On the Solid-Packing Constant for Circles

By Z. A. Melzak

Abstract. A solid packing of a circular disk U is a sequence of disjoint open circular subdisks $U_1, U_2, \cdots$ whose total area equals that of U. The Mergelyan-Wesler theorem asserts that the sum of radii diverges; here numerical evidence is presented that the sum of ath powers of the radii diverges for every $a < 1.306951$. This is based on inscribing a particular sequence of 19660 disks, fitting a power law for the radii, and relating the exponent of the power law to the above constant. ■

1. We shall be concerned here with solid packings of a closed circular disk U. Such a packing P consists of a sequence of open pairwise disjoint circular disks $U_1, U_2, \cdots$ which are subsets of U; P is called solid if the areas of U and $\bigcup_{n=1}^{\infty} U_n$ are the same. Let r be the radius of U and r_n that of U_n so that the condition for a solid packing is $r^2 = \sum_{n=1}^{\infty} r_n^2$; the Mergelyan-Wesler theorem [1], [2], asserts then that $\sum_{n=1}^{\infty} r_n$ diverges. Sums of the form $\sum_{n=1}^{\infty} r_n^a$ have been considered in [3]: for every solid packing P there is a number $e(P)$, called its exponent, given by

$$e(P) = \sup \left\{ x : \sum_{n=1}^{\infty} r_n^x \text{ diverges} \right\}$$

or by

$$e(P) = \inf \left\{ y : \sum_{n=1}^{\infty} r_n^y \text{ converges} \right\},$$

and one wishes now to study the set

$$B = \{u : u = e(P), P \text{ is a solid packing}\}.$$

Call a solid packing osculatory if its disks are determined in the following manner: U_1 is an arbitrary subdisk of U, and from then on each successive U_n is the biggest disk fitting into the as yet uncovered part of U. It is shown in [3] that all osculatory packings have the same exponent S which satisfies

$$1.035 < S < 1.999971, \tag{1}$$

and that there exist solid packings P such that $e(P) = 2$. Since an osculatory packing seems to minimize the exponent, the above suggests the conjecture

$$B = [S, 2]. \tag{2}$$

Our main interest here will be in a numerical determination of S, and we present numerical evidence which suggests that, approximately,

$$S = 1.306951.$$

2. In this section we deal with the configuration of three unit-radius circles A, B, C, pairwise externally tangent, and bounding a curvilinear triangle Z. Let the

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largest, 0th generation, circle C_1 be inscribed into Z, then inscribe the three first generation circles into the three curvilinear triangles of $Z - C_1$, then the nine second generation circles, and so on. Let the radii of all the circles from C_1 on, be $r_1, r_2, \cdots$, then the solid packing constant S of (1) is given by

$$S = \sup\left\{u\colon \sum_{n=1}^{\infty} r_n^u \text{ diverges}\right\}.$$

To be able to apply numerical methods we follow [4] and introduce the functions $f(y)$ and $m(y)$: $f(y)$ is the fraction of the area of Z covered by the circles of radius $\geqq y$, and $m(y)$ is their number. Then

$$f(y) = K_1 \int_y^1 x^2 dm(x)\,, \tag{3}$$

$$m(y) = K_2 \int_y^1 x^{-2} df(x)\,, \tag{4}$$

where K_1 and K_2 are constants, and the ath moment is

$$\sum_{n=1}^{\infty} r_n^a = K_3 \lim_{y\to+0} \int_y^1 x^a dm(x)\,. \tag{5}$$

The functions $f(x)$ and $m(x)$ can be numerically determined for suitable values of x, and it is found that $f(x)$ and $m(x)$ can be well fitted near $x = 0$ by power-laws:

