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On the solution of a vectorial radiative transfer equation in an arbitrary three-dimensional turbid medium with anisotropic scattering

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Abstract

The authors developed a numerical method of the boundary-value problem solution in the vectorial radiative transfer theory applicable to the turbid media with an arbitrary three-dimensional geometry. The method is based on the solution representation as the sum of an anisotropic part that contains all the singularities of the exact solution and a smooth regular part. The regular part of the solution could be found numerically by the finite element method that enables to extend the approach to the arbitrary medium geometry. The anisotropic part of the solution is determined analytically by the special form of the small-angle approximation. The method development is performed by the examples of the boundary-value problems for the plane unidirectional and point isotropic sources in a turbid medium slab.

Keywords: 3D vectorial radiative transfer; Small-angle approximation; Finite element method

1. Singularities of the solution of the radiative transfer equation (RTE)

Nowadays it is possible to consider the theory of the solution methods of the boundary-value problems of the RTE for media with a plane-parallel geometry fully completed [1]. At the same time it is necessary to note that there are no universal and effective solution methods of the three-dimensional (3D) problems in the transport theory [1]. Above all, it is connected with the properties of the RTE solution, which are determined by a physical model of the transport theory, that is, ray approximation. In particular, the radiance angular distribution has the singularities that essentially impede the RTE solution by any numerical method because of the physically selected direction of the radiation propagation in space.

These singularities are of a key character and require the development of special methods of RTE solution. Particularly, to eliminate these singularities, S. Chandrasekhar [2] suggested to subtract a direct non-scattered part of radiation from the solution and to formulate the equation for the rest smooth part that can be found numerically. However, in the case of a strong scattering anisotropy in all the natural media having suspended

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particles with diameters much larger than a light wave length, the radiation, scattered in small angles, is indistinguishable from the direct radiation. So, for this case, the Chandrasekhar method becomes ineffective. Furthermore, in the presence of a spatially confined source, which is equivalent to the sharp changes of the medium parameters in an arbitrary three-dimensional (3D) geometry, there are singularities in the radiance angular distribution not only in the direct radiation but also in the first two orders of scattering [3]. The presence of such singularities in the solution leads to the essential decrease of efficiency of any numerical methods of RTE solution. Nowadays there are two basic solution methods of RTE in the 3D medium geometry: Monte Carlo and SHDOM [4]; each of them is based on smoothing singularities in the radiance angular distribution that can result in a significant error in a number of cases.

2. Elimination of the anisotropic part of RTE solution

Let us consider the eventual development of the Chandrasekhar method by the example of the a well-investigated boundary-value problem of RTE for a turbid medium slab irradiated from the above by a plain unidirectional (PU) source:

$$\begin{cases} \mu \frac{\mathrm{d}L(\tau,\hat{\mathbf{l}})}{\mathrm{d}\tau} + L(\tau,\hat{\mathbf{l}}) = \frac{\Lambda}{4\pi} \oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}')\, L(\tau,\hat{\mathbf{l}}')\, \mathrm{d}\hat{\mathbf{l}}', \\ L(\tau,\hat{\mathbf{l}})\big|_{\tau=0,\mu 0} = \delta(\hat{\mathbf{l}}-\hat{\mathbf{l}}_0),\ L(\tau,\hat{\mathbf{l}})\big|_{\tau=\tau_0,\mu<0} = 0, \end{cases} \tag{1}$$

where $L(\tau,\hat{\mathbf{l}}) \equiv L(\tau,\mu,\varphi)$ is the radiance of light field at the optical depth τ in the sighting direction $\hat{\mathbf{l}} = \left\{\sqrt{1-\mu^2}\cos\varphi, \sqrt{1-\mu^2}\sin\varphi, \mu\right\}$, $\mu = (\hat{\mathbf{z}},\hat{\mathbf{l}})$; $\hat{\mathbf{l}}_0$ is the incident direction of the PU source, $\mu_0 = (\hat{\mathbf{z}},\hat{\mathbf{l}}_0)$; Λ is a single scattering albedo; $x(\hat{\mathbf{l}},\hat{\mathbf{l}}')$ is a scattering phase function; τ_0 is the slab optical depth. The axis OZ is directed downwards perpendicularly to the slab borders. Hereinafter, the single vectors are denoted by the symbol "^".

