Assume no degeneracy of the natural modes at s_α. Then the columns of $(f_{m,n})_1$ are all the same except for a scalar factor since $(\nu_m)_\alpha$ is unique except for a scalar factor. Call these constants μ_m so we can write

$$(f_{m,n})_1 = (\mu_1(\nu_m)_\alpha,\ \mu_2(\nu_m)_\alpha, \cdots\cdot,\ \mu_n(\nu_m)_\alpha, \cdots\cdot,\ \mu_N(\nu_m)_\alpha) \qquad (3.45)$$

Then we have

$$(\bar{\nu}_m)_\alpha \cdot (f_{m,n})_1 = [(\bar{\nu}_m) \cdot (\nu_m)](\mu_n)_\alpha \qquad (3.46)$$

defining a new N component vector so that $(f_{m,n})_1$ can be written in dyadic form as

$$(f_{m,n})_1 = (\nu_m)_\alpha (\mu_n)_\alpha \qquad (3.47)$$

where the outer vector product or dyadic product is used here. Thus we also have

$$(f_{m,n})_1 \cdot (\bar{\mu}_n)_\alpha = \left[(\bar{\mu}_n)_\alpha \cdot (\mu_n)_\alpha\right](\nu_m)_\alpha \qquad (3.48)$$

Now from equation 3.42 the rows of $(f_{m,n})_1$ are eigenvectors of the transpose of $(g_{n,m}(s_\alpha))$ so that we have

$$(\mu_n)_\alpha \cdot (g_{n,m}(s_\alpha)) = (0) \qquad (3.49)$$

From equation 3.40 we can then write the coupling coefficient as

$$c_\alpha(\vec{e}_1) = (\mu_n)_\alpha \cdot [\tilde{T}(s_\alpha)(\tilde{I}_n(s_\alpha))] \tag{3.50}$$

giving a simpler form of the result in terms of the eigenvector of the transpose of $(g_{n,m}(s_\alpha))$. Note that since we have assumed only one independent $(\nu_m)_\alpha$ there is only one independent $(\mu_m)_\alpha$.

Now similar statements can be made about the eigenvectors of $(f_{m,n})_1$ from equation 3.42. Specifically the columns of $(g_{n,m})$ must be eigenvectors of $(f_{m,n})_1$ and the rows of $(g_{n,m})$ must be eigenvectors of the transpose of $(f_{m,n})_1$. From equation 3.47 the eigenvectors of $(f_{m,n})_1$ are orthogonal to $(\mu_n)_\alpha$ and the eigenvectors of the transpose of $(f_{m,n})_1$ are orthogonal to $(\nu_m)_\alpha$.

We can calculate the transpose eigenvector $(\mu_n)_\alpha$ from equation 3.49 except for a scale factor. To find this scale factor we can use equation 3.46 to give

$$\left[(\bar{\nu}_m)_\alpha \cdot (\nu_m)_\alpha\right]\left[(\bar{\mu}_n)_\alpha \cdot (\mu_n)_\alpha\right] = (\bar{\nu}_m)_\alpha \cdot (f_{m,n})_1 \cdot (\bar{\mu}_n)_\alpha$$

$$= \delta_{-1}(\bar{\nu}_m)_\alpha \cdot (d_{m,n}(s_\alpha)) \cdot (\bar{\mu}_n)_\alpha \tag{3.51}$$

If we have found a $(\mu_n')_\alpha$ from equation 3.49 then

$$(\mu_n)_\alpha = \beta(\mu_n')_\alpha$$

$$\beta = \frac{1}{\Lambda_1}\left[(\bar{\nu}_m)_\alpha \cdot (\nu_m)_\alpha\right]^{-1}\left[(\bar{\mu}_n')_\alpha \cdot (\mu_n')_\alpha\right]^{-1}(\bar{\nu}_m)_\alpha \cdot (d_{m,n}(s_\alpha)) \cdot (\bar{\mu}_n')_\alpha \tag{3.52}$$

Another powerful result comes from taking the constant term in equation 3.36 as

$$(\delta_{n,m}) = (g_{n,m'})_1 \cdot (f_{m',m})_1 + (g_{n,m'})_o \cdot (f_{m'n}(s_\alpha))_e$$

$$= (f_{n,m'})_1 \cdot (g_{m',m})_1 + (f_{m',n}(s_\alpha))_e \cdot (g_{m',m})_o \tag{3.53}$$

Dot multiply on the right by the eigenvector $(\nu_m)_\alpha$ of $(g_{m',m})_o$ to make the second term vanish giving

$$(\nu_n')_\alpha = (f_{n,m'})_1 \cdot (g_{m',m})_1 \cdot (\nu_m)_\alpha \tag{3.54}$$

This defines a matrix which maps $(\nu_m)_\alpha$ back into itself. Having some $(\mu_n')_\alpha$ solving equation 3.49 we also have from dot multiplying on the left of equation 3.53 the result

$$(\mu_m')_\alpha = (\mu_n')_\alpha \cdot (g_{n,m'})_1 \cdot (f_{m',m})_1 \tag{3.55}$$

Thus multiplying equation 3.44 on the left by $(\mu_n')_\alpha \cdot (g_{n,m})_1 \cdot$ we have after rearranging terms

$$c_\alpha(\vec{e}_1) = \left[(\mu_n')_\alpha \cdot (g_{n,m})_1 \cdot (\nu_m)_\alpha\right] (\mu_n')_\alpha \cdot [\tilde{T}(s_\alpha)(\tilde{I}_n(s_\alpha))] \tag{3.56}$$

so that we can construct $(\mu_n)_\alpha$ as

$$(\mu_n)_\alpha = \left[(\mu_n')_\alpha \cdot (g_{n,m})_1 \cdot (\nu_m)_\alpha\right] (\mu_n')_\alpha \tag{3.57}$$

giving the additional result

$$(\mu_n)_\alpha \cdot (g_{n,m})_1 \cdot (\nu_m)_\alpha = 1 \tag{3.58}$$

Thus $(\mu_n)_\alpha$ can be explicitly constructed from $(g_{n,m})_0$ through its eigenvector and the eigenvector of its transpose plus another matrix $(g_{n,m})_1$ which also comes directly from $(g_{n,m}(s))$. Thus after one finds s_α as a zero of $\Delta(s)$ then if this zero of $\Delta(s)$ is simple and the natural mode nondegenerate, both the natural mode and the coupling coefficient can be found from $(g_{n,m}(s))$ through the special formulas above without having to calculate the coefficients in the expansion of the inverse matrix. If the zero of $\Delta(s)$ is not simple but of second order, third order, etc. then the formulas are somewhat more complex. Thus given an integral equation for the currents on the object and some matrix-vector representation of this which uses analytic elements with at most poles and assuming the approximate solution converges in the limit of large N, then there are various techniques to calculate the natural frequencies, natural modes, and coupling coefficients. Furthermore the coupling coefficients can have many forms, all of which give the correct contributions at the poles but can have various behavior as $|s| \to \infty$; this also affects the form of the entire function which is an additional term in the solution.

Let us now list some of the alternate forms that the coupling coefficients can take. While all forms give the correct damped sinusoid parts of the waveform at late times there is still the question of convergence, particularly at early times. We need to know more general results for the behavior of the entire functions (or behavior at infinity) for various types of objects as they enter into the singularity expansion. Of course convergence to the correct result can be determined in part by calculating the entire functions. One can observe the result for the matrix inversion times the incident vector for particular cases of $\vec{e}_1$ and p in frequency and/or time domains and compare to the sum of the first several poles to see where there is or is not convergence and what it takes to remedy the situation.

Consider then several types of coupling coefficients.

Type 1: Factor out the turn on time of the object.

This type of coupling coefficient is defined by

$$\tilde{T}(s) \equiv e^{st_o} \tag{3.59}$$

so that the coupling coefficients factor as restricted by equation 3.28 in the form

$$\tilde{\eta}_\alpha(\vec{e}_1,s) \equiv e^{-st_o} c_\alpha(\vec{e}_1) \tag{3.60}$$

This form is used in many cases in section 2. In appendix B the perfectly conducting sphere is shown to have this form of result with no entire function in addition to the pole expansion. There is an entire function e^{-st_o} which is common to all coupling coefficients. However whether we need an entire function as a separate term in the sum is not at all clear for more general objects. In time domain as discussed in section 2 the resulting pole terms go over to damped sinusoids (including cases of no oscillation) for simple poles with powers of t appearing for higher order poles. For $t < t_o$ there are no currents on the objects which is physically correct. At $t = t_o$ all modes turn on all over the object. Before any fields can reach a particular position on the object (associated with propagation by the shortest possible path which is not always a straight line) the modes with the transformed entire function must all add to zero (if the sum converges) at that position, so this is a test for this type of coupling coefficient representation.

Type 2: Factor out the time the incident wave function first turns on at the observer position.

This type of coupling coefficient is defined by

$$\tilde{T}(s) \equiv e^{st'}$$
$$\tilde{\eta}_\alpha(\vec{e}_1, s) \equiv e^{-st'} c_\alpha(\vec{e}_1) \qquad (3.61)$$

where

$$t' = \frac{\vec{e}_1 \cdot \vec{r}\,'}{c} \qquad (3.62)$$

with $\vec{r}\,'$ as the particular observer position on the object where the current, charge, etc. is to be calculated. For this type of definition then $c_\alpha(\vec{e}_1)$ is not the same for the whole object because the definition is changed to apply to each observer position separately. However the natural frequencies and natural modes are still the same for the whole object. For $t < t'$ at the observation position the fields, etc. must be zero and for most positions $t' > t_o$ for general objects. This form of coupling coefficient then may have some advantages for early time calculations. For the numerical problem the observation position is discrete, say $\vec{r}\,'_n$. Thus we would use

$$t'_n = \frac{\vec{e}_1 \cdot \vec{r}\,'_n}{c} \qquad (3.63)$$

Note the t'_n is the first time that the vector component I_n in the incident vector (I_n) turns on. This c_α is as easy to calculate as the one in equation 3.60 but it is calculated as many times as there are observer positions. In fact for $t > t'$ the two forms give exactly the same time domain waveform for simple poles. For $t_o < t < t'_n$ type 1 gives a non zero waveform for the particular pole contribution. Thus we can calculate the type 1 coefficient but just wait until time t'_n to turn it on.

To see this in general form take an arbitrary time t_a and make

$$\tilde{T}(s) = e^{st_a} \qquad (3.64)$$

Then

$$\tilde{n}_\alpha(\vec{e}_1,s) = e^{-st_a} c_\alpha(\vec{e}_1) = e^{-(s-s_\alpha)t_a} (\mu_{n_\alpha}) \cdot (\tilde{I}_n(s_\alpha)) \tag{3.65}$$

which when combined with the simple pole $(s - s_\alpha)^{-1}$ gives a time domain form as

$$L^{-1}\left[\tilde{n}_\alpha(\vec{e}_1,s)(s-s_\alpha)^{-n_\alpha}\right] = e^{s_\alpha t_a}(\mu_{n_\alpha}) \cdot (\tilde{I}_n(s_\alpha)) e^{s_\alpha(t-t_a)} u(t-t_a)$$

$$= (\mu_{n_\alpha}) \cdot (\tilde{I}_n(s_\alpha)) e^{s_\alpha t} u(t-t_a) \tag{3.66}$$

Thus for a simple pole the chosen starting time only shifts the turn on in the unit step. The waveform it multiplies stays the same. Therefore for $t > t_a$ all simple pole waveforms are the same independent of t_a. For higher order poles a term $(t - t_a)^{n_\alpha - 1}$ enters in but this can be expanded to leave $t^{n_\alpha - 1}$ as the leading power invariant. Furthermore considering all the pole contributions at s_α for $n_\alpha = 2, 3, \cdots$ the results can be manipulated to cancel some terms but we do not go into this here.

Type 3: Factor out the time that resultant fields can first exist at the observer position.

One can choose t_a as in equation 3.64 such that it is the first time that any field can reach the observer at $\vec{r}_n'$. This can be longer than the time t_n'. It can be calculated from geometrical optics. For example if the observer is on the surface of a perfectly conducting object and $\vec{e}_1$ is such that it must point through the object to reach the observer then the wave must come around the object and arrive at a time greater than t_n'. As another example a dielectric object with propagation velocity less than c may still have the wave propagating through the object reach an observer on a "shadowed" side first; this time is still greater than t_n'. Also for example the expansion of the response of the perfectly conducting sphere need not be started at time $t = t_o$ but t_o can be replaced by a more appropriate t_a in all the step functions.

Clearly if one chooses t_a to be greater than the time a signal first begins at an observer then there must be an entire function contribution for $t < t_a$ because the pole contributions would all be zero. Such a choice would not seem very useful.

Type 4: Expand the inverse matrix in poles but leave the entire function for the incident wave as a coefficient.

This approach is attractive for the case that the expansion of the inverse matrix $(f_{m,n}(s))$ as in equation 3.14 needs no additional entire function $(f_{m,n}(s))_e$ in the expansion or at least that this entire function is known and preferably has a simple form. Define a time

$$t_1(\vec{e}_1) = \max_{\vec{r}'}\left[\frac{\vec{e}_1\cdot\vec{r}'}{c}\right] \tag{3.67}$$

While the time t_o is the turn on time this new time t_1 is the time when the incident wave has reached every position on the object neglecting any scattered fields from the object. As such t_1 might be called the turn off time. Suppose we write the incident vector as

$$(\tilde{I}_n(\vec{e}_1,s)) = \left(b_n(\vec{e}_1)e^{-st'_n}\right) \tag{3.68}$$

so that the time delay in each component is explicitly displayed. Consider a special excitation function consisting of the nth component of this vector being as above but all other components zero. Then we would calculate an $\tilde{\eta}_{\alpha n}(\vec{e}_1,s)$ with $e^{-st'_n}$ as a factor. Repeat this for all n and add up the results on the basis that the equations are linear and superposition can thus be applied. We would then calculate our coupling coefficients for the case of simple poles as

$$\tilde{\eta}_\alpha(\vec{e}_1,s) = (\mu_n)_\alpha \cdot (\tilde{I}_n(\vec{e}_1,s)) = (\mu_n)_\alpha \cdot \left(b_n(\vec{e}_1)e^{-st'_n}\right) \tag{3.69}$$

For higher order poles derivatives of the incident vector with respect to s also come in. Now in the time domain we have for a simple pole

$$\tilde{\eta}_\alpha(\vec{e}_1,t) = (\mu_n)_\alpha \cdot (I_n(\vec{e}_1,t)) = (\mu_n)_\alpha \cdot (b_n(\vec{e}_1)\delta(t-t'_n)) \tag{3.70}$$

The pole gives a damped sinusoid $e^{s_\alpha t}$ for a simple pole and a factor $t^{n_\alpha - 1}$ for higher order poles. This is convoluted with η_α where each element of the incident vector makes its contribution at a time t'_n. As discussed before for $t < t_a$ where t_a can be arbitrary the time domain waveform, at least for simple poles, is the same regardless of t_a. This applies to our case of considering each component of the incident vector as separately non zero and adding up the results at the end. Thus for $t < t_o$ the pole contributions are zero. For $t_o < t < t_1$ the convolution as above is required which consists of turning on the contribution of each component of the incident vector at time t_n' and in time domain we have

$$L^{-1}\left[\tilde{\eta}_\alpha(\vec{e}_1,s)(s-s_\alpha)^{-1}\right] = (\mu_n)_\alpha \cdot \left(b_n(\vec{e}_1)u(t-t_n')e^{s_\alpha(t-t_n')}\right)$$

$$= (\mu_n)_\alpha \cdot \left(b_n(\vec{e}_1)e^{-s_\alpha t_n'}u(t-t_n')\right)e^{s_\alpha t} \tag{3.71}$$

For $t_1 < t$ this result goes to the form as in equation 3.66 giving the same late time behavior as the other types of coupling coefficient definition. This type 4 coupling coefficient gives a type of early time behavior which is different from the three previous types and thus gives another form of pole contribution from a convergence viewpoint.

We have shown several possible ways to define coupling coefficients. Any of these or combinations of them can be used to obtain the best representation in the sense of the fewest number of terms required in some region of frequency or time of interest. There are clearly many other cases one might consider. The ones discussed here are some that rather directly follow from physical considerations and/or give simpler results consistent with the requirement of having the correct coefficients at the poles. Note that types 1 through 3 for the coupling coefficients can all be calculated on a common basis. A c_α can be found for any choice of $\tilde{T}(s)$ in the form of a time advance. In particular $\tilde{T}(s)$ can be 1. The resulting c_α then applies to every position on the object. It is just the turn on time in the unit step which is shifted in types 1 through 3 based on a physical time of interest either for the whole object or for a local observation position on the object. Type 4 gives more complicated waveforms.

The considerations about the expansions in this section have been based on incident delta function plane waves. They can be carried over to other incident waveforms directly by the techniques outlined in section 2. The type of the incident wave may influence what one considers as the best form for the coupling coefficients. This is because the early time convergence of the expansion will be affected by the high frequency content of the incident waveform. As an example suppose we have an incident step function waveform. Then we can invert a term of the form $\tilde{\eta}_\alpha(\vec{e}_1,s)/[s_\alpha(s-s_\alpha)]$ into the time domain and add a static term or we can invert a term of the form $\tilde{\eta}_\alpha(\vec{e}_1,s)/[s(s-s_\alpha)]$ into the time domain which has a form like the time integral of the first form. This should give different convergence characteristics for early times and thus another form to be looked at in combination with types 1 through 4 coupling coefficients.

