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ON THE SIZE OF A MINIMAL VERTEX COVER IN A RANDOM SUBGRAPH OF THE n-CUBE

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Abstract

We describe and analyze a construction of a vertex cover (consisting of subcubes) in a random subgraph of the n-cube. The main idea of the construction is to select subcubes with minimal intersection into the vertex cover. We estimate the upper bound of such a vertex cover. Our analysis gives a theoretical justification for a heuristic that minimizes the disjunctive normal form of a random Boolean function by selecting conjunctions according to the strategy of minimal intersections.

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1. INTRODUCTION

We study randomly induced subgraphs of the n-cube Q_n. The model of random subgraphs is the following one: each edge is present in a subgraph with probability p $(0 < p < 1)$, independently on the presence of other edges. Connectedness of these graphs for $p \neq \frac{1}{2}$ was studied in [4]. Components of random subgraphs for $p = \frac{1}{2}$ were analyzed in [5; 11], and for $p = \frac{1+\varepsilon}{n}$ in [1]. The radius of a random subgraph of an n-cube for $p \geq \frac{1}{2}$ was estimated in [12]. Perfect matchings for $p \geq \frac{1}{2}$ were analyzed in [2; 6]. Structural properties of such random graphs were studied in [9]. An approach to vertex cover was presented in [10]. Randomly induced subgraphs of cubes are related to minimization of Boolean functions in the class of disjunctive normal forms. For detailed study of these aspects, see [7].

In the paper we build on the results obtained in [9] where we studied the number and distribution of maximal subcubes in the random subgraph of Q_n. We proved that the largest number of maximal subcubes has order approximately $\lambda \sim \log\log_{1/p} n = \log\log n - \log\log \frac{1}{p}$ (where log denotes the binary logarithm, and p is the edge probability), or alternatively that the number of maximal subcubes of order less than $\lambda_1 = \log\log_{1/p} n - \log\log\log_{1/p} n$ or greater than $\lambda_2 = \log\log_{1/p} n + 2$ is negligible comparing with the overall number of maximal subcubes. Based on values of these parameters we showed the lower bound for the vertex cover of a random graph by cubes is $2^n(1 - o(1))/(4\log_{1/p} n)$.

This paper describes the construction of the vertex cover of random subgraph of

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Q_n. We introduce the notion of subcube with given direction, and using this notion we define standard subcubes. The construction uses the fact that the intersection of standard subcubes contains at most one vertex. Simply said, we minimize the size of vertex cover by using subcubes with minimal overlap. Our analysis, together with results from [9] give a theoretical justification for a heuristic that minimizes the disjunctive normal form of a random Boolean function by selecting conjunctions according to the strategy described above. Similar ideas were used in [8].

2. PRELIMINARIES

Let G be a graph. The vertex set of G will be denoted by $V(G)$, and the edge set by $H(G)$.

Let Q_n be an n-cube graph consisting of 2^n vertices labeled by binary vectors of length n, and $n2^{n-1}$ edges joining vertices differing in exactly one coordinate. We denote by G^n the set of all subgraphs of Q_n with the complete set of vertices. Thus, every $G \in G^n$ has 2^n vertices.

A random graph is a graph obtained from Q_n by independent removal of edges. The probability that the edge is not removed is denoted by p, where p is a constant ($0 < p < 1$). We shall consider a probabilistic space (model) (G^n, P), where $P : G^n \to \langle 0, 1\rangle$ is a probabilistic function defined

$$P(G) = p^{|H(G)|}(1-p)^{|H(Q_n)|-|H(G)|}.$$

The probabilistic function P can be naturally extended to arbitrary subset M of G^n:

$$P(M) = \sum_{G \in M} P(G).$$

We call a subset $M \subseteq G^n$ a property of graphs. We shall say that the random graph has a property M, if $\lim_{n\to\infty} P(M) = 1$.