We present the solution of a boundary-value problem as the sum [5]

$$L(\tau,\hat{\mathbf{l}}) = L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}}) + \tilde{L}(\tau,\hat{\mathbf{l}}), \tag{2}$$

where $L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}})$ is the approximate RTE solution in the small-angle approximation (SA) [6]

$$L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} \exp\left[-\frac{(1-x_k)\tau}{\mu_0}\right] P_k(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}_0) \equiv \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} Z_k(\tau)\, P_k(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}_0), \tag{3}$$

where x_k are decomposition coefficients of a scattering phase function on spherical harmonics (SH):

$$x(\hat{\mathbf{l}},\hat{\mathbf{l}}') = \sum_{k=0}^{\infty} (2k+1) x_k\, P_k(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}'). \tag{4}$$

Since $L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}})$ contains [7] not only all the singularities but also an angle anisotropic part of the exact solution, the rest $\tilde{L}(\tau,\hat{\mathbf{l}})$ is a smooth function. To show that it does not present any difficulties, we use an arbitrary numerical method.

Taking into account the representation (2), the boundary-value problem (1) takes the form

$$\begin{cases} \mu \frac{\mathrm{d}\tilde{L}(\tau,\hat{\mathbf{l}})}{\mathrm{d}\tau} + \tilde{L}(\tau,\hat{\mathbf{l}}) = \frac{\Lambda}{4\pi} \oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}')\, \tilde{L}(\tau,\hat{\mathbf{l}}')\, \mathrm{d}\hat{\mathbf{l}}' + F(\tau,\hat{\mathbf{l}}), \\ \tilde{L}(\tau,\hat{\mathbf{l}})\big|_{\tau=0,\mu 0} = 0,\ \tilde{L}(\tau,\hat{\mathbf{l}})\big|_{\tau=\tau_0,\mu 0} = -\tilde{L}_{\mathrm{SA}}(\tau_0,\hat{\mathbf{l}}), \end{cases} \tag{5}$$

where the source function (the discrepancy of RTE solution in SA) looks like

$$F(\tau,\hat{\mathbf{l}}) = -\mu \frac{\mathrm{d}L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}})}{\mathrm{d}\tau} - L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}}) + \frac{\Lambda}{4\pi} \oint x(\hat{\mathbf{l}},\hat{\mathbf{l}}')\, L_{\mathrm{SA}}(\tau,\hat{\mathbf{l}}')\, \mathrm{d}\hat{\mathbf{l}}'. \tag{6}$$

The solution complexity of the boundary-value problem (5) is rooted in the complexity of the expression (6). However, using SA [6] essentially simplifies (6) on the basis of an addition theorem for the surface harmonic taking into account (3):

$$F(\tau,\hat{\mathbf{l}}) = \sum_{m=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\,F_k^m(\tau)\,Q_k^m(\mu)\mathrm{e}^{\mathrm{i}m\varphi}, \tag{7}$$

where $Q_k^m(\mu)=\sqrt{((k-m)!)/(k+m)!}\,P_k^m(\mu)$, $P_k^m(\mu)$, $P_k(\mu)$ are the semi-normalized, associated and ordinary Legendre polynomials, respectively; $b_k = 1-\Lambda x_k$,

$$F_k^m(\tau)=\frac{1}{2k+1}\frac{1}{\mu_0}\left[\sqrt{(k+1)^2-m^2}b_{k+1}\,Q_{k+1}^m(\mu_0)\,Z_{k+1}+\sqrt{k^2-m^2}b_{k-1}\,Q_{k-1}^m(\mu_0)\,Z_{k-1}(\tau)\right]-b_k\,Q_k^m(\mu_0)\,Z_k(\tau).$$