The remaining entire function in the general expansion is clearly a subject of much interest. Depending on which form of coupling coefficient is used this entire function clearly has a form which is different for different forms of coupling coefficients. If t_a is chosen as in type 3 but made later than the time a signal first begins (resultant field signal) at the observer then such an additional function must be non zero to give the only possible fields at early times before t_a. If t_a is chosen less than the first time resultant fields reach the observer then the pole terms must either sum to zero (if convergent) or have their sum cancelled by such a function. If, in specific problems being calculated, the sum does not go to zero for such early times then this other function must be non zero for such a case. An optimum choice of coupling coefficients might be one in which the remaining function is identically zero. This would have the beginning time for each mode no later than the first time resultant fields reach the observer, perhaps even at exactly this time. The "best" coupling coefficients may have more complicated forms for $\tilde{\eta}_\alpha$ than those used here. Perhaps using geometrical diffraction theory to consider asymptotic forms for $|s| \to \infty$ one can investigate the properties of the remaining entire function and/or impose tighter restrictions on the "best" form for the coupling coefficients.

IV. Some Possible Extensions of the Singularity Expansion Method and Some Areas for Further Investigation

Let us now consider some of the implications of the singularity expansion method for some general classes of objects. Most of the attention in this note has been given to objects with finite linear dimensions. This is clearly an important class of objects for electromagnetic interaction and scattering. The perfectly conducting sphere in appendix B is an example of this class of objects and can be used to suggest various general results for this class of objects. In fact some of the results and conjectures discussed in other sections were originally suggested to me from this example. The possible factorization of the coupling coefficients $\tilde{\eta}_\alpha(\vec{e}_1,s)$ into $c_\alpha(\vec{e}_1)e^{-st_0}$ for perfectly conducting objects of finite linear dimensions is a good example of such a conjecture. The fact that the perfectly conducting sphere has only poles in the singularity expansion while the perfectly conducting infinite length circular cylinder has branch cuts in its singularity expansion (associated with the cylindrical Hankel functions in its eigenfunction expansion) suggested that the expansion only in poles is associated with the finite dimensions of the object.

There are other finite sized perfectly conducting objects which can be studied analytically for their singularity expansions. Some examples might be the prolate spheroid, oblate spheroid, circular disk, etc. Such examples have less symmetry than the sphere and one would then expect less degeneracy of the natural modes. With the circular disk an edge would be introduced and one could see how this affected the natural modes.

For finite sized perfectly conducting objects with an axis of symmetry (objects of rotation) one can base a cylindrical (Ψ, ϕ, z) coordinate system on this axis and decompose the natural modes based on $\cos(m\phi')$ and $\sin(m\phi')$ for integer m while the coupling coefficients have factors $\cos(m\phi_1)$ and $\sin(m\phi_1)$ based on one of the angles of incidence. The integral equation over the surface reduces to a one dimensional integral equation for each m making the numerical solution simpler and the indexing of the natural modes also simpler. Drs. L. Marin and R. W. Latham (private communication) are already putting together numerical techniques to handle this case.

As suggested by Prof. C. Taylor (private communication) one can also look at thin wire approximations to simplify the singularity expansion analysis for such cases and perhaps obtain approximate analytic expressions for the natural frequencies, natural modes, and coupling coefficients. This would have the advantage of determining the approximate values of these quantities and suggesting an appropriate indexing system for the frequencies, modes, and coefficients for some rather complex object shapes, such as thin wire models of aircraft structures. Then in more detailed calculations of "fatter" structures one can use the thin wire results to help locate all the natural frequencies etc. because one expects the singularity expansion quantities to vary somewhat continuously as the object shape and dimensions are changed. Furthermore one could develop numerical techniques in which the approximate thin wire results are used as first terms in an iterative solution for the corresponding fat objects.

While we have been viewing the singularity expansion method from the viewpoint of the interaction or scattering problem there is no reason to expect this method to be limited to such problems. This method is fundamentally based on the expansion of analytic functions of the complex frequency s in terms of their singularities in the complex s plane. For example antenna problems in transmission and reception can be considered from this viewpoint. Prof. S. W. Lee (private communication) has looked at some features of the cylindrical antenna and this method appears to give some insight here and can even be used to relate this antenna problem to the interaction problem for a finite length perfectly conducting cylinder.

The results for finite sized perfectly conducting objects can be applied to other types of objects as well. Consider an aperture in a perfectly conducting plane as shown in figure 4.1A. By the Babinet principle this can be related to a complementary perfectly conducting disk.[11,12] Essentially by interchanging the roles of the electric and magnetic fields (rotating the polarization) and including a plane wave term for the reflection from the infinite plane the solution for the aperture scattering can be found from that for the disk scattering. Thus one can describe the deviation of the currents, fields, etc. from the continuous plane case by means of the natural frequencies, modes, and coupling coefficients of the complementary disk. Then these results can be applied to define natural frequencies, modes, and coupling coefficients; the modes can be formulated for change in fields in the aperture and/or changes in surface current and surface charge densities on the remainder of the perfectly conducting plane. Thus it is quite possible to define the singularity expansion for the lack of an object, i.e. a hole, at least in the case of a perfectly conducting plane.

Similar conclusions apply to a protrusion on a perfectly conducting plane as shown in figure 4.1B. This follows from image considerations. With the image of the protrusion included then the incident field can be split into symmetric and antisymmetric parts with respect to the perfectly conducting plane.[3] The interaction of each part with the equivalent object with a symmetry plane in place of the perfectly conducting plane can then be studied separately. However, due to the reflection of the incident wave at the perfectly conducting plane only an antisymmetric field distribution can exist and thus contribute to the result. Thus the change in the fields, currents, etc.

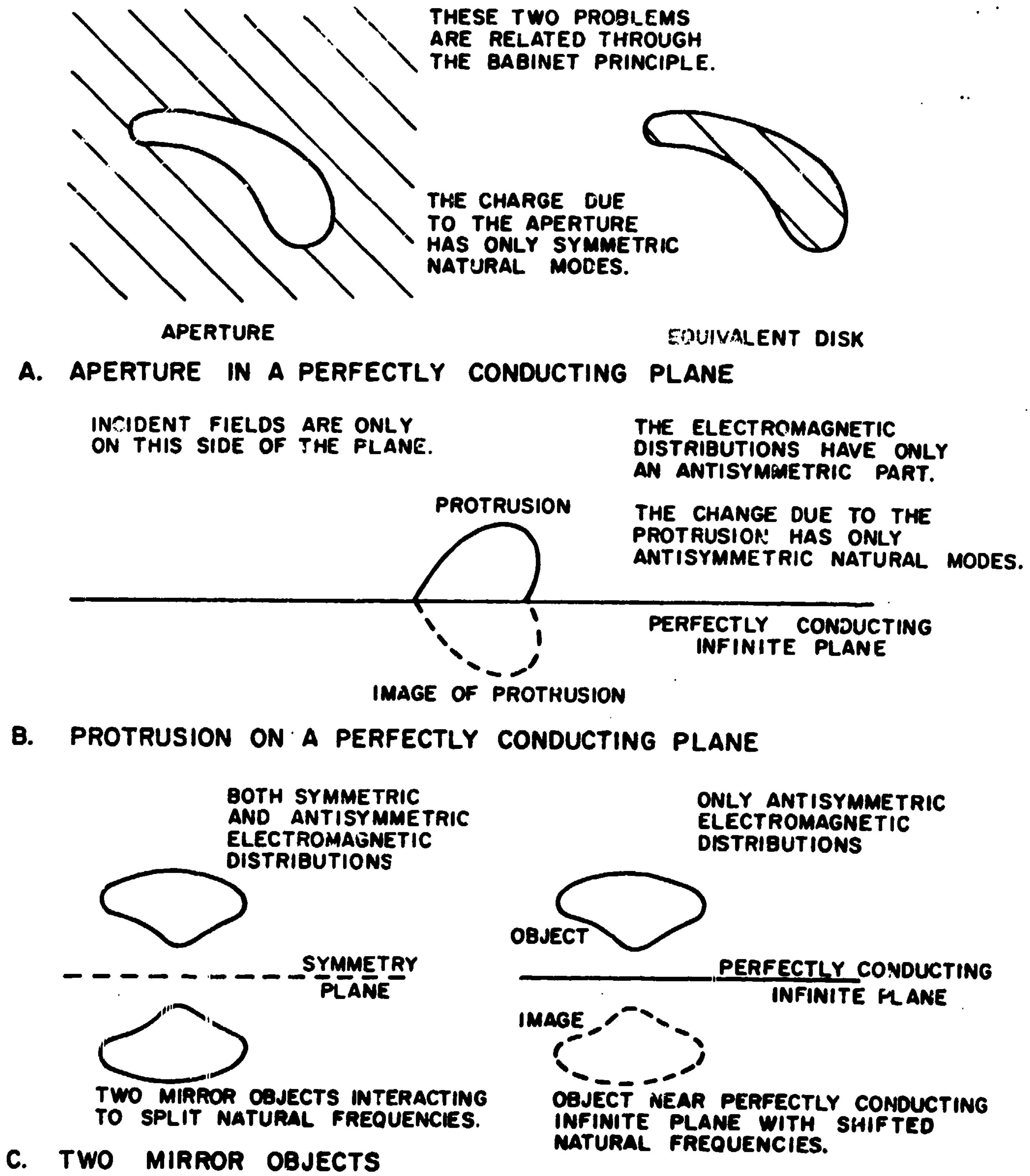

FIGURE 4.1 SOME "OBJECTS" WITH A PLANE OF SYMMETRY

can be described by the natural frequencies etc. of the protrusion with its image but only the antisymmetric natural modes are needed. This is in contrast to the aperture problem in a perfectly conducting plane where only the symmetric part of the incident wave interacts with the aperture making the aperture natural modes also symmetric.

This reasoning can be carried a little further in considering two finite size objects which are mirror images of each other with respect to a symmetry plane as shown in figure 4.1C. Such a pair of objects can be considered as one object for an expansion in natural frequencies, natural modes, and coupling coefficients. If the objects are far apart then the scattering from one will not be very large at the second when compared to the incident field. For large separation there is not very much interaction between the two and we can treat them as two separate objects with the same natural frequencies and the same natural modes except that the modes on the two bodies would be mirror images of one another. Now natural modes can be multiplied by any scalar merely in changing their normalization. Thus for the two objects considered as one we can define natural modes as symmetric and antisymmetric modes by taking sums and differences of the mirror modes. Now the symmetric and antisymmetric parts give an exact division of the natural modes on an object with a symmetry plane; there is no interaction between the two. Even for the two objects close together this is the case. As the objects are brought together one may typically expect a splitting of the natural frequencies in two, one with the symmetric and the other with the antisymmetric modes. This is analogous to the energy level splitting in quantum mechanics, say as two identical atoms are brought together. For a single finite size object near a perfectly conducting plane only antisymmetric modes contribute and so only natural frequencies associated with the antisymmetric modes are present. As the object nears the perfectly conducting plane one might typically expect a shift of the natural frequencies and not a splitting unless some symmetry in the object is destroyed in the process or there is some other degeneracy in the natural frequencies.

These results with perfectly conducting infinite planes suggest yet further results might be obtained for various perturbations on infinitely large perfectly conducting objects. The perturbation is regarded as the "object" and what is calculated is the change in the electromagnetic quantities associated with the introduction of this "object." If the perturbation is of finite size then we might expect its singularity expansion to comprise of natural frequencies, natural modes, and coupling coefficients. However more development is needed to understand this thoroughly. One might even extend this perturbation concept to perturbations on finite sized objects. Particularly if the perturbation is small compared to the object dimensions then one could make a singularity expansion for the object and use the results to define the incident fields for the perturbation which might in turn be approximately solved by another singularity expansion.

Of course there is the question of the singularity expansion for infinite or semi infinite objects. We know that the perfectly conducting circular cylinder has branch cut contributions in its singularity expansion. There are many other shapes besides spheres and cylinders which can be treated from the viewpoint of eigenfunction expansions.[13] This can be used to help divide up the terms in the singularity expansion by treating each term in the eigenfunction expansion separately for its singularity characteristics. Prof. Garbacz has developed a method for calculating eigenfunctions associated with general geometries for lossless objects.[7] Perhaps these expansions can be used by studying each term to determine its singularity expansion and thereby aid in developing or even indexing the terms in a singularity expansion for such objects. In studying infinite objects such as general cylinders (say irregular but of some maximum "radius") or semi infinite objects such as general cones (say contained within some maximum "half cone angle") one may find some general properties of the singularity expansion associated with certain features of the general geometries. This in turn may give some guidance on how to approach the singularity expansions (and index them) for specific cases of such objects.

Note that some objects which are finite in size may have properties of infinite bodies such as branch cuts in the singularity expansion. For example take a perfectly conducting object of finite size located between two infinite parallel perfectly conducting plates. This can be replaced by an equivalent problem involving an infinite number of images extending infinitely far away. This is basically a segmented infinite object. For the case of a thin wire of finite length between parallel plates some frequency domain results (on the $i\omega$ axis of the s plane) exhibit peculiar step and slope discontinuities.[14] This may be associated with new terms such as branch contributions in the singularity expansion. Perhaps the case of infinitely repeated objects (and/or images) in one, two, and three directions can be specially treated so as to obtain some general results for the singularity expansions for such problems. Since translational symmetry is present for such repeated objects perhaps group theory considerations can be applied to obtain general results for this type of problem.

Other important classes of objects involve lossy media of infinite size, such as half spaces with finite non zero conductivity. Objects of finite size may be close enough to such media to affect their response characteristics, thereby altering their singularity expansions. Furthermore, infinite objects such as wires may be in proximity to semi infinite lossy half spaces; this could introduce yet additional features in the singularity expansion.

Another whole class of problems concerns the analysis of experimental data, say from tests using EMP simulators. Using numerical Laplace transforms, numerical Hilbert transforms, etc. features of the singularity expansion of the experimental data can be found within the limits of accuracy of the experimental data and the numerical techniques. It would seem that various approaches to this problem are possible depending on the kind of experimental data and type of object being considered.

Clearly there are numerous topics in the singularity expansion method involving classes of objects, numerical techniques, etc. which need extensive development. In this note we have for the most part considered finite size objects. Even for this important though limited class of objects much needs to be done. For example the natural modes come from the coefficients of the poles in the expansion. The natural modes may be orthogonal over the volume or surface of the object. Such is the case for the sphere but what about in general? Perhaps the topology of the object can be used to help index the singularity expansion quantities and can be used to identify whether or not certain kinds of terms are present. The object symmetries have much influence on the degeneracy of natural frequencies and modes. Group theory should then be useful in understanding the degeneracy and splitting up the resulting modes as well as indexing the natural frequencies, natural modes, and coupling coefficients.

A finite size object need not be perfectly conducting. Suppose that it is composed of linear passive media. Prof. C. T. Tai has suggested (private communication) that the types of theoretical considerations applied in circuit theory can be applied to the singularity expansion to obtain new general results. Considerations like conservation of energy are important here in constraining the allowable forms of the solutions. Perhaps some properties of the pole expansion can be deduced such as the permissable order of the poles. The perfectly conducting sphere has only simple poles. It seems safe to conjecture that this is true of all finite size perfectly conducting objects; as of yet I have not found any case to the contrary. The case of lossy objects may admit more general pole types. The properties of the coupling coefficients also need investigation for these more general finite size objects. Whether or not the linear passive media are also reciprocal should also have important impact on the properties of the singularity expansion.

Numerical calculations for objects such as finite length perfectly conducting cylinders can be used to test various numerical techniques for calculating the singularity expansion quantities and suggest improvement on them. Dr. F. Tesche is already doing such calculations showing some cases of very rapid convergence for the step function response. Furthermore such calculations can test conjectures of a general nature and if calculations are performed for various different objects in the class of interest and the conjecture proves correct in all cases then one has high confidence in the general validity of the conjecture. Consideration of various example objects has been very useful to me in suggesting new results and testing old conjectures. It would seem that one important driving force in the future development of this technique will be the calculation of the object response of various important objects of practical interest.

An interesting and important question for finite objects concerns the uniqueness of the form of the singularity expansion. Clearly the natural frequencies and modes for finite $|s_\alpha|$ are well defined, but there are some possible alternatives in defining $\tilde{\eta}_\alpha(\vec{e}_1,s)$ and $c_\alpha(\vec{e}_1)$, the forms of the coupling coefficients. Specifically $\tilde{\eta}_\alpha$ can be an entire function of s (no singularities in the finite s plane). This gives some flexibility in choosing the form of the coupling coefficients. Of course the choice is not completely arbitrary. The final resulting current density, charge density, etc. are unique quantities and all exact representations of them must amount to the same thing. The individual terms in a series expansion can be altered as long as the sum remains the same. This then raises the question of what is the "best" form in which to express the coupling coefficients. A form like $c_\alpha(\vec{e}_1)e^{-st_o}$ clearly has much to recommend it for its simplicity. However there may be other terms needed if such a form is used. This problem is associated with the time during which the incident delta function wave is sweeping over the body. Since the object response must be zero before a field can reach any particular point on the object (with this time calculable from geometrical diffraction theory considerations) and since t_o is the time the first point on the object is excited, then all the terms in the expansion must sum to zero (if the sum converges) for times between t_o and the time that an excitation can reach a point of interest on the object. In order to best define the expansion for early times so as to obtain the most rapid convergence then some other definition of the coupling coefficients may be appropriate. The possible alternative forms of the coupling coefficients seems to me to be an issue of fundamental importance in the whole theory and practical utility of the singularity expansion for finite size objects, especially for early times. For infinite bodies the terms analogous to these coupling coefficients may also have similar questions associated with them. Much research is needed then on coupling coefficient representation both in terms of general considerations and specific examples.