A graph K is a subgraph of G, denoted by $K \subseteq G$, if $V(K) \subseteq V(G)$ and $H(K) \subseteq H(G)$. For a graph $G \in G^n$, we shall say that K is contained in G, if $K \subseteq G$.

A real-valued random variable X is a measurable real-valued function on a probability space, $X : (G^n, P) \to \mathbb{R}$. All random variables in this paper are non-negative integer random variables. Let X be a random variable. The expectation, and the variance of the random variable X will be denoted by $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, respectively. The variance of a random variable X can be expressed as follows: $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$.

Let X be a non-negative random variable with the expected value $\mathrm{E}(X)$ and let $t > 0$. Then we have (Markov's inequality):

$$\Pr[X \geq t \cdot \mathrm{E}(X)] \leq \frac{1}{t}.$$

Now, let X be a real-valued random variable with the expected value $\mathrm{E}(X)$ and the variance $\mathrm{Var}(X)$, and let $d > 0$. Then we have (Chebyshev's inequality):

$$\Pr[|X - \mathrm{E}(X)| \geq d] \leq \frac{\mathrm{Var}(X)}{d^2}.$$

We say that a_n is asymptotically equal to b_n, notation $a_n \sim b_n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. The symbol $\log x$ denotes the binary logarithm of x. We shall often use the logarithm to the base $\frac{1}{p}$. To simplify the notation, we put $b = \frac{1}{p}$ and write $\log_b x$ instead of $\log_{1/p} x$.

Finally, we denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the floor and the ceiling of x, respectively.

3. VERTEX COVER

We analyze the size of the minimal vertex cover in the probabilistic space (G^n, P). Let $p_v(G)$ be a random variable denoting the size of the minimal vertex cover of $G \in G^n$. The main result of this section is the upper bound of $p_v(G)$. First, let us define the notions of subcube and vertex cover formally.

Definition 3.1. A subcube of order k, or k-subcube (for $0 \leq k \leq n$), is a k-cube subgraph of Q_n.

Every k-subcube K is uniquely determined by some set $l = \{l_1, l_2, \ldots, l_{n-k}\} \subseteq \{1, 2, \ldots, n\}$ of fixed coordinates together with some vertex $\alpha \in Q_n$. The subcube K contains those vertices which coordinates differ from α at indices contained in l. We say that the subcube K has direction l.

Trivially, for every l there exist exactly 2^{n-k} subcubes of order k with direction l, and the set of all k-subcubes can be partitioned into $\binom{n}{k}$ subsets consisting from subcubes of given direction. Trivially, there exist exactly $\binom{n}{k}2^{n-k}$ subcubes of order k in Q_n.

Definition 3.2. The set of graphs $\mathcal{G} = \{G_1, \ldots, G_m\}$ is a vertex cover of a graph $G \in G^n$, if $\bigcup_{i=1}^{m} V(G_i) = V(G)$ and each G_i is a subcube contained in G. The number m is called the size of the vertex cover. The smallest possible m is the size of minimal vertex cover.

In order to estimate the size of the vertex cover later, it is necessary to estimate the number of subcubes of a given direction contained in $G \in G^n$. Let $i_{n,l}(G)$ be a random variable denoting the number of subcubes with direction l contained in $G \in G^n$.

We start by computing the value $\mathrm{E}(i_{n,l}(G))$ in Lemma 3.3, and estimating the value $\mathrm{Var}(i_{n,l}(G))$ in Lemma 3.4. These results allow us to estimate $i_{n,l}$ in Lemma 3.5.

Lemma 3.3. $\mathrm{E}(i_{n,l}(G)) = 2^{n-k} p^{k2^{k-1}}$.