3. Determination of the smooth solution part by the SH method

In view of the SA analytical form, the SH method is offered in [5] to be used as the numerical method for the solution (5). In the SH method, the angular dependences of all the functions in RTE are represented as the decomposition on the Legendre polynomials

$$\tilde{L}(\tau,\mu,\varphi)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\frac{2k+1}{4\pi}\,C_k^m(\tau)\,Q_k^m(\mu)\,\mathrm{e}^{-\mathrm{i}m\varphi}, \tag{8}$$

which gives the infinite set of the connected ordinary differential equations. Since $\tilde{L}(z,\hat{\mathbf{l}})$ is a smooth function, the number N_m of the series terms (8) is finite, which allows reducing this connected equation set to the matrix form [5,8]

$$\overset{\leftrightarrow m}{A}\frac{\mathrm{d}}{\mathrm{d}\tau}\vec{C}^m(\tau)+\overset{\leftrightarrow}{D}\,\vec{C}^m(\tau)=\left(\overset{\leftrightarrow m}{A}\Big/\mu_0-\overset{\leftrightarrow}{1}\right)\overset{\leftrightarrow}{D}\,\overset{\leftrightarrow m}{Q}\,\vec{Z}(\tau)+a_{N+1}^m\,\vec{Z}_{N+1}(\tau), \tag{9}$$

where

$$\left[\overset{\leftrightarrow m}{A}\right]_{i,i+1}=\frac{\sqrt{i^2-m^2}}{2i-1},\ \vec{C}=\left[C_{i-1}^m(\tau)\right],\ \left[\overset{\leftrightarrow m}{A}\right]_{i,i-1}=\frac{\sqrt{(i-1)^2-m^2}}{2i-1},\ \vec{Z}_{N+1}=\left[\underbrace{0\ldots0}_{N},Z_{N+1}\right],$$

$$\overset{\leftrightarrow}{D}=\mathrm{diag}((1-x_{i-1})/\mu_0),\ \overset{\leftrightarrow m}{Q}=\mathrm{diag}(Q_{i-1}^m(\mu_0)),\ a_{N+1}^m=\frac{\sqrt{(N+1)^2-m^2}}{2N+1}\frac{b_{N+1}}{\mu_0}\,Q_{N+1}^m(\mu_0),\ \vec{Z}=[Z_{i-1}].$$

Hereinafter, the matrices are denoted by the double arrow above the symbol. A column vector is denoted with the unary right arrow, and a row vector with the unary left arrow. To simplify the notation, we will omit an azimuth coefficient m, where it is obvious.

The matrix analytical solution of the set (9) that is equivalent to the set of the (N_m-m) linear algebraic equations with the $2(N_m-m)$ unknowns was suggested in [5]:

$$-\vec{C}(0)+\mathrm{e}^{-\overset{\leftrightarrow}{B}\tau_0}\,\vec{C}(\tau_0)=\frac{1}{\mu_0}\int_0^{\tau_0}\mathrm{e}^{-\overset{\leftrightarrow}{B}t}\left(\overset{\leftrightarrow}{1}-\mu_0\overset{\leftrightarrow}{A}^{-1}\right)\overset{\leftrightarrow}{D}\,\overset{\leftrightarrow}{Q}\,\vec{Z}(t)\,\mathrm{d}t+a_{N+1}^m\int_0^{\tau_0}\mathrm{e}^{-\overset{\leftrightarrow}{B}t}\overset{\leftrightarrow}{A}^{-1}\,\vec{Z}_{N+1}(t)\,\mathrm{d}t, \tag{10}$$

where $\overset{\leftrightarrow}{B}=\overset{\leftrightarrow}{A}^{-1}\overset{\leftrightarrow}{D}$.