In past years there has been some consideration of the natural frequencies of some simple objects and to some extent the natural modes have also been investigated for such objects. Let us mention some examples. Thin wire natural frequencies have received some attention.[15] Natural frequencies have been discussed in the context of antenna resonances.[16] Natural frequencies and modes of a sphere have been discussed;[17] this forms a starting point for our discussion of the singularity expansion of the perfectly conducting sphere in appendix B. Prolate spheroidal geometry has also been considered for natural frequencies and modes.[18] Since the perfectly conducting sphere has shown so many interesting results it would seem a good idea to look at the prolate and oblate spheroids to see to what extent the general form of the results carries over to these geometries. For example spheroidal geometries can allow one to look at some analytic results for forms of the coupling coefficients. These and other investigations, even though limited in some respects, at least solve some portions of terms in the singularity expansion for some objects. As such they can shed some light on some details of the singularity expansion for such objects and give a start for obtaining the full singularity expansions. They also give some guidance about what problems can be profitably considered for examples to develop general results for the solution representations in the singularity expansion method.

V. Summary

This note is intended to introduce a new way of looking at many kinds of EMP interaction problems, although it has bearing on scattering problems as well. From an EMP time domain interaction viewpoint this approach has the potential for directly calculating the amplitudes, frequencies, damping constants, and phases of the damped sinusoidal oscillations that are commonly seen as major portions of interaction waveforms on systems under test. The idea is to then construct a large portion or even all of such waveforms as a sum of such damped sinusoids.

The general technique can aptly be called the singularity expansion method because it is based on representing the functions of the complex frequency s in terms of their singularities in the complex s plane. In the time domain the individual terms are the inverse Laplace transforms of the singularity terms. While for general objects we can expect branch cut contributions the results for finite size objects using well behaved media include only poles for the singularities in the finite s plane. This simplifies the form of the terms considerably and allows one to factor the terms into natural frequencies, natural modes, and coupling coefficients. The natural frequencies and modes are independent of the incident wave parameters while the incident wave parameters enter into the coupling coefficients for the delta function response. The incident wave can also have singularities in the finite s plane but these can be separated out so that the response can be generally written as the sum of an object part and a waveform part.

There are various matrix techniques for solving integral equations numerically. In this note we have considered these from a general viewpoint, not specifying which integral equation is being approximated. This shows some general ways to calculate natural frequencies, natural modes, and coupling coefficients. The actual numerical procedures that one could use are numerous and need to be considered for various problems to determine the most efficient and accurate techniques.

There are various theoretical problems associated with the convergence of the matrix representations which need to be considered for the integral equations for finite objects. Preferably continuous operators over the body geometry can be developed to analytically represent the terms in the singularity expansion. The question of the singularities at infinity or additional entire functions for finite objects needs treatment. Of course the completeness of the singularity expansion with some allowable chosen form for the coupling coefficients can be readily checked for any given boundary value problem by comparison to the solution by standard numerical techniques. This method can be used to determine better forms of coupling coefficients.

There has been some work done in the past on the natural frequencies of objects and less work done on natural modes. This work can serve as useful starting points for the singularity expansions of some classical geometries which can be used for test problems. There are the essentially new questions of the pole order and coupling coefficients for finite size bodies. This note has included a common object of past investigations, the perfectly conducting sphere. There are analytic forms for coupling coefficients and there are only simple poles in the finite s plane with no additional entire function required for the delta function response. Investigation of other common objects considered previously should also give some valuable insight into appropriate forms for coupling coefficients and questions such as pole order.

I hope that this note has given the reader some insight into what the singularity expansion method is all about, particularly with regard to finite size objects. It appears to be quite powerful for some kinds of EMP interaction problems. Several investigators are already performing some studies of both general problems and specific examples using this type of expansion. I would hope then that the near future will see some significant additions both to the theory of the method and problems solved using the method.

"Off with her head!" the Queen shouted at the top of her voice. Nobody moved.

"Who cares for you?" said Alice (she had grown to her full size by this time). "You're nothing but a pack of cards!"

At this the whole pack rose up into the air, and came flying down upon her; ...

"Wake up, Alice dear!" said her sister. "Why, what a long sleep you've had!"

"Oh, I've had such a curious dream!" said Alice, and she told her sister, as well as she could remember them, all these strange Adventures of hers that you have just been reading about; ...

(Lewis Carroll, Alice in Wonderland)

Appendix A: The Natural Frequency $s = 0$

As mentioned in section 2 it is possible for finite sized bodies to have static current and/or charge distributions. However these do not couple to the incident wave and thus do not start or stop at a time such as t_o. These kind of static solutions can be added as separate terms to any response of the object to the incident wave. This is not the same term as the static response of the object as in equations 2.46 and 2.52.

The natural modes for $s = 0$ are similar to others on the $i\omega$ axis in that they correspond to no power going to infinite radius. For static current and charge distributions the fields at large r for dipole and higher terms decay like r^{-3} (no radiated power); the r^{-1} electric monopole term is a radial electric field (no radiated power). For other poles on the $i\omega$ axis not at $s = 0$ any radiation field like r^{-1} at infinity would constitute radiated power and thus damping the mode (i.e. making $\Omega < 0$). Thus such modes when expanded over some sphere containing the object in terms of divergenceless spherical wave functions must give no terms behaving like r^{-1} at infinity, whereas all such functions do for $s \neq 0$ on the $i\omega$ axis. Thus the fields for such modes are contained in some volume of finite dimensions and might logically be called the cavity modes.

What are some of the characteristics of these static natural modes? Since we are dealing with the case of $s = 0$ the wave equation reduces to the Laplace equation and the electric and magnetic fields are decoupled. Thus we first distinguish between electrostatic natural modes and magnetostatic natural modes. As shown in figure A1A there are various types of examples as in 1 a perfectly conducting object with net charge Q_1; this gives rise to an electrostatic natural mode surface charge density ρ_{s1}. As in 2 an insulating dielectric can have a net charge Q_2 giving rise to the same kind of field at large r as in 1; the charge distribution throughout the body is not constrained by the surface shape as in 1 and static electric fields in the body are possible so that the distribution is arbitrary to some extent. As in 3 conductors (perfect or imperfect) can be combined with insulating dielectrics to allow not only a net charge but allow a volume charge distribution in some parts but not in others. Of course all the above mentioned cases can be combined together, say as laid out in figure A1A as multiple objects in a volume of finite dimensions so that the whole ensemble has a net charge $Q_1 + Q_2 + \cdots$ and an associated volume and surface charge distribution. It is not necessary for there to be a net charge, or even for there to be any fields for large r. The charged objects could be contained in a closed conducting shell and the net charge of all (including the shell) made zero; the natural electrostatic mode would still have a non trivial charge distribution.

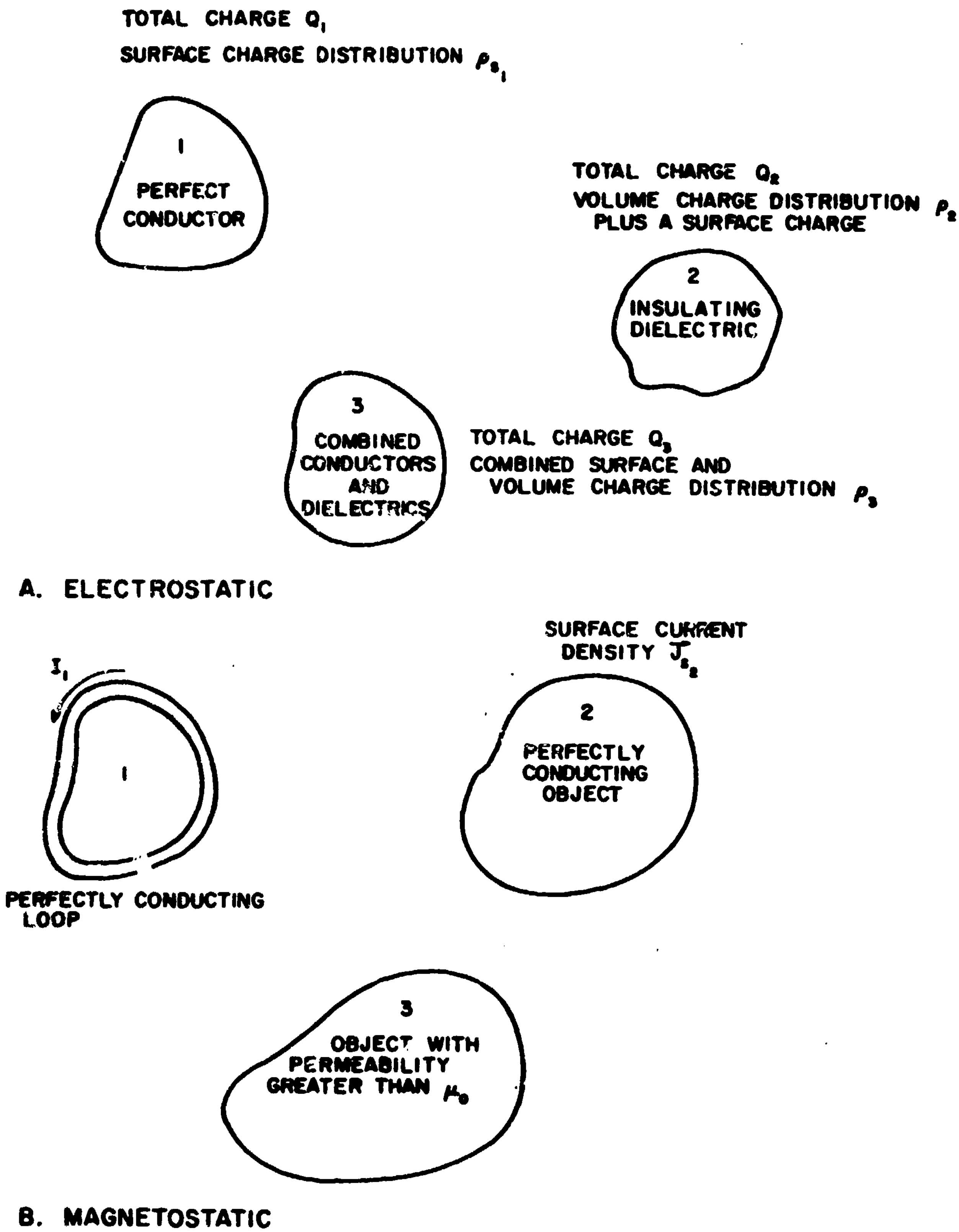

FIGURE A1. STATIC NATURAL MODES

In figure A1B we illustrate some magnetostatic natural modes. Excluding magnetic charge from Maxwell's equations we have no magnetic monopole term to give an r^{-1} magnetic field at large r. However, we can do next best and get a magnetic dipole term from a closed perfectly conducting loop as in 1. If the loop is perfectly conducting then the total magnetic flux through the loop cannot change because this would imply an electric field tangential to the perfect conductors which is impossible by hypothesis. A perfectly conducting object as in 2 can have a surface current density $\vec{J}_{s2}$ when immersed in a magnetic field such as from the loop 1. A permeable object as in 3 can have a magnetization (with an equivalent volume current density distribution) induced by a magnetic field from the loop 1. One might also consider permanent magnets but since we wish to remain with linear Maxwell's equations we may wish to exclude such things.

Practically speaking we do not normally deal with perfect conductors but they still make a useful idealization for many problems. Thus the magnetostatic natural modes still are useful concepts. In a practical case these would not be exactly at $s = 0$ but have Ω slightly negative. Of course for the case of superconductors such magnetostatic modes do exist and have been observed; they are even quantized. This leads to another phenomenon in which the magnetostatic fields are excluded from superconductors except for a thin surface layer. This is analogous to the case of magnetostatic fields excluded from perfectly conducting objects as discussed above. However, there may be cases of highly conducting objects for which it is useful to think of a magnetostatic mode with magnetic field penetrating what is thought of as a perfect conductor for purposes at hand.

Having considered the electrostatic and magnetostatic natural modes there is no reason why one cannot combine them and have both associated with some object or collection of objects contained in some volume of finite dimensions. Note that the static natural modes can easily be degenerate. For example one can change total charge (Q_1, Q_2, etc.) on each of several discrete conductors as well as change the currents (I_1, I_2, etc.) circulating around perfectly conducting loops (or equivalently separate holes through perfectly conducting objects). All of these apply at the natural frequency $s = 0$. Stated briefly the natural modes for $s = 0$ are any electrostatic and/or magnetostatic modes which are not associated with any incident field.

Appendix B: Example to Illustrate the Singularity Expansion: The Perfectly Conducting Sphere

As an aid to understanding the form of the actual singularity expansions we consider an example chosen so that the various terms in the expansion may be more readily expressed in terms of common functions. For this purpose we choose the perfectly conducting sphere. This example shows all simple poles in the expansion and poles on both the negative Ω axis and in conjugate pairs with $\Omega < 0$; the internal cavity modes and an electrostatic mode with poles on the $i\omega$ axis which have zero coupling coefficients are not included. The natural modes are degenerate and we choose those for the surface current and charge densities to correspond to an appropriate set of spherical harmonics. The coupling coefficients have a time advance which factors out leaving dependence only on direction of incidence and polarization. Again this example is directed toward explicitly exhibiting the form of the singularity expansion and the kinds of general results that ensue. For numerical purposes the singularity expansion may not be the mose useful in the case of the sphere. However it can be used to more readily suggest general results which also apply to more complex shapes. Other interesting results can be found from considering other spherical problems involving surface resistance and finite volume conductivity, permittivity, and permeability but we do not go into these variations in this note.

Consider then the problem of a plane wave incident on a perfectly conducting sphere as illustrated in figure B1. We have a spherical (r, θ, ϕ) coordinate system and unit vectors $\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\phi$ which can also be listed with a prime to indicate the object coordinates. Let the sphere have radius a and let the incident plane wave be described as in section 2. The unit vectors for the incident plane wave are illustrated in figure B1. As in an earlier note[4] the unit vectors for the plane wave are expanded as

$$\begin{aligned}
\vec{e}_1 &= \sin(\theta_1)\cos(\phi_1)\vec{e}_x + \sin(\theta_1)\sin(\phi_1)\vec{e}_y + \cos(\theta_1)\vec{e}_z \\
\vec{e}_2 &= -\cos(\theta_1)\cos(\phi_1)\vec{e}_x - \cos(\theta_1)\sin(\phi_1)\vec{e}_y + \sin(\theta_1)\vec{e}_z \qquad \text{(B1)} \\
\vec{e}_3 &= \sin(\phi_1)\vec{e}_x - \cos(\phi_1)\vec{e}_y
\end{aligned}$$

where θ_1 is the angle of $\vec{e}_1$ with respect to the reference z axis (arbitrarily chosen) and ϕ_1 is the orientation of the projection of $\vec{e}_1$ on the x, y plane with respect to the x axis (arbitrarily chosen). The second unit vector $\vec{e}_2$ is chosen in a plane parallel to $\vec{e}_1$ and the z axis while $\vec{e}_3$ is then parallel to the x, y plane.