Proof. We define random variable η_K for each k-subcube K of Q_n with direction l (for $0 \leq k \leq n$) in the following way:

$$\eta_K(G) = \begin{cases} 1, & \text{if } K \subseteq G; \\ 0, & \text{otherwise.} \end{cases}$$

Then $i_{n,l}(G) = \sum_K \eta_K(G)$, where the sum is evaluated for all k-subcubes of Q_n

with direction l – the number of these subcubes is 2^{n-k}. Thus, we get:

$$\begin{aligned} \mathrm{E}(i_{n,l}(G)) &= \mathrm{E}(\sum_K \eta_K(G)) = \sum_K \mathrm{E}(\eta_K) \\ &= 2^{n-k}\,\Pr[K \subseteq G] = p^{k2^{k-1}} \\ &= 2^{n-k} p^{k2^{k-1}}. \end{aligned}$$

□

LEMMA 3.4. $\mathrm{Var}(i_{n,l}(G)) \le 2^{n-k} p^{k2^{k-1}}$.

PROOF. The first step is to compute $\mathrm{E}(i^2_{n,l}(G))$. Let η_K be the random variable defined in the proof of Lemma 3.3. Then $\mathrm{E}(i^2_{n,l}(G) = \sum_{K,L} \mathrm{E}(\eta_K \cdot \eta_L)$, where the sum is evaluated for all ordered pairs (K, L) – k-subcubes of Q_n with direction l. Thus, we get:

$$\mathrm{E}(i^2_{n,l}(G)) = \sum_{K,L} \Pr[K \subseteq G \ \wedge \ L \subseteq G] = \sum_{K,L} \Pr[K \cup L \subseteq G] = p^{|H(K \cup L)|}.$$

Since two subcubes of equal order are either equal or have empty intersection, we distinguish two cases:

(1) if $K = L$, then there is exactly 2^{n-k} such ordered pairs, and $|H(K \cup L)| = |H(K)| = k \cdot 2^{k-1}$;
(2) if $K \neq L$ (i.e. $K \cap L = \emptyset$), then there is exactly $2^{n-k}(2^{n-k} - 1)$ such ordered pairs, and $|H(K \cup L)| = |H(K)| + |H(L)| = 2k \cdot 2^{k-1}$.

Putting this together we have:

$$\mathrm{E}(i^2_{n,l}(G)) = 2^{n-k} \cdot p^{k2^{k-1}} + 2^{n-k}(2^{n-k} - 1) \cdot p^{k2^k}.$$

The estimation of $\mathrm{Var}(i_{n,l}(G))$ is then straightforward:

$$\begin{aligned} \mathrm{Var}(i_{n,l}(G)) &= \mathrm{E}(i^2_{n,l}(G)) - \mathrm{E}^2(i_{n,l}(G)) \\ &= 2^{n-k} \cdot p^{k2^{k-1}} + 2^{n-k}(2^{n-k} - 1) \cdot p^{k2^k} - \left(2^{n-k} p^{k2^{k-1}}\right)^2 \\ &= 2^{n-k} \cdot p^{k2^{k-1}} \cdot (1 - p^{k2^{k-1}}) \\ &\le 2^{n-k} \cdot p^{k2^{k-1}} \end{aligned}$$

□

LEMMA 3.5 NUMBER OF SUBCUBES WITH GIVEN DIRECTION. *Let $\psi(n)$ be an arbitrary increasing function. Then, with probability converging to* 1 *as $n \to \infty$, the following inequality holds for any $G \in G^n$:*

$$2^{n-k} p^{k2^{k-1}} - \varepsilon < i_{n,l}(G) < 2^{n-k} p^{k2^{k-1}} + \varepsilon,$$

where $\varepsilon = \psi(n)\sqrt{2^{n-k} p^{k2^{k-1}}}$.