$$f(x) = 1 - A_1 x^{a_1}\,, \qquad m(x) = A_2 x^{-a_2}\,. \tag{6}$$

By (3), (4), (5) this leads at once to $S = 2 - a_1 = a_2$. We shall work with the function $m(x)$ since it is somewhat easier to count the number of circles of an osculatory packing, whose radii exceed a given bound, than to add the squares of their radii. Computation of the radii proceeds readily on the basis of Soddy's formula [5], which states that if a, b, c are the curvatures of three pairwise externally tangent circles, then the curvature of the smaller of the two circles tangent to the three is $a + b + c + 2(ab + ac + bc)^{1/2}$. In the order of descending magnitude the first 19660 circles inscribed into Z were examined, the results are displayed in Table 1. There the index n ranges from 1 to 20, Num(n) is the number of circles whose radii are $\geqq (1000n)^{-1}$, and Fit(n) is the function obtained by fitting (by least squares) the best power-law of the form $N(n) = An^b$ to the data Num(n). It turns out that

$$N(n) = 3926.48 \cdot n^{1.306951}\,.$$

3. The solid packings P, exponents $e(P)$, constant S, and the conjecture (2) are all in reference to a packing of a circular disk by similar disks. However, a considerable generalization is possible. Let U be an arbitrary plane convex body and let K be the interior of another convex body. It is assumed that the boundary of K does not contain a pair of parallel straight segments. Consider the family F of all homothetic images of the closure of K, then F covers U in the sense of Vitali and by Vitali's covering theorem [6], there exists a countable subset of F, consisting of subsets $U_1, U_2, U_3, \cdots$ of U, whose interiors are pairwise disjoint and satisfy the solid packing condition:

$$\textbf{Area}\ (U) = \sum_{n=1}^{\infty} \text{Area}\ (U_n)\,.$$

We call the sequence $U_1, U_2, \cdots$ a solid K-packing P of U. Next, one defines the exponent $e(K, P)$ by

$$e(K, P) = \sup \left\{ x \colon \sum_{n=1}^{\infty} [\text{diam } (U_n)]^x \text{ diverges} \right\}$$

and the set $B(K)$ by

$$B(K) = \{u : u = e(K, P), P \text{ is a solid } K\text{-packing}\} .$$

Both are easily shown to be independent of U. The equivalent of the Mergelyan-Wesler theorem holds here and we have $e(K, P) \geqq 1$. By the methods of [3] one can show that there exist solid packings P such that $e(K, P) = 2$. Further, solid K-packings analogous to the osculatory ones can be defined, and shown to have the same exponent $S(K)$. This leads to the generalization of the conjecture (2):

$$B(K) = [S(K), 2] . \tag{7}$$

The packing constants $S(K)$ do not appear to be easy to calculate except when K is a triangle; we have then

$$S(K) = \log 3/\log 2 = 1.585 \cdots .$$

It might perhaps be conjectured that as K varies over all admissible convex bodies, $S(K)$ attains its minimum for a circle, and its maximum for a triangle. The conjecture appears to be contradicted if we take for K a "round" square, for instance, the region bounded by the locus of the equation $x^{10} + y^{10} = 1$, for, one might argue, such figures "almost" fit together to form an economical packing. However, the near fit breaks down as soon as one has to fit the interstices with smaller homothetic images of K.

TABLE 1

n	Num (n)	Fit (n)
1	388	393
2	973	972
3	1672	1650
4	2428	2404
5	3220	3218
6	4066	4083
7	4951	4995
8	5947	5947
9	6955	6937
10	7984	7961
11	9058	9017
12	10075	10103
13	11230	11217
14	12373	12358
15	13450	13504
16	14677	14714
17	15970	15927
18	17095	17162
19	18433	18419
20	19660	19696

The packing constants $S(K)$ appear to measure, in a certain sense, the efficiency of a convex set K to form a plane packing, and they certainly merit further numerical and theoretical study.

4. The computational work was done on the Bell Telephone Laboratories' machine GE-645, the programming presented no particular difficulties, and the total central processor running time was 90 seconds. The author wishes to thank D. Bzowy and M. D. McIlroy for suggestions and help in preparing the problem for the machine.

University of British Columbia
Vancouver 8, B. C., Canada

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