$\vec{e}_1$ DIRECTION OF INCIDENCE

$\vec{e}_2$ POLARIZATION PARALLEL TO PLANE OF $\vec{e}_1$ AND $\vec{e}_z$

$\vec{e}_3$ POLARIZATION PARALLEL TO x, y PLANE

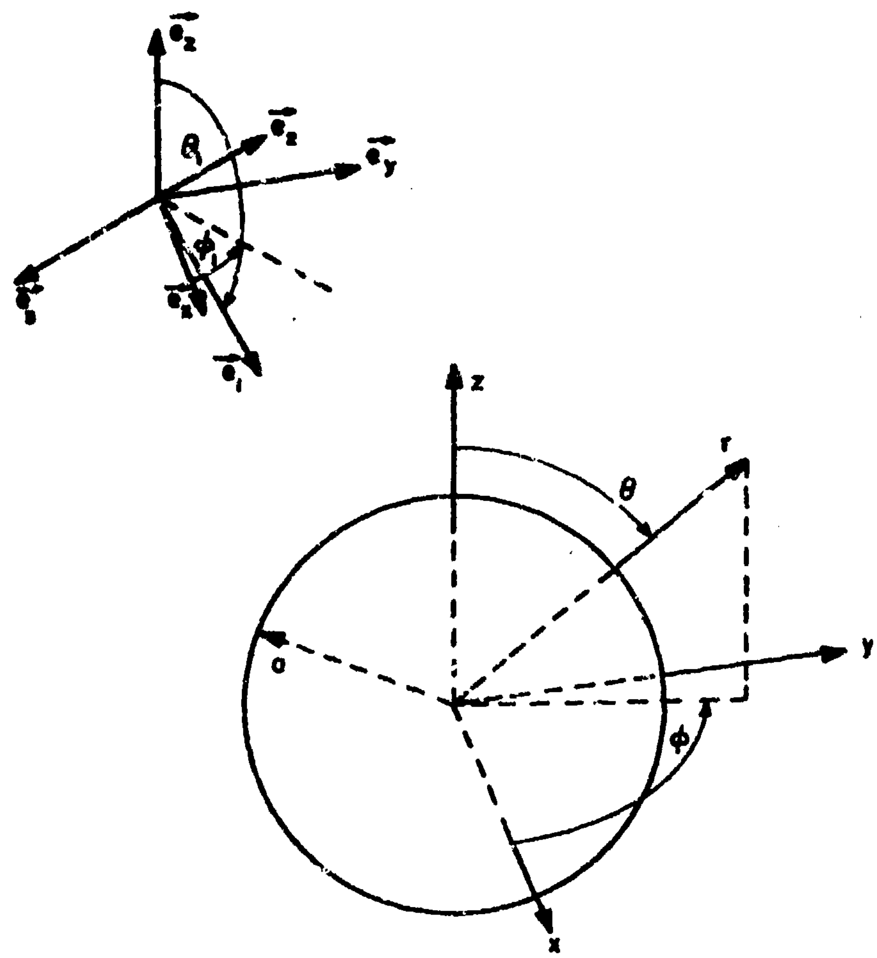

FIGURE B1. PLANE WAVE INCIDENT ON PERFECTLY CONDUCTING SPHERE

The cartesian (x, y, z) and spherical (r, θ, ϕ) coordinate systems are related as

$$x = r\,\sin(\theta)\,\cos(\phi)$$

$$y = r\,\sin(\theta)\,\sin(\phi) \qquad \text{(B2)}$$

$$z = r\,\cos(\theta)$$

and similarly for primed coordinates. Expanding the plane wave unit vectors in spherical coordinates gives

$$\vec{e}_1 = [\cos(\theta_1)\cos(\theta) + \sin(\theta_1)\sin(\theta)\cos(\phi-\phi_1)]\vec{e}_r$$

$$+ [-\cos(\theta_1)\sin(\theta) + \sin(\theta_1)\cos(\theta)\cos(\phi-\phi_1)]\vec{e}_\theta$$

$$- \sin(\theta_1)\sin(\phi-\phi_1)\vec{e}_\phi$$

$$\vec{e}_2 = [\sin(\theta_1)\cos(\theta) - \cos(\theta_1)\sin(\theta)\cos(\phi-\phi_1)]\vec{e}_r$$

$$- [\sin(\theta_1)\sin(\theta) + \cos(\theta_1)\cos(\theta)\cos(\phi-\phi_1)]\vec{e}_\theta \qquad \text{(B3)}$$

$$+ \cos(\theta_1)\sin(\phi-\phi_1)\vec{e}_\phi$$

$$\vec{e}_3 = -\sin(\theta)\sin(\phi-\phi_1)\vec{e}_r$$

$$-\cos(\theta)\sin(\phi-\phi_1)\vec{e}_\theta$$

$$-\cos(\phi-\phi_1)\vec{e}_\phi$$

where we can expand some of the terms in the forms

$$\cos(\phi-\phi_1) = \cos(\phi_1)\cos(\phi) + \sin(\phi_1)\sin(\phi)$$

$$\text{(B4)}$$

$$\sin(\phi-\phi_1) = \cos(\phi_1)\sin(\phi) - \sin(\phi_1)\cos(\phi)$$

Having the direction of incidence and two polarizations expressed in spherical coordinates we can go on to express the response of the perfectly conducting sphere to the two delta function plane waves $\tilde{u}_p$ as in equations 2.17. This can be followed by finding the delta function responses for the surface current density and surface charge density.

For the incident delta function plane wave we first need spherical harmonics and vector wave functions in which to express the expansion in spherical coordinates. In spherical coordinates we have the common differential operators as

$$\nabla F = \vec{e}_r \frac{\partial}{\partial r} F + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} F + \vec{e}_\phi \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} F$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} F_\phi$$

$$\begin{aligned} \nabla \times \vec{F} = {} & \vec{e}_r \left[\frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\phi) - \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} F_\theta\right] \\ & + \vec{e}_\theta \left[\frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi} F_r - \frac{1}{r} \frac{\partial}{\partial r}(r F_\phi)\right] \\ & + \vec{e}_\phi \left[\frac{1}{r} \frac{\partial}{\partial r}(r F_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} F_r\right] \end{aligned} \tag{B5}$$

where $\vec{F}$ is a general vector and F a general scalar. Other operator such as Laplacian ($\nabla^2 \vec{F}$ and $\nabla^2 F$) can be constructed using the three in equations B5. These operators are suggestive of ones that could be defined to operate with respect to the surface coordinates θ, ϕ on a unit sphere. Of course these operate on scalar and vector quantities which are functions of these surface coordinates (or considered as only functions of such coordinates). Using a subscript s to denote these operators we have

$$\nabla_s F \equiv \vec{e}_\theta \frac{\partial}{\partial \theta} F + \vec{e}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F$$

$$\nabla_s \cdot \vec{F} \equiv \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\theta) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_\phi$$

$$\begin{aligned} \nabla_s \times \vec{F} \equiv {} & \vec{e}_r \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) F_\phi) - \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_\theta\right] \\ & + \vec{e}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial \phi} F_r \end{aligned} \tag{B6}$$

$$- \vec{e}_\phi \frac{\partial}{\partial \theta} F_r$$

Note that in the process of removing all r factors and derivatives with respect to r the ∇_s operator leaves the units unchanged instead of multiplying the units of $\vec{F}$ or F by meter^{-1}.

Let us now consider the spherical harmonics. The scalar spherical harmonics can be written as

$$Y_{n,m,{e \atop o}}(\theta,\phi) = P_n^m(\cos(\theta)) \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \qquad m = 0, 1, 2, \cdots, n \quad \text{(B7)}$$

The subscript ${e \atop o}$ meaning even or odd, indicating that one is to be chosen corresponding to whether $\cos(m\phi)$ or $\sin(m\phi)$ respectively is intended. The Legendre functions are given their standard definition such that for $-1 \le \xi \le 1$ the $P_n^m(\xi)$ have the definition[19]

$$P_n^m(\xi) \equiv (-1)^m (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_n(\xi)$$

$$P_n(\xi) \equiv P_n^o(\xi) \equiv \frac{1}{2^n n!} \frac{d^n}{d\xi^n}(\xi^2-1)^n \quad \text{(B8)}$$

For convenience a subscript σ (for symmetry) can be used to indicate e or o or as an index for sums over both. Using the Kronecker delta notation defined by

$$\delta_{\sigma_1,\sigma_2} = \begin{cases} 1 \text{ for } \sigma_1 = \sigma_2 \\ 0 \text{ for } \sigma_1 \neq \sigma_2 \end{cases} \quad \text{(B9)}$$

we can write the spherical harmonics as

$$Y_{n,m,\sigma}(\theta,\phi) = P_n^m(\cos(\theta))[\delta_{e,\sigma}\cos(m\phi) + \delta_{o,\sigma}\sin(m\phi)] \quad \text{(B10)}$$

where we use letters e, o for the arguments with obvious meaning. If desired $\pm$ and - or +1 and -1 could be used to denote even and odd respectively being some kind of parity value.

Having considered the scalar spherical harmonics we now need the vector spherical harmonics, three kinds of them. In a manner similar to previous usage[20] we define three types of vector spherical harmonics. This definition differs slightly from one of our previous notes[4] but our present definitions seem more natural. The first kind have only an r component and are simply defined as

$$\vec{P}_{n,m,\sigma}(\theta,\phi) \equiv Y_{n,m,\sigma}(\theta,\phi)\vec{e}_r \tag{B11}$$

The second kind has only θ and ϕ components and is defined as

$$\vec{Q}_{n,m,\sigma}(\theta,\phi) \equiv \nabla_s Y_{n,m,\sigma}(\theta,\phi)$$

$$= \vec{e}_\theta \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi) + \vec{e}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} Y_{n,m,\sigma}(\theta,\phi) \tag{B12}$$

which can be written out as

$$\vec{Q}_{n,m,{e \atop o}}(\theta,\phi) = \vec{e}_\theta \frac{dP_n^m(\cos(\theta))}{d\theta} \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} + \vec{e}_\phi \frac{P_n^m(\cos(\theta))}{\sin(\theta)} m \begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix}$$

$$= \left[\frac{n(n-m+1)}{2n+1} \frac{P_{n+1}^m(\cos(\theta))}{\sin(\theta)} - \frac{(n+1)(n+m)}{2n+1} \frac{P_{n-1}^m(\cos(\theta))}{\sin(\theta)}\right] \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$+ \vec{e}_\phi m \frac{P_n^m(\cos(\theta))}{\sin(\theta)} \begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix} \tag{B13}$$

The third kind also has only θ and ϕ components and is defined as

$$\vec{R}_{n,m,\sigma}(\theta,\phi) \equiv \nabla_s \times [\vec{e}_r Y_{n,m,\sigma}(\theta,\phi)]$$

$$= \vec{e}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} Y_{n,m,\sigma}(\theta,\phi) - \vec{e}_\phi \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi) \tag{B14}$$

which can be written out as

$$\vec{R}_{n,m,{}^{e}_{o}}(\theta,\phi) = \vec{e}_\theta \frac{P_n^m(\cos(\theta))}{\sin(\theta)} m \begin{Bmatrix} -\sin(m\phi) \\ \cos(m\phi) \end{Bmatrix} - \vec{e}_\phi \frac{dP_n^m(\cos(\theta))}{d\theta} \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix}$$

$$= \vec{e}_\theta m \frac{P_n^m(\cos(\theta))}{\sin(\theta)} \begin{Bmatrix} -\sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}$$

$$+ \vec{e}_\phi \left[- \frac{n(n-m+1)}{2n+1} \frac{P_{n+1}^m(\cos(\theta))}{\sin(\theta)} + \frac{(n+1)(n+m)}{2n+1} \frac{P_{n-1}^m(\cos(\theta))}{\sin(\theta)} \right] \begin{Bmatrix} \cos(m\phi) \\ \sin(m\phi) \end{Bmatrix} \tag{B15}$$

Some useful relations hold among the three kinds of vector spherical harmonics as

$$\begin{aligned} \vec{Q}_{n,m,\sigma}(\theta,\phi) &= \vec{e}_r \times \vec{R}_{n,m,\sigma}(\theta,\phi) \\ \vec{R}_{n,m,\sigma}(\theta,\phi) &= -\vec{e}_r \times \vec{Q}_{n,m,\sigma}(\theta,\phi) \\ \vec{Q}_{n,m,\sigma}(\theta,\phi) &= \nabla_s[\vec{e}_r \cdot \vec{P}_{n,m,\sigma}(\theta,\phi)] \\ &= \vec{e}_r \times [\nabla_s \times \vec{P}_{n,m,\sigma}(\theta,\phi)] \\ \vec{R}_{n,m,\sigma}(\theta,\phi) &= \nabla_s \times \vec{P}_{n,m,\sigma}(\theta,\phi) \end{aligned} \tag{B16}$$

These also have various relations to the scalar spherical harmonics as

$$\begin{aligned} \vec{P}_{n,m,\sigma}(\theta,\phi) &= \vec{e}_r Y_{n,m,\sigma}(\theta,\phi) \\ \vec{Q}_{n,m,\sigma}(\theta,\phi) &= \nabla_s Y_{n,m,\sigma}(\theta,\phi) \\ &= \vec{e}_r \times [\nabla_s \times [\vec{e}_r Y_{n,m,\sigma}(\theta,\phi)]] \\ \vec{R}_{n,m,\sigma}(\theta,\phi) &= \nabla_s \times [\vec{e}_r Y_{n,m,\sigma}(\theta,\phi)] \\ &= -\vec{e}_r \times \nabla_s Y_{n,m,\sigma}(\theta,\phi) \end{aligned} \tag{B17}$$

Note that the three types of vector spherical harmonics are mutually orthogonal at each point on a unit sphere for the same set of indices. They also are mutually orthogonal in an integral sense on the unit sphere for any combination of index sets as

$$\int_0^{\pi}\int_0^{2\pi}\vec{P}_{n,m,\sigma}(\theta,\phi)\cdot\vec{Q}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta = 0$$

$$\int_0^{\pi}\int_0^{2\pi}\vec{P}_{n,m,\sigma}(\theta,\phi)\cdot\vec{R}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta = 0 \qquad \text{(B18)}$$

$$\int_0^{\pi}\int_0^{2\pi}\vec{Q}_{n,m,\sigma}(\theta,\phi)\cdot\vec{R}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta = 0$$

For the same kinds of vector spherical harmonics we have orthogonality relationships on the unit sphere as

$$\int_0^{\pi}\int_0^{2\pi}\vec{P}_{n,m,\sigma}(\theta,\phi)\cdot\vec{P}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta$$

$$= [1+[\delta_{e,\sigma}-\delta_{o,\sigma}]\delta_{o,m}]\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n,n'}\delta_{m,m'}\delta_{\sigma,\sigma'}$$

$$\int_0^{\pi}\int_0^{2\pi}\vec{Q}_{n,m,\sigma}(\theta,\phi)\cdot\vec{Q}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta \qquad \text{(B19)}$$

$$= \int_0^{\pi}\int_0^{2\pi}\vec{R}_{n,m,\sigma}(\theta,\phi)\cdot\vec{R}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta$$

$$= [1+[\delta_{e,\sigma}-\delta_{o,\sigma}]\delta_{o,m}]2\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n,n'}\delta_{m,m'}\delta_{\sigma,\sigma'}$$

As a next step we need vector wave functions for spherical coordinates. One part of these functions comes from spherical Bessel functions which are functions of $\gamma r = ikr = sr/c$. It is from these that the radial and complex frequency dependences are formed. These are commonly expressed as

$$j_n(kr)\ , \quad y_n(kr)$$

$$h_n^{(1)}(kr) = j_n(kr) + iy_n(kr) \qquad \text{(B20)}$$

$$h_n^{(2)}(kr) = j_n(kr) - iy_n(kr)$$

where the j_n are used for cases of no singularity at $r = 0$, $h_n^{(1)}$ are used for incoming waves, and $h_n^{(2)}$ are used for outgoing waves satisfying the radiation condition at infinity. For $\xi \to 0$ we have[19]

$$j_n(\xi) = \frac{\xi^n}{(2n+1)!!}[1+O(\xi^2)] \qquad \text{(B21)}$$

$$h_n^{(2)}(\xi) = i(\xi)^{-n-1}(2n-1)!![1+O(\xi^2)]$$

where the double factorial is defined by

$$m!! \equiv \begin{cases} m(m-2)\cdots(4)(2) & \text{for } m \text{ even} \\ m(m-2)\cdots(3)(1) & \text{for } m \text{ odd} \end{cases} \qquad \text{(B22)}$$

$$1!! = 0!! = (-1)!! = 1$$

Now for present purposes we define spherical Bessel functions with argument $\zeta = i\xi$ and we define two kinds

$$f_n^{(1)}(\zeta) \equiv i_n(\zeta) \qquad \text{(B23)}$$

$$f_n^{(2)}(\zeta) \equiv k_n(\zeta)$$

The first kind is used to expand the incident wave (and is like $j_n(kr)$) and the second to expand the scattered fields (and is like $h_n^{(2)}(kr)$); the functions like y_n and $h_n^{(1)}$ can be formed as a linear combination of the $f_n^{(\ell)}$ for $\ell = 1, 2$. We wish these functions to be real for real γ_r so that the complex conjugate relationship of equation 2.4 will apply and zeros of the functions will have complex conjugate symmetry. Let us then constrain $\zeta \to 0$ the asymptotic forms

$$i_n(\zeta) = \frac{\zeta^n}{(2n+1)!!}[1+O(\zeta^2)]$$

$$k_n(\zeta) = \zeta^{-n-1}(2n-1)!![1+O(\zeta^2)] \qquad \text{(B24)}$$

from which we can make the identification

$$i_n(\zeta) = i^n j_n(\xi) = i^n j_n(-i\zeta)$$

$$k_n(\zeta) = i^{-n-2}h_n^{(2)}(\xi) = i^{-n-2}h_n^{(2)}(-i\zeta) \qquad \text{(B25)}$$

$$= -i^{-n}h_n^{(2)}(\xi)$$

For these functions we need a Wronskian relation with respect to the argument ζ as

$$W\{i_n(\zeta),k_n(\zeta)\} \equiv i_n(\zeta)k_n'(\zeta) - i_n'(\zeta)k_n(\zeta) = -\zeta^{-2} \qquad \text{(B26)}$$

where the prime with the Bessel function indicates differentiation with respect to the argument. Another related expression is

$$i_n(\zeta)[\zeta k_n(\zeta)]' - k_n(\zeta)[\zeta i_n(\zeta)]' = -\zeta^{-1} \qquad \text{(B27)}$$

which in Wronskian form is

$$W\{\zeta i_n(\zeta),\zeta k_n(\zeta)\} = -1 \qquad \text{(B28)}$$

These are useful for simplifying coefficients in the field, current, and charge expansions.