PROOF. We substitute the results of Lemma 3.3 and Lemma 3.4 into Chebyshev inequality for random variable $i_{n,l}$. Moreover, we set $\varepsilon = \psi(n)\sqrt{2^{n-k} p^{k2^{k-1}}}$. Since

$\lim_{n\to\infty} 1/\psi(n) = 0$ we get

$$\lim_{n\to\infty} \Pr[|i_{n,l}(G) - \mathrm{E}(i_{n,l}(G))| \geq \varepsilon] \leq \lim_{n\to\infty} \frac{\mathrm{Var}(i_{n,l})}{\varepsilon^2} = 0.$$

Hence

$$\lim_{n\to\infty} \Pr[|i_{n,l}(G) - \mathrm{E}(i_{n,l}(G))| < \varepsilon] = 1.$$

□

We will construct our vertex cover by employing only so-called "standard" subcubes (with carefully chosen directions).

Definition 3.6. The following sets are the standard directions for k-subcubes in Q_n:

$$\begin{aligned} l_1 &= \{k+1, k+2, \ldots, n\} \\ l_2 &= \{1, 2, \ldots, k, 2k+1, 2k+2, \ldots, n\} \\ l_3 &= \{1, 2, \ldots, 2k, 3k+1, 3k+2, \ldots, n\} \\ &\ldots \\ l_m &= \{1, 2, \ldots, (m-1)k, mk+1, mk+2, \ldots, n\}, \end{aligned}$$

where $mk \leq n$. We will refer to a k-subcube with direction l_i $(1 \leq i \leq n)$ as a standard subcube.

It can be easily seen that the intersection of any two standard k-subcubes is either an empty set or a single vertex. To construct as small vertex cover as possible our idea is to choose subcubes with minimal overlaps. Hence, we use the standard k-subcubes (the exact value of k will be determined later). Certainly, once k is fixed there can be vertices in $G \in G^n$ that cannot be covered by k-subcubes. We cover these vertices by 0-subcubes. In order to assess the quality of such a vertex cover, i.e. estimate its size, we have to estimate the number of G's vertices covered by standard k-subcubes, as well as the number of vertices not covered by these subcubes. Let $z_n(G)$ be a random variable denoting the number of vertices not covered by any standard k-subcube contained in $G \in G^n$.

LEMMA 3.7. $\mathrm{E}(z_n(G)) = 2^n(1 - p^{k2^{k-1}})^m$.

PROOF. Let $\eta_\alpha(G)$ be a random variable (an indicator) denoting whether a vertex $\alpha \in V(G)$ is covered by some standard subcube contained in G:

$$\eta_\alpha(G) = \begin{cases} 1, & \text{if } \alpha \text{ is not covered by any standard subcube contained in } G \\ 0, & \text{otherwise.} \end{cases}$$

Since $z_n(G) = \sum_{\alpha\in V(G)} \eta_\alpha(G)$, we get:

$$\mathrm{E}(z_n(G)) = \mathrm{E}\Bigg(\sum_{\alpha\in V(G)} \eta_\alpha(G)\Bigg) = \sum_{\alpha\in V(G)} \mathrm{E}(\eta_\alpha(G)) = \sum_{\alpha\in V(G)} \Pr[\eta_\alpha(G) = 1].$$

We compute the probability $\Pr[\eta_\alpha(G) = 1]$. Let us denote by $K_{l_i(\alpha)}$, for $1 \leq i \leq m$, a standard k-subcube containing vertex α. For a given vertex α there exist m

distinct standard subcubes $K_{l_i(\alpha)}$ that are contained in G. The intersection of every pair of distinct subcubes $K_{l_i(\alpha)}$ contains exactly the vertex α. Hence, the vertex α is not covered by any standard k-subcube contained in G if and only if none of the standard subcubes $K_{l_i(\alpha)}$ is contained in G (and these events are independent). Therefore:

$$\begin{aligned}\Pr[\eta_\alpha(G)=1] &= \Pr[\forall i \in \{1,\dots,m\}:\ K_{l_i(\alpha)} \nsubseteq G] \\ &= \prod_{i=1}^{m} \Pr[K_{l_i(\alpha)} \nsubseteq G] \\ &= \prod_{i=1}^{m} (1-\Pr[K_{l_i(\alpha)} \subseteq G])\end{aligned}$$