These spherical Bessel functions can be written as combinations of polynomials and exponentials giving them a simpler form for their exact representation. This is important for finding the poles for our singularity expansion. From a standard reference[19] the spherical Hankel functions can be written for $n = 0, 1, 2, \cdots$ as

$$h_n^{(1)}(\xi) = i^{-n-1}\xi^{-1}e^{i\xi}\sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(-i2\xi)^{-\beta}$$

$$h_n^{(2)}(\xi) = i^{n+1}\xi^{-1}e^{-i\xi}\sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(i2\xi)^{-\beta} \tag{B29}$$

From these we can construct the other spherical Bessel functions as

$$j_n(\xi) = \frac{1}{2}\left[h_n^{(1)}(\xi) + h_n^{(2)}(\xi)\right]$$

$$y_n(\xi) = -\frac{i}{2}\left[h_n^{(1)}(\xi) - h_n^{(2)}(\xi)\right] \tag{B30}$$

With $\zeta = i\xi$ we then can write our k functions as

$$k_n(\zeta) = i^{-n-2}h_n^{(2)}(\xi) = \frac{e^{-\zeta}}{\zeta}\sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(2\zeta)^{-\beta} \tag{B31}$$

with the resulting simplification of the expression this last form for the spherical Bessel functions for outgoing waves has some useful advantages. Next consider the spherical Bessel functions which are analytic at $\zeta = 0$; these can now be written as

$$i_n(\zeta) = i^n j_n(\xi) = \frac{i^n}{2}\left[h_n^{(1)}(\xi) + h_n^{(2)}(\xi)\right]$$

$$= \frac{e^{\zeta}}{2\zeta}\sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(-2\zeta)^{-\beta} + (-1)^{n+1}\frac{e^{-\zeta}}{2\zeta}\sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(2\zeta)^{-\beta}$$

$$= \cosh(\zeta)\sum_{\beta=0}^{n} 2[(-1)^{\beta} + (-1)^{n+1}]\frac{(n+\beta)!}{\beta!(n-\beta)!}(2\zeta)^{-\beta-1}$$

$$+ \sinh(\zeta)\sum_{\beta=0}^{n} 2[(-1)^{\beta} + (-1)^{n+1}]\frac{(n+\beta)!}{\beta!(n-\beta)!}(2\zeta)^{-\beta-1} \tag{B32}$$

The finite sums in equations B31 and B32 can be expressed as ratios of polynomials in ζ. For the k_n functions the numerator and denominator both have zeros (in conjugate complex pairs except on the Re[ζ] axis). For the i_n functions we only have zeros in the complex ζ plane; the i_n are then entire functions. All the inverse powers of ζ cancel when the exponentials are expanded as power series in ζ.

Having the Bessel functions in the forms we desire we now consider the spherical vector wave functions. These are closely related to the spherical harmonics. As a building block we have the spherical scalar wave functions as

$$\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \Xi^{(\ell)}_{n,m,\sigma}(\gamma r,\theta,\phi) \equiv f^{(\ell)}_n(\gamma r)Y_{n,m,\sigma}(\theta,\phi) \tag{B33}$$

which can be written out as

$$\Xi^{(\ell)}_{n,m,{e \atop o}}(\gamma\vec{r}) = f^{(\ell)}_n(\gamma r)P^m_n(\theta,\phi)\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \quad m = 0, 1, 2, \cdots, n \tag{B34}$$

where $\ell = 1, 2$ refer to i_n and k_n respectively. Coefficients times this when summed over all possible indices satisfy the scalar wave equation which for each function we can write in operator form as

$$[\nabla^2 - \gamma^2]\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = 0 \tag{B35}$$

From the solution of the scalar wave equation one constructs as usual the solutions of the vector wave equation, and these are of three kinds. The first kind have zero curl but non zero divergence and are defined by

$$\vec{L}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \frac{1}{\gamma}\nabla\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$

$$= \left[\vec{e}_r\frac{\partial}{\partial(\gamma r)} + \frac{1}{\gamma r}\nabla_s\right]\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$

$$= f^{(\ell)'}_n(\gamma r)\vec{P}_{n,m,\sigma}(\theta,\phi) + \frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\vec{Q}_{n,m,\sigma}(\theta,\phi) \tag{B36}$$

which can be written out in components as

$$L^{(\ell)}_{r_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = f^{(\ell)'}_n(\gamma r)P^m_n(\cos(\theta))\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$L^{(\ell)}_{\theta_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = \frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\,\frac{dP^m_n(\cos(\theta))}{d\theta}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \tag{B37}$$

$$L^{(\ell)}_{\phi_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = \frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\,\frac{P^m_n(\cos(\theta))}{\sin(\theta)}\,m\begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix}$$

The prime is used to indicate derivatives of the functions with respect to the argument (γr here). The second kind have zero divergence but non zero curl and are defined by

$$\begin{aligned}
\vec{M}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) &\equiv \nabla \times \left[\vec{r}\,\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})\right] \\
&= -\vec{r} \times \nabla\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \\
&= -\vec{e}_r \times \nabla_s\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \\
&= [-\vec{e}_r \times \nabla_s Y_{n,m,\sigma}(\vartheta,\phi)]\,f^{(\ell)}_n(\gamma r) \\
&= f^{(\ell)}_n(\gamma r)\,[-\vec{e}_r \times \vec{Q}_{n,m,\sigma}(\theta,\phi)] \\
&= f^{(\ell)}_n(\gamma r)\,\vec{R}_{n,m,\sigma}(\theta,\phi)
\end{aligned} \tag{B38}$$

The components are

$$M^{(\ell)}_{r_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = 0$$

$$M^{(\ell)}_{\theta_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = f^{(\ell)}_n(\gamma r)\frac{P^m_n(\cos(\theta))}{\sin(\theta)}\,m\begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix} \tag{B39}$$

$$M^{(\ell)}_{\phi_{n,m,{e \atop o}}}(\gamma\vec{r}) = -f^{(\ell)}_n(\gamma r)\frac{dP^m_n(\cos(\theta))}{d\theta}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

The third kind also have non zero curl but zero divergence and are defined by

$$\vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \frac{1}{\gamma}\nabla\times\vec{M}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$

$$= \frac{1}{\gamma}\nabla\times\left[\nabla\times\left[\vec{r}\,\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi)\right]\right]$$

$$= -\frac{\vec{r}}{\gamma}\nabla^2\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi) + \frac{1}{\gamma}(\vec{r}\cdot\nabla)\nabla\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi)$$

$$+ \frac{2}{\gamma}\nabla\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi)$$

$$= -\vec{e}_r\gamma r\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi) + \frac{\partial}{\partial(\gamma r)}\nabla\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi) + \frac{2}{\gamma}\nabla\Xi^{(\ell)}_{n,m,\sigma}(\theta,\phi)$$

$$= \left\{-\gamma r f^{(\ell)}_n(\gamma r) + \gamma r f^{(\ell)''}_n(\gamma r) + 2f^{(\ell)'}_n(\gamma r)\right\}\vec{P}_{n,m,\sigma}(\theta,\phi)$$

$$+ \left\{\gamma r\left[\frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\right]' + 2\,\frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\right\}\vec{Q}_{n,m,\sigma}(\theta,\phi)$$

$$= n(n+1)\frac{f^{(\ell)}_n(\gamma r)}{\gamma r}\vec{P}_{n,m,\sigma}(\theta,\varphi) + \frac{\left[\gamma r f^{(\ell)}_n(\gamma r)\right]'}{\gamma r}\vec{Q}_{n,m,\sigma}(\theta,\phi) \quad \text{(B40)}$$

where we have used the differential equation for the spherical Bessel functions and some vector calculus identities.[11] The components are

$$N^{(\ell)}_{r_{n,m,{e \atop o}}}(\gamma\vec{r}) = n(n+1)\frac{f^{(\ell)}_n(\gamma r)}{\gamma r}P^m_n(\cos(\theta))\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$N^{(\ell)}_{\theta_{n,m,{e \atop o}}}(\gamma\vec{r}) = \frac{\left[\gamma r f^{(\ell)}_n(\gamma r)\right]'}{\gamma r}\frac{dP^m_n(\cos(\theta))}{d\theta}\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \quad \text{(B41)}$$

$$N^{(\ell)}_{\phi_{n,m,\substack{e\\o}}}(\gamma\vec{r}) = \frac{\left[\gamma r f_n^{(\ell)}(\gamma r)\right]'}{\gamma r} \frac{P_n^m(\cos(\theta))}{\sin(\theta)} \begin{Bmatrix} -\sin(m\phi) \\ \cos(m\phi) \end{Bmatrix}$$

Note that all three kinds of vector wave functions satisfy the vector wave equation in Laplacian form which we can summarize as

$$[\nabla^2 - \gamma^2] \begin{Bmatrix} \vec{L} \\ \vec{M} \\ \vec{N} \end{Bmatrix} = \vec{0} \tag{B42}$$

However from the operator identity

$$\nabla\times\nabla\times \;\equiv\; \nabla\nabla\cdot \;-\; \nabla^2 \tag{B43}$$

and noting from their definitions that

$$\nabla\nabla\cdot \begin{Bmatrix} \vec{M} \\ \vec{N} \end{Bmatrix} = \vec{0}$$

$$\nabla\nabla \cdot \vec{L} = \frac{1}{\gamma} \nabla[\nabla^2 \Xi] = \gamma\nabla\Xi = \gamma^2 \vec{L} \tag{B44}$$

we can write a curl curl wave equation for only the second and third kinds of vector wave functions as

$$[\nabla\times\nabla\times \; + \gamma^2] \begin{Bmatrix} \vec{M} \\ \vec{N} \end{Bmatrix} = 0 \tag{B45}$$

The three kinds of vector wave functions have some interrelations as

$$\vec{M}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = -\gamma\vec{r} \times \vec{L}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$

$$\vec{M}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = -\frac{1}{\gamma} \nabla \times \vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$

$$\vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = \frac{1}{\gamma} \nabla \times \vec{M}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \tag{B46}$$

$$\vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = -\nabla \times \left[\vec{r} \times \vec{L}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})\right]$$

Since the vector spherical wave functions have their θ, ϕ dependence expressed in terms of the vector spherical harmonics then these vector wave functions have certain orthogonality properties on the unit sphere based on those for the spherical harmonics (equations B18 and B19). However since two spherical vector harmonics are used for the $\vec{L}$ and $\vec{N}$ functions the orthogonality relations for the vector wave functions on the unit sphere are not as simple and convenient.

Returning to our dyadic plane wave from equation 2.13 in Laplace form for propagation in the direction $\vec{e}_1$ as

$$\overset{\leftrightarrow}{I}_1 \equiv \overset{\leftrightarrow}{I} e^{-\gamma\vec{e}_1\cdot\vec{r}} \tag{B47}$$

where $\overset{\leftrightarrow}{I}$ is the unit dyadic which can be expressed many ways such as

$$\begin{aligned}
\overset{\leftrightarrow}{I} &= (\delta_{b_1,b_2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y + \vec{e}_z\vec{e}_z \\
&= \vec{e}_1\vec{e}_1 + \vec{e}_2\vec{e}_2 + \vec{e}_3\vec{e}_3 \\
&= \vec{e}_r\vec{e}_r + \vec{e}_\theta\vec{e}_\theta + \vec{e}_\phi\vec{e}_\phi
\end{aligned} \tag{B48}$$

This dyadic plane wave is expanded in our spherical vector wave functions as[20]

$$\begin{aligned}
\overset{\leftrightarrow}{I}_1 = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o} [2-\delta_{o,m}](-1)^n(2n+1)\frac{(n-m)!}{(n+m)!}\Big\{-\vec{P}_{n,m,\sigma}(\theta_1,\phi_1)\vec{L}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) \\
+\frac{n}{n(n+1)}\left[\vec{R}_{n,m,\sigma}(\theta_1,\phi_1)\vec{M}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) - \vec{Q}_{n,m,\sigma}(\theta_1,\phi_1)\vec{N}^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right]\Big\}
\end{aligned} \tag{B49}$$

where θ_1 and ϕ_1 are angles giving the direction of $\vec{e}_1$ as used previously. Note that for $n = 0$ the summation is not extended over the $\vec{M}$ and $\vec{N}$ functions which are identically zero.

This dyadic plane wave expansion is related to integral representations for the spherical vector wave functions for $\ell = 1$ as[20]

$$\vec{L}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) = \frac{(-1)^{n+1}}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{-\gamma\vec{e}_1\cdot\vec{r}}\,\vec{P}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1$$

$$= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{\gamma\vec{e}_1\cdot\vec{r}}\,\vec{P}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1$$

$$\vec{M}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) = \frac{(-1)^{n}}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{-\gamma\vec{e}_1\cdot\vec{r}}\,\vec{R}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1$$

$$= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{\gamma\vec{e}_1\cdot\vec{r}}\,\vec{R}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1 \tag{B50}$$

$$\vec{N}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) = \frac{(-1)^{n+1}}{4\pi}\int_0^{\pi}\int_0 \quad {}^{-\gamma\vec{e}_1\cdot\vec{r}}\,\vec{Q}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1$$

$$= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} e^{\gamma\vec{e}_1\cdot\vec{r}}\,\vec{Q}_{n,m,\sigma}(\theta_1,\phi_1)\sin(\theta_1)\,d\phi_1 d\theta_1$$

Thus the spherical vector wave functions for $\ell = 1$ can be considered as weighted integrals over plane waves travelling all possible directions of propagation $\vec{e}_1$.

Having an expansion for the dyadic plane wave we can find the two delta function plane waves for $p = 2, 3$ as

$$\tilde{\vec{u}}_p \equiv \vec{e}_p \cdot \tilde{\vec{I}}_1 \tag{B51}$$

For these we need $\vec{e}_p$ expressed in terms of $\vec{e}_{r1}$, $\vec{e}_{\theta 1}$, $\vec{e}_{\phi 1}$ unit vectors which we can see by referring to figure B1 are

$$\vec{e}_1 = \vec{e}_{r_1}$$

$$\vec{e}_2 = -\vec{e}_{\theta_1} \tag{B52}$$

$$\vec{e}_3 = -\vec{e}_{\phi_1}$$

Then considering the terms in the expansion coefficients we have

$$\vec{e}_2 \cdot \vec{P}_{n,m,{e \atop o}}(\theta_1,\phi_1) = 0$$

$$\vec{e}_2 \cdot \vec{Q}_{n,m,{e \atop o}}(\theta_1,\phi_1) = -\frac{\partial}{\partial\theta_1} Y_{n,m,{e \atop o}}(\theta_1,\phi_1) = -\frac{dP_n^m(\cos(\theta_1))}{d\theta_1}\begin{Bmatrix}\cos(m\phi_1)\\ \sin(m\phi_1)\end{Bmatrix}$$

$$\vec{e}_2 \cdot \vec{R}_{n,m,{e \atop o}}(\theta_1,\phi_1) = -\frac{1}{\sin(\theta_1)}\frac{\partial}{\partial\phi_1} Y_{n,m,{e \atop o}}(\theta_1,\phi_1)$$

$$= -\frac{P_n^m(\cos(\theta_1))}{\sin(\theta_1)} m \begin{Bmatrix}-\sin(m\phi_1)\\ \cos(m\phi_1)\end{Bmatrix} \qquad \text{(B53)}$$

$$\vec{e}_3 \cdot \vec{P}_{n,m,{e \atop o}}(\theta_1,\phi_1) = 0$$

$$\vec{e}_3 \cdot \vec{Q}_{n,m,{e \atop o}}(\theta_1,\phi_1) = -\frac{1}{\sin(\theta_1)}\frac{\partial}{\partial\phi_1} Y_{n,m,{e \atop o}}(\theta_1,\phi_1)$$

$$= -\frac{P_n^m(\cos(\theta_1))}{\sin(\theta_1)} m \begin{Bmatrix}-\sin(m\phi_1)\\ \cos(m\phi_1)\end{Bmatrix}$$

$$\vec{e}_3 \cdot \vec{R}_{n,m,{e \atop o}}(\theta_1,\phi_1) = \frac{\partial}{\partial\theta_1} Y_{n,m,{e \atop o}}(\theta_1,\phi_1) = \frac{dP_n^m(\cos(\theta_1))}{d\theta_1}\begin{Bmatrix}\cos(m\phi_1)\\ \sin(m\phi_1)\end{Bmatrix}$$

and for completeness

$$\vec{e}_1 \cdot \vec{P}_{n,m,{e \atop o}}(\theta_1,\phi_1) = Y_{n,m,{e \atop o}}(\theta_1,\phi_1) = P_n^m(\cos(\theta_1))\begin{Bmatrix}\cos(m\phi_1)\\ \sin(m\phi_1)\end{Bmatrix} \qquad \text{(B54)}$$

$$\vec{e}_1 \cdot \vec{Q}_{n,m,{e \atop o}}(\theta_1,\phi_1) = 0$$

$$\vec{e}_1 \cdot \vec{R}_{n,m,{e \atop o}}(\theta_1,\phi_1) = 0$$

For p = 2, 3 the delta function plane waves (transformed) can be written as

$$\tilde{\vec{u}}_2 = \vec{e}_2 e^{-\gamma\vec{e}_1\cdot\vec{r}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\left[a'_{n,m,\sigma}\vec{M}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) + b'_{n,m,\sigma}\vec{N}^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right]$$

$$\tilde{\vec{u}}_3 = \vec{e}_3 e^{-\gamma\vec{e}_1\cdot\vec{r}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\left[b'_{n,m,\sigma}\vec{M}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) - a'_{n,m,\sigma}\vec{N}^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right] \tag{B55}$$

where

$$a'_{n,m,{e \atop o}} = [2-\delta_{o,m}](-1)^{n+1}\frac{(2n+1)}{n(n+1)}\frac{(n-m)!}{(n+m)!}\, m\, \frac{P_n^m(\cos(\theta_1))}{\sin(\theta_1)}\begin{Bmatrix}-\sin(m\phi_1)\\ \cos(m\phi_1)\end{Bmatrix}$$

$$b'_{n,m,{e \atop o}} = [2-\delta_{o,m}](-1)^{n}\frac{(2n+1)}{n(n+1)}\frac{(n-m)!}{(n+m)!}\frac{dP_n^m(\cos(\theta_1))}{d\theta_1}\begin{Bmatrix}\cos(m\phi_1)\\ \sin(m\phi_1)\end{Bmatrix} \tag{B56}$$

The prime is used with these coefficients to differentiate these from the a_α. Note that we have

$$\frac{1}{\gamma}\nabla \times \left[\vec{e}_2 e^{-\gamma\vec{e}_1\cdot\vec{r}}\right] = \vec{e}_3 e^{-\gamma\vec{e}_1\cdot\vec{r}}$$

$$\frac{1}{\gamma}\nabla \times \left[\vec{e}_3 e^{-\gamma\vec{e}_1\cdot\vec{r}}\right] = -\vec{e}_2 e^{-\gamma\vec{e}_1\cdot\vec{r}} \tag{B57}$$

which is associated with the curl relations between the $\vec{M}$ and $\vec{N}$ functions. Furthermore any divergenceless electric field expansion ($\vec{E}$) can be converted to a magnetic field expansion ($\vec{H}$) by dividing by the wave impedance Z of the medium and changing $\vec{M} \to -\vec{N}$ and $\vec{N} \to +\vec{M}$. To go from $\vec{H}$ to $\vec{E}$ multiply by Z and change $\vec{M} \to +\vec{N}$ and $\vec{N} \to -\vec{M}$.