The last probability can be computed easily: $\Pr[K_{l_i(\alpha)} \subseteq G] = p^{|H(K_{l_I(\alpha)})|} = p^{k2^{k-1}}$, because the subcube has order k. Substituting back we get:

$$\mathrm{E}(z_n) = \sum_{\alpha \in V(G)} \prod_{i=1}^{m} (1-p^{k2^{k-1}}) = 2^n(1-p^{k2^{k-1}})^m.$$

□

Recall, the random variable $p_n(G)$ denotes the size of a minimal vertex cover of $G \in G^n$. The following theorem provides the main result of this section, the upper bound of $p_v(G)$.

THEOREM 3.8. *With probability converging to* 1 *as* $n \to \infty$*, the following inequality holds for any* $G \in G^n$*:*

$$p_v(G) \leq \frac{2^n(1+o(1))(\log\log_b n)^2}{\log_b n}.$$

PROOF. The upper bound of $p_v(G)$, the size of minimal vertex cover of $G \in G^n$, is obtained by estimating the size of the following vertex cover:

(1) The cover contains all standard k-subcubes contained in G.
(2) The remaining (not covered) vertices are covered by 0-subcubes, i.e. by isolated vertices.

Thus, we take $\sum_{j=1}^{m} i_{n,l_j}(G)$ standard subcubes (with directions l_j) in the first step, and $z_n(G)$ 0-subcubes in the second step. We use the estimations from Lemmas 3.5 and 3.7. Let $G \in G^n$ be a graph such that:

$$\begin{aligned} i_{n,l_j} &< 2^{n-k}\cdot p^{k2^{k-1}} + \psi(n)\sqrt{2^{n-k}\cdot p^{k2^{k-1}}}, \quad \text{for } j=1,\dots,m; \\ z_n &< \varphi(n)\,\mathrm{E}(z_n(G)), \quad \text{where } \varphi(n) \text{ will be determined later.}\end{aligned}$$

With the aid of Markov's inequality for $z_n(G)$ (when $t = \varphi(n)$) we can estimate the size of vertex cover:

$$\begin{aligned} &\sum_{j=1}^{m} i_{n,l_j}(G) + z_n(G) \\ &\qquad < m \cdot \left(2^{n-k} \cdot p^{k2^{k-1}} + \psi(n)\sqrt{2^{n-k} \cdot p^{k2^{k-1}}}\right) + \varphi(n) \cdot 2^n (1 - p^{k2^{k-1}})^m \end{aligned} \tag{1}$$

More precisely, the probability that (1) does not hold for the graph G is at most $\frac{m}{\psi^2(n)} + \frac{1}{\varphi(n)}$. Hence, for a random graph the inequality holds if $\lim_{n\to\infty} \frac{m}{\psi^2(n)} + \frac{1}{\varphi(n)} = 0$. The estimate for $p_v(G)$ will be obtained by suitable choice of parameters $k, m, \varphi(n), \psi(n)$, while satisfying conditions $mk \leq n$ (according to the definition of standard subcubes), and $\lim_{n\to\infty} \frac{m}{\psi^2(n)} + \frac{1}{\varphi(n)} = 0$ (according to the previous discussion). We are "guided" by the lower bound of $p_v(G)$ from [9], and particularly by the fact that the largest number of maximal subcubes contained in a random graph has order approximately $\lambda \sim \log\log_b n$. Hence we use λ-subcubes in the first step of our vertex cover construction. Let

$$\begin{aligned} k &= \lambda; \\ m &= \left\lfloor \lambda \cdot \left(\frac{1}{p}\right)^{\lambda 2^{\lambda-1}} \right\rfloor; \\ \psi(n) &= n. \end{aligned}$$