Now define two sets of coefficients for q = 1, 2 as

$$A_{1,n,m,\sigma,p} = \begin{cases} a'_{n,m,\sigma} & \text{for } p = 2 \\ b'_{n,m,\sigma} & \text{for } p = 3 \end{cases}$$

$$A_{2,n,m,\sigma,p} = \begin{cases} b'_{n,m,\sigma} & \text{for } p = 2 \\ -a'_{n,m,\sigma} & \text{for } p = 3 \end{cases} \tag{B58}$$

Then for p = 2, 3 we can write our unit incident plane wave as

$$\vec{u}_p = \vec{e}_p e^{-\gamma\vec{e}_1\cdot\vec{r}} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\left[A_{1,n,m,\sigma,p}M^{(1)}_{n,m,\sigma}(\gamma\vec{r}) + A_{2,n,m,\sigma,p}N^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right] \tag{B59}$$

Our incident plane wave electric field is written as

$$\tilde{\vec{E}}_{inc}(\vec{r},s) = E_o[\tilde{f}_2(s)\tilde{\vec{u}}_2 + \tilde{f}_3(s)\tilde{\vec{u}}_3] \tag{B60}$$

In the presence of our perfectly conducting sphere of radius a we have a scattered electric field as

$$\tilde{\vec{E}}_{sc}(\vec{r},s) = E_o\left[\tilde{f}_2(s)\tilde{\vec{u}}_2^{(sc)} + \tilde{f}_3\tilde{\vec{u}}_3^{(sc)}\right] \tag{B61}$$

where for p = 2,3 the scattered electric field response functions are

$$\tilde{\vec{u}}_p^{(sc)} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\left[A^{(sc)}_{1,n,m,\sigma,p}\tilde{\vec{M}}^{(2)}_{n,m,\sigma}(\gamma\vec{r}) + A^{(sc)}_{2,n,m,\sigma,p}\tilde{\vec{N}}^{(2)}_{n,m,\sigma}(\gamma\vec{r})\right] \tag{B62}$$

Constraining the tangential electric field to be zero on r = a requires for the tangential components

$$\vec{e}_r \times \left[A_{1,n,m,\sigma,p}\vec{M}^{(1)}_{n,m,\sigma}(\gamma a\vec{e}_r) + A^{(sc)}_{1,n,m,\sigma,p}\vec{M}^{(2)}_{n,m,\sigma}(\gamma a\vec{e}_r)\right] = \vec{0}$$

$$\vec{e}_r \times \left[A_{2,n,m,\sigma,p}\vec{N}^{(1)}_{n,m,\sigma}(\gamma a\vec{e}_r) + A^{(sc)}_{2,n,m,\sigma,p}\vec{N}^{(2)}_{n,m,\sigma}(\gamma a\vec{e}_r)\right] = \vec{0} \tag{B63}$$

This gives equations for the coefficients as

$$A^{(sc)}_{1,n,m,\sigma,p} = -\frac{i_n(\gamma a)}{k_n(\gamma a)} A_{1,n,m,\sigma,p}$$

$$A^{(sc)}_{2,n,m,\sigma,p} = -\frac{[\gamma a i_n(\gamma a)]'}{[\gamma a k_n(\gamma a)]'} A_{2,n,m,\sigma,p} \tag{B64}$$

The surface current and charge densities (equations 2.18, 2.19, 2.25, and 2.26) are written as

$$\tilde{\vec{J}}_s(\vec{r}\,',s) = \frac{E_o}{Z_o}\left[\tilde{f}_2(s)\tilde{\vec{U}}_2^{(\vec{J}_s)}(\vec{r}\,',s) + \tilde{f}_3(s)\tilde{\vec{U}}_3^{(\vec{J}_s)}(\vec{r}\,',s)\right]$$

$$\tilde{\rho}_s(\vec{r}\,',s) = \epsilon_o E_o\left[\tilde{f}_2(s)\tilde{U}_2^{(\rho_s)}(\vec{r}\,',s) + \tilde{f}_3(s)\tilde{U}_3^{(\rho_s)}(\vec{r}\,',s)\right] \tag{B65}$$

To find the surface current and charge densities we need to evaluate just outside r = a the expressions

$$\tilde{\vec{J}}_s(\vec{r}\,',s) = \vec{e}_r\,' \times \tilde{\vec{H}}(\vec{r}\,',s)$$

$$\tilde{\rho}_s(\vec{r}\,',s) = \epsilon_o\vec{e}_r\,' \cdot \tilde{\vec{E}}(\vec{r}\,',s) \tag{B66}$$

The surface current density response functions are then

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',s) = \vec{e}_r\,' \times \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\Big[-A_{1,n,m,\sigma,p}\vec{N}^{(1)}_{n,m,\sigma}(\gamma a\vec{e}_r\,')$$
$$- A^{(sc)}_{1,n,m,\sigma,p}\vec{N}^{(2)}_{n,m,\sigma}(\gamma a\vec{e}_r\,') + A_{2,n,m,\sigma,p}\vec{M}^{(1)}_{n,m,\sigma}(\gamma a\vec{e}_r\,')$$
$$+ A^{(sc)}_{2,n,m,\sigma,p}\vec{M}^{(2)}_{n,m,\sigma}(\gamma a\vec{e}_r\,')\Big]$$

$$= \vec{e}_r' \times \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} \left\{ -A_{1,n,m,\sigma,p} \left[\frac{[\gamma a i_n(\gamma a)]'}{\gamma a} - \frac{i_n(\gamma a)}{k_n(\gamma a)} \frac{[\gamma a k_n(\gamma a)]'}{\gamma a} \right] \vec{Q}_{n,m,\sigma}(\theta',\phi') + A_{2,n,m,\sigma,p} \left[i_n(\gamma a) - \frac{[\gamma a i_n(\gamma a)]'}{[\gamma a k_n(\gamma a)]'} k_n(\gamma a) \right] \vec{R}_{n,m,\sigma}(\theta',\phi') \right\} \tag{B67}$$

From the Wronskian relations for the spherical Bessel functions this reduces to

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}',s) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} \left\{ A_{1,n,m,\sigma,p} \vec{R}_{n,m,\sigma}(\theta',\phi') \frac{1}{(\gamma a)^2 k_n(\gamma a)} - A_{2,n,m,\sigma,p} \vec{Q}_{n,m,\sigma}(\theta',\phi') \frac{1}{\gamma a [\gamma a k_n(\gamma a)]'} \right\} \tag{B68}$$

The surface charge density response functions are then

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} \left\{ A_{1,n,m,\sigma,p} \vec{e}_r' \cdot \vec{M}^{(1)}_{n,m,\sigma}(\gamma a \vec{e}_r') + A^{(sc)}_{1,n,m,\sigma,p} \vec{e}_r' \cdot \vec{M}^{(2)}_{n,m,\sigma}(\gamma a \vec{e}_r') + A_{2,n,m,\sigma,p} \vec{e}_r' \cdot \vec{N}^{(1)}_{n,m,\sigma}(\gamma a \vec{e}_r') + A^{(sc)}_{2,n,m,\sigma,p} \vec{e}_r' \cdot \vec{N}^{(2)}_{n,m,\sigma}(\gamma a \vec{e}_r') \right\}$$

$$= \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} A_{2,n,m,\sigma,p} \left[n(n+1) \frac{i_n(\gamma a)}{\gamma a} - \frac{[\gamma a i_n(\gamma a)]'}{[\gamma a k_n(\gamma a)]'} n(n+1) \frac{k_n(\gamma a)}{\gamma a} \right] Y_{n,m,\sigma}(\theta',\phi') \tag{B69}$$

Note that only the q = 2 terms contribute to the surface charge density. Using a Wronskian relation we have

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} -n(n+1)A_{2,n,m,\sigma,p}$$

$$Y_{n,m,\sigma}(\theta',\phi') \frac{1}{(\gamma a)^2 [\gamma a k_n(\gamma a)]'} \tag{B70}$$

Note that since n = 1, 2, 3, ··· there is no pole at $\gamma a = 0$ in either the surface current density or surface charge density response functions.

Now that we have explicit representations of the response functions for surface current and charge densities in terms of known functions we can identify various terms with the terms in the singularity expansion. Let us start with the natural frequencies. These are the zeros defined by

$$k_n\left(s_\alpha \frac{a}{c}\right) = 0 \ , \quad \left[s_\alpha \frac{a}{c} k_n\left(s_\alpha \frac{a}{c}\right)\right]' = 0 \tag{B71}$$

There are then two classes of natural frequencies which can be labelled by q = 1, 2 depending on which of equations B71 they satisfy. Clearly n is another index and for each n there are some number of natural frequencies which we index by n'. Thus the index set α as applied to labelling the natural frequencies can be written as q, n, n' and we have

$$k_n\left(s_{1,n,n'} \frac{a}{c}\right) = 0 \ , \quad \left[s_{2,n,n'} \frac{a}{c} k_n\left(s_{2,n,n'} \frac{a}{c}\right)\right]' = 0$$
$$s_\alpha \equiv s_{q,n,n'} \tag{B72}$$

Since from equation B31 we have

$$k_n(\zeta) = \frac{e^{-\zeta}}{\zeta} \sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!}(2\zeta)^{-\beta} \tag{B73}$$

Then let us write the spherical Bessel function terms in equations B68 and B69 as

$$\frac{1}{(\gamma a)^2 k_n(\gamma a)} \equiv e^{\gamma a} B_{1,n}(\gamma a) = e^{-st_o} B_{1,n}(\gamma a)$$

$$\frac{1}{\gamma a[\gamma a k_n(\gamma a)]'} \equiv e^{\gamma a} B_{2,n}(\gamma a) = e^{-st_o} B_{2,n}(\gamma a) \quad \text{(B74)}$$

$$\frac{1}{(\gamma a)^2 [\gamma a k_n(\gamma a)]'} \equiv e^{\gamma a} B_{3,n}(\gamma a) = e^{-st_o} \frac{B_{3,n}(\gamma a)}{\gamma a}$$

where

$$t_o = -\frac{a}{c} \quad \text{(B75)}$$

Since the B functions are all ratios of polynomials in γa then we can make a pole expansion of them. Note that t_o is just the turn on time when the incident wave first touches the sphere so that the coupling coefficients are factored as in equation 2.21; the perfectly conducting sphere is then an example of this factoring.

The ratios of polynomials are written as

$$B_{1,n}(\zeta) \equiv \frac{1}{C_{1,n}(\zeta)} = \left\{ \zeta \sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!} (2\zeta)^{-\beta} \right\}^{-1}$$

$$B_{2,n}(\zeta) \equiv \frac{1}{C_{2,n}(\zeta)} = \left\{ -\zeta \sum_{\beta=0}^{n} \frac{(n+\beta)!}{\beta!(n-\beta)!} (2\zeta)^{-\beta} - 2\zeta \sum_{\beta=1}^{n} \beta \frac{(n+\beta)!}{\beta!(n-\beta)!} (2\zeta)^{-\beta-1} \right\}^{-1} \quad \text{(B76)}$$

$$B_{3,n}(\zeta) \equiv \frac{1}{C_{3,n}(\zeta)} = \frac{B_{2,n}(\zeta)}{\zeta} = \frac{1}{\zeta C_{2,n}(\zeta)}$$

In terms of the spherical Bessel functions these rational functions (i.e. polynomial ratios) can be written as

$$B_{1,n}(\zeta) = \frac{e^{-\zeta}}{\zeta^2 k_n(\zeta)}$$

$$B_{2,n}(\zeta) = \frac{e^{-\zeta}}{\zeta[\zeta k_n(\zeta)]'} \tag{B77}$$

$$B_{3,n}(\zeta) = \frac{e^{-\zeta}}{\zeta^2[\zeta k_n(\zeta)]'}$$

Let us make a pole expansion of these rational functions where $B_{1,n}$ has the poles for $q = 1$ and $B_{2,n}$ and $B_{3,n}$ both have poles for $q = 2$. Note that $B_{1,n}$ has the form ζ^{n-1} divided by a polynomial of degree n. $B_{2,n}$ has the form ζ^n divided by a polynomial of degree n + 1. $B_{3,n}$ has the form ζ^{n-1} divided by a polynomial of degree n + 1. Since $n \geq 1$ then $B_{1,n}$ and $B_{2,n}$ are zero for $\zeta \to 0$ while $B_{3,n}$ is a constant (for n = 1) or zero (for $n \geq 2$) for $\zeta \to 0$. Thus there are no poles at s = 0 (consistent with physical requirements). For $s \to \infty$ all three rational functions go to zero; thus there are no poles at $s = \infty$ and no constant terms in the expansions.

Using the rational functions $C_{1,n}$, $C_{2,n}$, and $C_{3,n}$ we can then write our pole expansions around the $s_\alpha = s_{q,n,n'}$ simple zeros of the C functions. Since the number of zeros of a polynomial is equal to the degree of the polynomial then for q = 1 we have n values for s_α and for q = 2 we have n + 1 values of s_α. Define

$$\lambda(b) \equiv \text{largest integer} \leq b \tag{B78}$$

Then we have a range for our index n' as

<u>for q = 1</u>

$$-\lambda\left(\frac{n}{2}\right) \leq n' \leq \lambda\left(\frac{n}{2}\right) \quad \text{with } n' \neq 0 \text{ if n is even}$$

(B79)

<u>for q = 2</u>

$$-\lambda\left(\frac{n+1}{2}\right) \leq n' \leq \lambda\left(\frac{n+1}{2}\right) \quad \text{with } n' \neq 0 \text{ if n is odd}$$

For the s_α we then automatically have the convenient relation possible between n' and -n' indices as

$$B_{1,n}(\zeta) = \frac{e^{-\zeta}}{\zeta^2 k_n(\zeta)}$$

$$B_{2,n}(\zeta) = \frac{e^{-\zeta}}{\zeta[\zeta k_n(\zeta)]'} \tag{B77}$$

$$B_{3,n}(\zeta) = \frac{e^{-\zeta}}{\zeta^2[\zeta k_n(\zeta)]'}$$

Let us make a pole expansion of these rational functions where $B_{1,n}$ has the poles for $q = 1$ and $B_{2,n}$ and $B_{3,n}$ both have poles for $q = 2$. Note that $B_{1,n}$ has the form ζ^{n-1} divided by a polynomial of degree n. $B_{2,n}$ has the form ζ^n divided by a polynomial of degree $n + 1$. $B_{3,n}$ has the form ζ^{n-1} divided by a polynomial of degree $n + 1$. Since $n \geq 1$ then $B_{1,n}$ and $B_{2,n}$ are zero for $\zeta \to 0$ while $B_{3,n}$ is a constant (for $n = 1$) or zero (for $n \geq 2$) for $\zeta \to 0$. Thus there are no poles at $s = 0$ (consistent with physical requirements). For $s \to \infty$ all three rational functions go to zero; thus there are no poles at $s = \infty$ and no constant terms in the expansions.

Using the rational functions $C_{1,n}$, $C_{2,n}$, and $C_{3,n}$ we can then write our pole expansions around the $s_\alpha = s_{q,n,n'}$ simple zeros of the C functions. Since the number of zeros of a polynomial is equal to the degree of the polynomial then for $q = 1$ we have n values for s_α and for $q = 2$ we have $n + 1$ values of s_α. Define

$$\lambda(b) \equiv \text{largest integer} \leq b \tag{B78}$$

Then we have a range for our index n' as

<u>for q = 1</u>

$$-\lambda\left(\frac{n}{2}\right) \leq n' \leq \lambda\left(\frac{n}{2}\right) \quad \text{with } n' \neq 0 \text{ if } n \text{ is even}$$

(B79)

<u>for q = 2</u>

$$-\lambda\left(\frac{n+1}{2}\right) \leq n' \leq \lambda\left(\frac{n+1}{2}\right) \quad \text{with } n' \neq 0 \text{ if } n \text{ is odd}$$

For the s_α we then automatically have the convenient relation possible between n' and -n' indices as

$$s_{q,n,-n'} = \bar{s}_{q,n,n'}$$

$$\mathrm{Im}[s_{1,n,0}] = 0 \quad \text{for } n \text{ odd} \tag{B80}$$

$$\mathrm{Im}[s_{2,n,0}] = 0 \quad \text{for } n \text{ even}$$

It is known[19] that the zeros of k_n lie approximately on an arc in the left half of the s plane joining $sa/c = -in$ to $sa/c = +in$ and passing through $sa/c = -.66n$. The zeros of $[\zeta k_n(\zeta)]'$ behave similarly. Then a convenient way to identify the s_α with specific n' is to start with the most negative value of n' from equations B79 and assign it to the s_α with the most negative $\mathrm{Im}[s_\alpha]$ and progressively work up to the most positive $\mathrm{Im}[s_\alpha]$.

Thus our pole expansions may be written as

$$B_{1,n}\left(\frac{sa}{c}\right) = \sum_{n'=-\lambda\left(\frac{n}{2}\right)}^{\lambda\left(\frac{n}{2}\right)} \frac{D_{1,n,n'}}{s-s_{1,n,n'}} \quad \text{with } n' \neq 0 \text{ if } n \text{ is even}$$

$$B_{2,n}\left(\frac{sa}{c}\right) = \sum_{n'=-\lambda\left(\frac{n+1}{2}\right)}^{\lambda\left(\frac{n+1}{2}\right)} \frac{D_{2,n,n'}}{s-s_{2,n,n'}} \quad \text{with } n' \neq 0 \text{ if } n \text{ is odd} \tag{B81}$$

$$B_{3,n}\left(\frac{sa}{c}\right) = \sum_{n'=-\lambda\left(\frac{n+1}{2}\right)}^{\lambda\left(\frac{n+1}{2}\right)} \frac{D_{3,n,n'}}{s-s_{2,n,n'}} \quad \text{with } n' \neq 0 \text{ if } n \text{ is odd}$$

where

$$D_{1,n,n'} = \frac{c}{a}\left[\frac{d}{d\zeta} C_{n,1}(\zeta)\right]^{-1}\Bigg|_{\zeta=s_{1,n,n'}\frac{a}{c}} = \bar{D}_{1,n,-n'}$$

$$D_{2,n,n'} = \frac{c}{a}\left[\frac{d}{d\zeta} C_{n,2}(\zeta)\right]^{-1}\Bigg|_{\zeta=s_{2,n,n'}\frac{a}{c}} = \bar{D}_{2,n,-n'}$$

$$D_{3,n,n'} = \frac{c}{a}\left[\frac{d}{d\zeta} C_{n',3}(\zeta)\right]^{-1}\Bigg|_{\zeta=s_{2,n,n'}\frac{a}{c}} = \frac{c}{a}\left[\zeta \frac{d}{d\zeta} C_{n,2}\right]^{-1}\Bigg|_{\zeta=s_{2,n,n'}\frac{a}{c}} \qquad \text{(B82)}$$

$$= \bar{D}_{3,n,-n'} = \frac{c}{s_{2,n,n'}a} D_{2,n,n'}$$

To see that the poles must be simple poles note that the zeros of the C functions are all simple zeros because they are the zeros of $k_n(\zeta)$ and $[\zeta k_n(\zeta)]'$. Observe the differential equation for the spherical Bessel functions is[19]

$$\zeta^2 f_n^{(\ell)''}(\zeta) + 2\zeta f_n^{(\ell)'}(\zeta) - [\zeta^2+n(n+1)]f_n^{(\ell)}(\zeta) = 0 \qquad \text{(B83)}$$

Suppose ζ has a zero at $\zeta_o \neq 0$. Then since f_n is analytic at this zero we can write a convergent power series expansion in a neighborhood of $\zeta = \zeta_o$. If the zero is higher than first order, say $(\zeta - \zeta_o)^2$, then both f_n and f_n' are zero at ζ_o, but this forces f_n'' to also be zero at ζ_o so the zero had to be at least $(\zeta - \zeta_o)^3$ as the leading term in the power series. Then divide through by $\zeta - \zeta_o$, but f_n and f_n' are still zero making f_n'' still zero. This process continues to make all terms in the power series zero and the function then identically zero. Thus the zeros are all simple for $\zeta \neq 0$. Similarly $\zeta k_n(\zeta)$ satisfies the Riccati-Bessel equation[19]

$$\frac{\zeta^2}{\zeta^2+n(n+1)}\left[\zeta f_n^{(\ell)}(\zeta)\right]'' - \zeta f_n^{(\ell)}(\zeta) = 0 \qquad \text{(B84)}$$

Differentiating gives a differential equation for $[\zeta k_n(\zeta)]'$ as

$$\frac{\zeta^2}{\zeta^2+n(n+1)}\left[\zeta f_n^{(\ell)}(\zeta)\right]''' + \frac{2\zeta n(n+1)}{[\zeta^2+n(n+1)]^2}\left[\zeta f_n^{(\ell)}(\zeta)\right]'' - \left[\zeta f_n^{(\ell)}(\zeta)\right]' = 0$$

(B85)

Clearing the denominators this equation has the same form as equation B83 and so for $\zeta \neq 0$ all zeros must be simple. Therefore the perfectly conducting sphere has only simple poles in its surface current density and surface charge density response functions.

To see some of the numbers we can write out the first few terms, say for $n = 1$, from

$$C_{1,1} = \zeta + 1$$

$$C_{2,1} = \frac{C_{3,1}}{\zeta} = -\frac{1}{\zeta}[\zeta^2 + \zeta + 1] \qquad \text{(B86)}$$

$$\frac{d}{d\zeta}C_{1,1} = 1 , \quad \frac{d}{d\zeta}C_{2,1} = -1 + \zeta^{-2} , \quad \frac{d}{d\zeta}C_{3,1} = -2\zeta - 1$$

From these results we can construct a table allowing q = 3 for the surface charge density as

q	n'	$s_{q,1,n'}\frac{a}{c}$	$\frac{a}{c} D_{q,1,n'}$
1	0	-1	1
2	1	$-\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$	$-\frac{1}{2} \mp i \frac{\sqrt{3}}{6}$
3	1 (like q=2)	$-\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$ (like q=2)	$-i \frac{\sqrt{3}}{3}$

Table B1. Pole expansion terms for n = 1

If one wishes these natural frequencies and D coefficients can be generated to obtain any number of terms in the expansions.

For n = 1, 2, 3 the zeros of the $C_{q,n}$ polynomials can be found from formulas for the zeros of up to quartic polynomials.[19] There is disagreement in a few cases of the natural frequencies with the numbers in Stratton.[17] However, the present results appear to be more accurate and are confirmed by Dr. Marin (private communication). The zeros have been substituted in the polynomials to check that in fact the results are closest to the true zeros to the number of places listed. Figure B2 shows the positions in the complex sa/c plane (normalized s plane). The division of the natural frequencies into q = 1 and q = 2 varieties has a physical basis in that only the q = 2 poles contribute to the surface charge density. Another way to view this is from a property of the vector spherical harmonics as

$$\nabla_s \times \vec{Q}_{n,m,\sigma}(\theta,\phi) = \nabla_s \times [\nabla_s Y_{n,m,\sigma}(\theta,\phi)] = \vec{0}$$
$$\qquad \text{(B87)}$$
$$\nabla_s \cdot \vec{R}_{n,m,\sigma}(\theta,\phi) = \nabla_s \cdot [\nabla_s \times [\vec{e}_r Y_{n,m,\sigma}(\theta,\phi)]] = 0$$

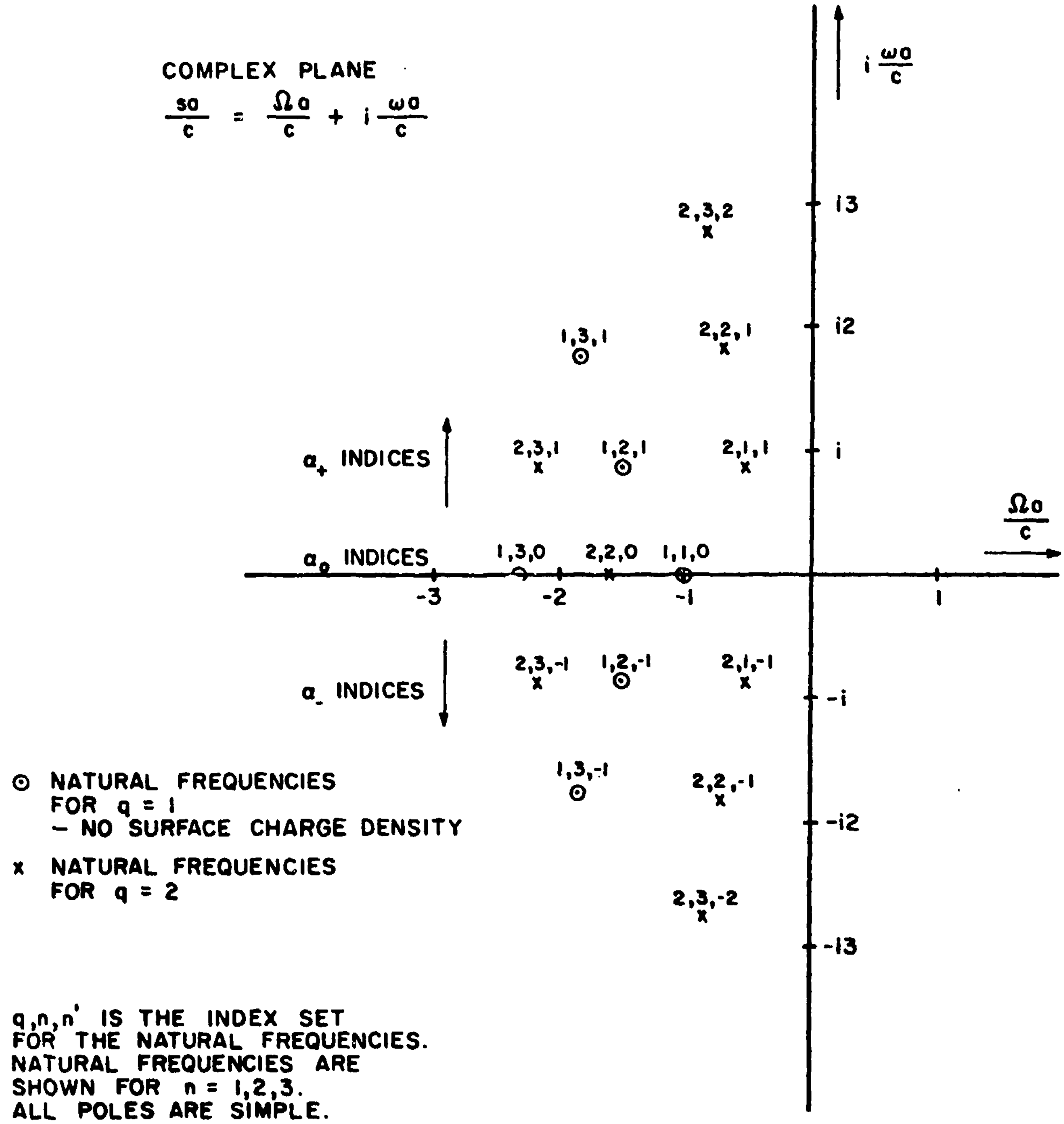

FIGURE B2. NATURAL FREQUENCIES OF THE PERFECTLY CONDUCTING SPHERE FOR USE WITH EXTERIOR INCIDENT WAVE

q	n	n'	$s_{q,n,n'}\frac{a}{c}$
1	1	0	-1
2	1	1	-.500 + i.866
1	2	1	-1.500 + i.866
2	2	1	-.702 + i1.807
		0	-1.596
1	3	1	-1.839 + i1.754
		0	-2.322
2	3	2	-.843 + i2.758
		1	-2.157 + i.871

Table B2. Natural frequencies for n = 1, 2, 3

from which we can divide the surface current density into solenoidal terms (precisely q = 1) and irrotational terms (precisely q = 2). (See ref. 6 for more elaboration of this point in the general case.) However, the n, n' division may not give the best indexing. Referring to figure B2 there are various possible paths through the complex s plane which one might trace to connect poles with the same q index. For a given n (and q) there are many modal distributions generated by varying m and σ, all applying to the entire set of natural frequencies generated by varying n'. This is a very degenerate situation in both natural frequencies and modes. Perhaps more insight into the division of the indices for the natural frequencies and modes can be gained from a group theory investigation of the symmetry properties. Symmetry planes and axes can be used to divide up natural modes, and thereby natural frequencies as well. A diagram as in figure B2 is useful in that it can suggest ways of grouping natural frequencies, even for objects more complex than a perfectly conducting sphere. Note that the pattern of the natural frequencies tends to fill up the left half of the s plane. This two dimensional pole distribution may be associated with the distributed nature of the body; we are dealing with surface current and charge densities. For cases that the currents are idealized as on one dimensional paths then the pole distribution should be much less dense and localized to "discrete paths" in the complex s plane; in any event there would be one less index needed and not say n and m both for the modes or perhaps not n and n' both for the frequencies.

Next we consider the natural modes of the sphere. This is the part of the singularity expansion where the object coordinates are expressed. For the surface current density the natural modes are readily identified in equation B68 as

$$\vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,') = \vec{\nu}_{q,n,m,\sigma}^{(\vec{J}_s)}(\vec{r}\,')$$

$$= \begin{cases} \vec{R}_{n,m,\sigma}(\theta',\phi') & \text{for } q = 1 \\ \vec{Q}_{n,m,\sigma}(\theta',\phi') & \text{for } q = 2 \end{cases} \tag{B88}$$

Furthermore the surface charge density natural modes are readily identified in equation B69 as

$$\nu_\alpha^{(\rho_s)}(\vec{r}\,') = \nu_{q,n,m,\sigma}^{(\rho_s)}(\vec{r}\,')$$

$$= \begin{cases} 0 & \text{for } q = 1 \\ Y_{n,m,\sigma}(\theta',\phi') & \text{for } q = 2 \end{cases} \tag{B89}$$

Thus for the surface charge density we can drop the q index in the summation understanding that only q = 2 is used for the natural frequencies and modes. Back in equation 2.27 we observed a relation between the natural modes for the surface current density and surface charge density as

$$\nu_\alpha^{(\rho_s)} = -a_\alpha \nabla' \cdot \vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,') \tag{B90}$$

where a_α is an arbitrary constant depending on how one has defined the natural modes since the modes can be multiplied by any non zero complex constant. For the sphere problem we can write this in terms of the divergence on the unit sphere as

$$\nu_\alpha^{(\rho_s)} = -\frac{a_\alpha}{a} \nabla_s' \cdot \vec{\nu}_\alpha^{(\vec{J}_s)}(\vec{r}\,') \tag{B91}$$

Now we have

$$\begin{aligned} \nabla_s' \cdot \vec{R}_{n,m,\sigma}(\theta',\phi') &= \nabla_s' \cdot [\nabla_s \times [\vec{e}_r Y_{n,m,\sigma}(\theta',\phi')]] \equiv 0 \\ \nabla_s' \cdot \vec{Q}_{n,m,\sigma}(\theta',\phi') &= \nabla_s'^2 Y_{n,m,\sigma}(\theta',\phi') \end{aligned} \tag{B92}$$

so that only q = 2 natural modes have non zero surface charge density. From the fact that the Ξ functions defined in equation B34 satisfy the scalar wave equation as in equation B35, then noting that from the separation equation for the radial functions in spherical coordinates as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} f_n^{(\ell)}(\gamma r)\right) - \left[\gamma^2 + \frac{n(n+1)}{r^2}\right] f_n^{(\ell)}(\gamma r) = 0 \qquad \text{(B93)}$$

equation B35 can then be written as

$$0 = [\nabla^2-\gamma^2]\Xi_{n,m,\sigma}^{(\ell)}(\gamma\vec{r})$$

$$= [\nabla^2-\gamma^2] f_n^{(\ell)}(\gamma r) Y_{n,m,\sigma}(\theta,\phi)$$

$$= \frac{n(n+1)}{r^2} f_n^{(\ell)}(\gamma r) Y_{n,m,\sigma}(\theta,\phi) + f_n^{(\ell)}(\gamma r)\nabla^2 Y_{n,m,\sigma}(\theta,\phi) \qquad \text{(B94)}$$

from which we find

$$\nabla_s' \cdot \vec{Q}_{n,m,\sigma}(\theta',\phi') = \nabla_s'^2 Y_{n,m,\sigma}(\theta',\phi')$$

$$= -n(n+1) Y_{n,m,\sigma}(\theta',\phi') \qquad \text{(B95)}$$

Thus we have

$$a_\alpha \equiv a_n = n(n+1)a \qquad \text{(B96)}$$

where only q = 2 is relevant.