Clearly, $\lim_{n\to\infty} \frac{m}{\psi^2(n)} = \lim_{n\to\infty} \frac{m}{n^2} = 0$, and our second condition is satisfied if we choose $\varphi(n)$ such that $\lim_{n\to\infty} \frac{1}{\varphi(n)} = 0$. The first condition ($mk \leq n$) can be shown by employing the following inequalities from [9]:

$$\left(\frac{1}{p}\right)^{(\lambda+1)2^{\lambda-1}} < n \leq \left(\frac{1}{p}\right)^{(\lambda+2)2^{\lambda}}.$$

We get

$$mk \leq \lambda^2 \cdot \left(\frac{1}{p}\right)^{\lambda 2^{\lambda-1}} \leq \lambda^2 \cdot n^{\frac{\lambda}{\lambda+1}} \leq n.$$

The parameter m was chosen such that we can use the following bound:

$$(1 - p^{\lambda 2^{\lambda-1}})^m \leq e^{-mp^{\lambda 2^{\lambda-1}}} \leq e^{\frac{-m\lambda}{m+1}} = 2^{\frac{-m\lambda \log e}{m+1}}.$$

For sufficiently large n we use $\frac{m \log e}{m+1} \geq 1$, and obtain final bound:

$$(1 - p^{\lambda 2^{\lambda-1}})^m \leq 2^{-\lambda}.$$

Similarly, for sufficiently large n we get:

$$\psi(n)\sqrt{2^{n-k} \cdot p^{k2^{k-1}}} = o(2^{n-k} \cdot p^{k2^{k-1}}).$$

Let us put all these bounds and estimates together. If $\varphi(n)$ is chosen in such a way that $\lim_{n\to\infty} \frac{1}{\varphi(n)} = 0$, then for a random graph G:

$$\begin{aligned}
p_v(G) &< m \cdot \left(2^{n-k} \cdot p^{k2^{k-1}} + \psi(n)\sqrt{2^{n-k} \cdot p^{k2^{k-1}}}\right) + \varphi(n) \cdot 2^n (1 - p^{k2^{k-1}})^m \\
&\leq m \cdot \left(2^{n-k} \cdot p^{k2^{k-1}} (1 + o(1))\right) + \varphi(n) \cdot 2^n (1 - p^{k2^{k-1}})^m \\
&\leq \left(\frac{1}{p}\right)^{\lambda 2^{\lambda-1}} \cdot \lambda \cdot \left(2^{n-k} \cdot p^{k2^{k-1}} (1 + o(1))\right) + +\varphi(n) \cdot 2^n (1 - p^{k2^{k-1}})^m \\
&\leq 2^{n-\lambda} \lambda (1 + o(1)) + \varphi(n) \cdot 2^n \cdot 2^\lambda \\
&\leq 2^n (1 + o(1)) \cdot \left(\frac{\lambda + \varphi(n)}{2^\lambda}\right)
\end{aligned}$$

Using the following estimate from [9]:

$$\log\log_b n - \log\log\log_b n < \lambda \leq \lceil \log\log_b n - \log\log\log_b n + 1 \rceil,$$

and putting $\varphi(n) = \log\log_b n(\log\log\log_b n - 2)$ (trivially $\lim_{n\to\infty} \frac{1}{\varphi(n)} = 0$) we are able to finish the proof:

$$\begin{aligned}
p_v(G) &\leq 2^n (1 + o(1)) \cdot \left(\frac{\lambda + \varphi(n)}{2^\lambda}\right) \\
&\leq 2^n (1 + o(1)) \cdot \left(\frac{\log\log_b n(\log\log_b n - \log\log\log_b n + 2) + \varphi(n)}{\log_b n}\right) \\
&\leq \frac{2^n (1 + o(1))(\log\log_b n)^2}{\log_b n}.
\end{aligned}$$

□

Similar idea, used here for obtaining lower and upper bound of the size of minimal vertex cover, can be applied for estimating the size of a minimal edge cover.

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