Having the modes we now only need the coupling coefficients, these are everything that remains in equations B68 and B69. Since the poles are all simple the basic equations we are matching come from equations 2.34 giving the plane wave response functions as

$$\tilde{U}_p^{(\vec{J}_s)}(\vec{r}',s) = e^{-st_o}\sum_{q=1}^{2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\sum_{\substack{n'=-\lambda\left(\frac{n+q-1}{2}\right)\\ n'\neq 0 \text{ for } n+q \text{ odd}}}^{\lambda\left(\frac{n+q-1}{2}\right)}\left[c_{q,n,n',m,\sigma,p}(\theta_1,\phi_1)\right.$$

$$\left.\vec{\nu}_{q,n,m,\sigma}^{(\vec{J}_s)}(\theta',\phi')\frac{1}{s-s_{q,n,n'}}\right]$$

(B97)

$$\tilde{U}_p^{(\rho_s)}(\vec{r}',s) = e^{-st_o}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\sum_{\substack{n'=-\lambda\left(\frac{n+1}{2}\right)\\ n'\neq 0 \text{ for } n \text{ odd}}}^{\lambda\left(\frac{n+1}{2}\right)}\left[\frac{c}{s_{q,n,n'}a_n}\right.$$

$$\left.c_{2,n,n',m,\sigma,p}(\theta_1,\phi_1)\nu_{2,n,m,\sigma}^{(\rho_s)}(\theta',\phi')\frac{1}{s-s_{2,n,n'}}\right]$$

The coupling coefficients are then for q = 1, 2

$$c_{q,n,n',m,\sigma,p}(\theta_1,\phi_1) = (-1)^{q-1}A_{q,n,m,\sigma,p}D_{q,n,n'}$$

$$= \begin{cases} a'_{n,m,\sigma}D_{q,n,n'} & \text{for } q = 1,\ p = 2 \\ b'_{n,m,\sigma}D_{q,n,n'} & \text{for } q = 1,\ p = 3 \\ -b'_{n,m,\sigma}D_{q,n,n'} & \text{for } q = 2,\ p = 2 \\ a'_{n,m,\sigma}D_{q,n,n'} & \text{for } q = 2,\ p = 3 \end{cases} \qquad \text{(B98)}$$

The D coefficients are evaluated from equations B82 and a few are listed in table B1. The a'_α and b'_α coefficients are found explicitly in equations B56. Note that the surface charge density expansion uses only q = 2 for the c_α and the D_α. While we can calculate $D_{3,n,n'}$ as well it is simply $D_{2,n,n'}\ c/(s_{2,n,n'}a)$; using the results for a_α with this and the $n(n+1)$ coefficient in equation B70 for the surface charge density one can see that the same answer for the surface charge density expansion results, thereby giving a check.

Equations B97 then explicitly give the singularity expansion for the surface current and charge densities. The natural frequencies are the zeros of the C_α in equation B76; the natural modes are in equations B88 and B89; the a_α are given by equation B96; the turn-on time t_o is given in equation B75; the coupling coefficients are given by equation B98 together with equations B82 and B56.

Now that we have the singularity expansions for the delta function response functions we can consider arbitrary waveforms by taking their Laplace transforms and splitting the response into a part associated with the singularities of the perfectly conducting sphere and a part associated with the waveform singularities as developed in section II. For convenience let the incident wave be a step function. Then from equations 2.43 the surface current density expansion is

$$\tilde{\vec{V}}_p^{(\vec{J}_s)}(\vec{r}\,',s) \equiv \tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) + \tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}\,',s)$$

$$\tilde{\vec{V}}_{p_w}^{(\vec{J}_s)}(\vec{r}\,',s) = \frac{e^{-st_o}}{s}\,\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',0)$$

$$\tilde{\vec{V}}_{p_o}^{(\vec{J}_s)}(\vec{r}\,',s) = e^{-st_o}\sum_{q=1}^{2}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\sum_{\substack{n'=-\lambda\left(\frac{n+q-1}{2}\right)\\ n'\neq 0 \text{ for } n+q \text{ odd}}}^{\lambda\left(\frac{n+q-1}{2}\right)}\left[\frac{1}{s_{q,n,n'}}\,c_{q,n,n',m,\sigma,p}(\theta_1,\phi_1)\,\vec{v}_{q,n,m,\sigma}^{(\vec{J}_s)}(\theta',\phi')\frac{1}{s-s_{q,n,n'}}\right] \tag{B99}$$

and from equations 2.50 the surface charge density step response is written as

$$\tilde{V}_p^{(\rho_s)}(\vec{r}\,',s) \equiv \tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) + \tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s)$$

$$\tilde{V}_{p_w}^{(\rho_s)}(\vec{r}\,',s) = \frac{e^{-st_o}}{s}\,\tilde{U}_p^{(\rho_s)}(\vec{r}\,',0) \tag{B100}$$

$$\tilde{V}_{p_o}^{(\rho_s)}(\vec{r}\,',s) = e^{-st_o}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\sum_{\substack{n'=-\lambda\left(\frac{n+1}{2}\right)\\ n'\neq 0 \text{ for } n \text{ odd}}}^{\lambda\left(\frac{n+1}{2}\right)}\left[\frac{c}{s_{2,n,n'}^2 a_n}\right.$$

$$\left. c_{2,n,n',m,\sigma,p}(\theta_1,\phi_1)\,\nu_{2,n,m,\sigma}^{(\rho_s)}(\theta',\phi')\frac{1}{s-s_{2,n,n'}}\right]$$

The static surface current density response is

$$\tilde{\vec{U}}_p^{(\vec{J}_s)}(\vec{r}\,',0) = \vec{\vec{U}}_s^{(\vec{J}_s)}(\vec{r}\,')\cdot[\vec{e}_1\times\vec{e}_p]$$

$$= \sum_{m=0}^{1}\sum_{\sigma=e,o} A_{1,1,m,\sigma,p}\vec{R}_{1,m,\sigma}(\theta',\phi') \tag{B101}$$

where for m = 0, 1 we have

$$A_{1,1,m,{e \atop o},p} = \begin{cases} -\frac{3}{2}\,\vec{e}_3\cdot\vec{Q}_{1,m,\sigma} & \text{for } p = 2 \\ \frac{3}{2}\,\vec{e}_2\cdot\vec{Q}_{1,m,\sigma} & \text{for } p = 3 \end{cases}$$

$$= -\frac{3}{2}[\vec{e}_1\times\vec{e}_p]\cdot\vec{Q}_{1,m,\sigma}(\theta_1,\phi_1) \tag{B102}$$

from which the dyadic surface current density static response function can be written as

$$\vec{\vec{U}}_s^{(\vec{J}_s)}(\vec{r}\,') = \sum_{m=0}^{1}\sum_{\sigma=e,o} -\frac{3}{2}\,\vec{R}_{1,m,\sigma}(\theta',\phi')\vec{Q}_{1,m,\sigma}(\theta',\phi')$$

$$= \sum_{\sigma=e,o}\left[-\frac{3}{2}\,\vec{R}_{1,0,\sigma}(\theta',\phi')\vec{Q}_{1,0,\sigma}(\theta_1,\phi_1)\right.$$

$$\left. -\frac{3}{2}\,\vec{R}_{1,1,\sigma}(\theta',\phi')\vec{Q}_{1,1,\sigma}(\theta_1,\phi_1)\right] \tag{B103}$$

With the direction of the static magnetic field (which is $\vec{e}_1 \times \vec{e}_p$ for our plane wave problem) dot multiplied on the right we obtain the static surface current density response. The static surface charge density is

$$\tilde{U}_p^{(\rho_s)}(\vec{r}\,',0) = \tilde{U}_s^{(\rho_s)}(\vec{r}\,') \cdot \vec{e}_p$$

$$= \sum_{m=0}^{1} \sum_{\sigma=e,o} 2A_{2,1,m,\sigma,p} Y_{1,m,\sigma}(\theta',\phi') \qquad \text{(B104)}$$

where for m = 0, 1 we have

$$2A_{2,1,m,\sigma,p} = \begin{cases} 3\vec{e}_2 \cdot \vec{Q}_{1,m,\sigma}(\theta_1,\phi_1) & \text{for } p = 2 \\ 3\vec{e}_3 \cdot \vec{Q}_{1,m,\sigma}(\theta_1,\phi_1) & \text{for } p = 3 \end{cases}$$

$$= 3\vec{e}_p \cdot \vec{Q}_{1,m,\sigma}(\theta_1,\phi_1) \qquad \text{(B105)}$$

from which the vector surface charge density response function can be written as

$$\tilde{U}_s^{(\rho_s)}(\vec{r}\,') = \sum_{m=0}^{1} \sum_{\sigma=e,o} 3Y_{1,m,\sigma}(\theta',\phi')\vec{Q}_{1,m,\sigma}(\theta_1,\phi_1)$$

$$= \sum_{\sigma=e,o} 3Y_{1,0,\sigma}(\theta',\phi')\vec{Q}_{1,0,\sigma}(\theta_1,\phi_1) + 3Y_{1,1,\sigma}(\theta',\phi')\vec{Q}_{1,1,\sigma}(\theta_1,\phi_1) \qquad \text{(B106)}$$

Dot multiplying this by $\vec{e}_p$ (the direction of the static electric field in our plane wave problem) gives the static charge density response. Thus for the perfectly conducting sphere the plane wave delta function response can be converted to the plane wave step function response of both surface current and charge densities by multiplying each term by $1/s_{q,n,n'}$ and adding a static term (consisting of a few simple known functions) with a unit step function turning on at time t_o.

This basically completes the singularity expansion of the response of a perfectly conducting sphere to an incident plane wave for simple waveforms. As long as the incident waveform can be expressed only in terms of poles the delta function response as in equations B97 can be combined with the waveform poles and through partial fraction expansion as discussed in section 2 the response can be split into a waveform part and an object part. Also as discussed in section II the time domain response is easily and directly obtained from the pole expansion since the frequency dependence of the coupling coefficients factors out as a common delay term e^{-st_o}. For the case of the step function response the Laplace form as in equations B99 and B100 can be immediately converted to time domain using equations 2.64 and 2.65 respectively since the poles are all simple poles. Note also that the natural mode functions are real and the a_α are also real; both of these can then be moved out (together with c) from the Re function in equations 2.64 and 2.65 leaving only the natural frequencies, coupling coefficients and oscillatory exponentials for $n' > 0$ as the only complex terms inside the Re function. Of course the c_α coupling coefficients can be written in the form $a_{n,m,\sigma}D_\alpha$ or $b_{n,m,\sigma}D_\alpha$ and only the D_α coefficients are complex.

As an example of the time domain response consider just a few terms in the step function response, say for $p = 2$, $n = 1$, $\phi_1 = 0$, and $\theta_1 = \pi/2$ so that we have a vertically polarized wave propagating parallel to the x axis. Then we have

$$\vec{V}_2^{(\vec{J}_s)}(\vec{r}',t) = \vec{U}_s^{(\vec{J}_s)}(\vec{r}') \cdot \vec{e}_3 u(t-t_o)$$

$$+ \frac{1}{s_{1,1,0}} c_{1,1,0,1,o,2}\left(\frac{\pi}{2},0\right) \vec{\nu}_{1,1,1,o}^{(\vec{J}_s)}(\theta',\phi') e^{\Omega_{1,1,0}(t-t_o)} u(t-t_o)$$

$$+ 2\mathrm{Re}\left[\frac{1}{s_{2,1,1}} c_{2,1,1,0,e,2}\left(\frac{\pi}{2},0\right) e^{i\omega_{2,1,1}(t-t_o)}\right]\left[\vec{\nu}_{2,1,0,e}^{(\vec{J}_s)}(\theta',\phi')\right.$$

$$\left. e^{\Omega_{2,1,1}(t-t_o)} u(t-t_o)\right] + \sum_{n>1}$$

(B107)

$$V_2^{(\rho_s)}(\vec{r}',t) = \vec{U}_s^{(\rho_s)}(\vec{r}') \cdot \vec{e}_2 u(t-t_o)$$

$$\mathrm{Re}\left[\frac{c}{s_{2,1,1}^2 a} c_{2,1,1,0,e,2}\left(\frac{\pi}{2},0\right) e^{i\omega_{2,1,1}(t-t_o)}\right]\left[\nu_{2,1,0,e}^{(\rho_s)}(\theta',\phi')\right.$$

$$\left. e^{\Omega_{2,1,1}(t-t_o)} u(t-t_o)\right] + \sum_{n>1}$$

where the last terms indicate the remainder terms. Writing these out we have

$$\vec{V}_2^{(\vec{J}_s)}(\vec{r}',t) = -\frac{3}{2}\vec{R}_{1,1,o}(\theta',\phi')u\left(t+\frac{a}{c}\right)$$

$$+\frac{3}{2}\vec{R}_{1,1,o}(\theta',\phi')e^{-\left(\frac{ct}{a}+1\right)}u\left(t+\frac{a}{c}\right)$$

$$-3\mathrm{Re}\left[-i\frac{\sqrt{3}}{3}e^{i\frac{\sqrt{3}}{2}\left(\frac{ct}{a}+1\right)}\right]\vec{Q}_{1,0,e}(\theta',\phi')e^{-\frac{1}{2}\left(\frac{ct}{a}+1\right)}u\left(t+\frac{a}{c}\right)$$

$$+\sum_{n>1} \qquad \text{(B108)}$$

$$V_2^{(\rho_s)}(\vec{r}',t) = 3Y_{1,0,e}(\theta',\phi')u\left(t+\frac{a}{c}\right)$$

$$-\left[\frac{3}{2}\mathrm{Re}\left[\frac{-i\frac{\sqrt{3}}{3}}{-\frac{1}{2}+i\frac{\sqrt{3}}{3}}e^{i\frac{\sqrt{3}}{2}\left(\frac{ct}{a}+1\right)}\right]Y_{1,0,e}(\theta',\phi')\right.$$

$$\left.e^{-\frac{1}{2}\left(\frac{ct}{a}+1\right)}u\left(t+\frac{a}{c}\right)\right]+\sum_{n>1}$$

These can be summarized as

$$\vec{V}_2^{(\vec{J}_s)}(\vec{r}',t) = -\frac{3}{2}\vec{R}_{1,1,o}(\theta',\phi')\left[1-e^{-\left(\frac{ct}{a}+1\right)}\right]u\left(t+\frac{a}{c}\right)$$

$$+\sqrt{3}\vec{Q}_{1,0,e}(\theta',\phi')e^{-\frac{1}{2}\left(\frac{ct}{a}+1\right)}\sin\left(\frac{\sqrt{3}}{2}\left(\frac{ct}{a}+1\right)\right)u\left(t+\frac{a}{c}\right)+\sum_{n>1}$$

(B109)

$$V_2^{(\rho_s)}(\vec{r}\,',t) = Y_{1,0,e}(\theta',\phi')\left[3 + \frac{\sqrt{3}}{2} e^{-\frac{1}{2}(\frac{ct}{a}+1)} \cos\left(\frac{\sqrt{3}}{2}(\frac{ct}{a}+1) - \frac{\pi}{6}\right)\right] u(t + \frac{a}{c})$$

$$+ \sum_{n>1}$$

For reference we have

$$\vec{R}_{1,1,o}(\theta',\phi') = -\cos(\phi')\vec{e}_\theta\,' + \cos(\theta')\sin(\phi')\vec{e}_\phi\,'$$

$$\vec{Q}_{1,0,e}(\theta',\phi') = -\sin(\theta')\cos(\phi')\vec{e}_\theta\,' \qquad \text{(B110)}$$

$$Y_{1,0,e}(\theta',\phi') = \cos(\theta')$$

For comparison to these results for the step response of a sphere one can consider the numerical results for the step response graphed in another note.[21] Consider the case in that note that the perfectly conducting plane is infinitely far away from the perfectly conducting sphere. Note that the basic ringing period agrees closely with $4\pi/\sqrt{3}$ (in units of ct/a) and that in one period the amplitude of the ringing decays by approximately $e^{-2\pi/\sqrt{3}}$ and that even the coefficients of the ringing terms in equations B109 and B110 give about the correct amplitudes for the oscillations. In the referenced note only the total current crossing the equator $\theta' = \pi/2$ is considered, and so the comparison has to neglect the first term in equation B109 which gives no contribution in this case. Note that for times $-a/c < t < a/c$ we do not expect the first terms only to accurately describe the surface current and charge densities because the terms for $n > 1$ have not had a chance to decay to zero amplitude. Even so, the first few terms give a simple description with some features of the surface current and charge densities even at such early times. As time goes on the first few terms asymptotically give the exact results.

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