All the integrals in Eq. (10) can be formulated analytically [5]. The missing (N_m-m) equations are given by the boundary conditions, which in [5] are selected in Marshak's form:

$$[\overset{\leftrightarrow}{1}\quad\overset{\leftrightarrow}{G}]\,\overset{\leftrightarrow}{P}\,\vec{C}(0)=\vec{0},\quad[\overset{\leftrightarrow}{1}\quad-\overset{\leftrightarrow}{G}]\,\overset{\leftrightarrow}{P}\,\vec{C}(\tau_0)=-\vec{Y}(\tau_0), \tag{11}$$

where $[\overset{\leftrightarrow}{G}]_{jl}=(4l-3)\int_0^1 Q_{2j-1}^m(\mu)\,Q_{2l-2}^m(\mu)\,\mathrm{d}\mu$; $\overset{\leftrightarrow}{P}$ is the matrix that sorts out a vector into its even and odd parts; $\vec{Y}=\left[Z_{2j-1}(\tau)\,Q_{2j-1}^m(\mu_0)-\sum_{l=0}^{\infty}G_{jl}^m\,Z_{2l}(\tau)\,Q_{2l}^m(\mu_0)\right]$. Square brackets in Eq. (11) represent, similar to Matlab, a concatenation of two quadric $N/2\times N/2$ matrices into one rectangular $N/2\times N$ matrix.

Expressions (9) and (11) are in fact a closed set of the linear algebraic equations. Solving it makes possible to determine the desired coefficients. The set matrix condition is quickly worsened with the increase in the slab optical depth. To eliminate this effect, it is necessary to use the scale transformation [8].

Fig. 1 shows the comparison of the calculation of the radiance angular distribution of the reflected slab radiation by the offered method with that by the classical SH method [8] using Chandrasekhar's representation. The calculations are carried out for a scattering phase function in the Henyey–Greenstein form with one parameter g. M in Fig. 1 is the number of zenith terms in the series (8).

The presented diagrams visually demonstrate the essential convergence acceleration made by the suggested method that does not require any procedure of the solution oscillations smoothing that is of great importance for the classical approach [8].

The weak spots of the offered method are the boundary conditions in Marshak's form impeding the generalization for the arbitrary non-isotropic reflection law of the slab borders that is of considerable practical interest. The best solution of the indicated problem is the transition to Mark's boundary conditions, under which the equality of the radiance angular distribution to an exterior flux for the fixed directions is required:

$$[\overleftrightarrow{0} \quad \overleftrightarrow{G}_1]\,\vec{C}(0) = \vec{0}, \quad [\overleftrightarrow{0} \quad \overleftrightarrow{G}_2]\,\vec{C}(\tau_0) = -\vec{L}_{SA}(\tau_0), \tag{12}$$

where

$$[\overleftrightarrow{G}_1]_{ik} = \frac{2k+1}{4\pi}\,P_k(\mu_i)\Big|_{\mu_i>0}, \; [\overleftrightarrow{G}_2]_{ik} = \frac{2k+1}{4\pi}\,P_k(\mu_i)\Big|_{\mu_i<0}, \; \vec{L}_{SA}(\tau_0) = [L_{SA}(\tau_0,\mu_i)\big|_{\mu_i<0}],$$

and μ_i are the roots of the polynomial $P_{N+1}(\mu_i) = 0$.

The calculation comparison of the angular distribution of the reflected radiation by both approaches is shown in Fig. 1. The precision of the both approaches is approximately identical; however, in the case of Mark's boundary conditions, it is easy to take into account an arbitrary reflection law from the slab borders.

Further development of the method is possible for the case of 3D medium geometry, but the statement of the SH method has considerable analytical difficulties for the arbitrary geometry. Actually, the development of Mark's boundary conditions is a discrete ordinates method (DOM) where not only the boundary conditions but RTE is also stated for the fixed space directions.

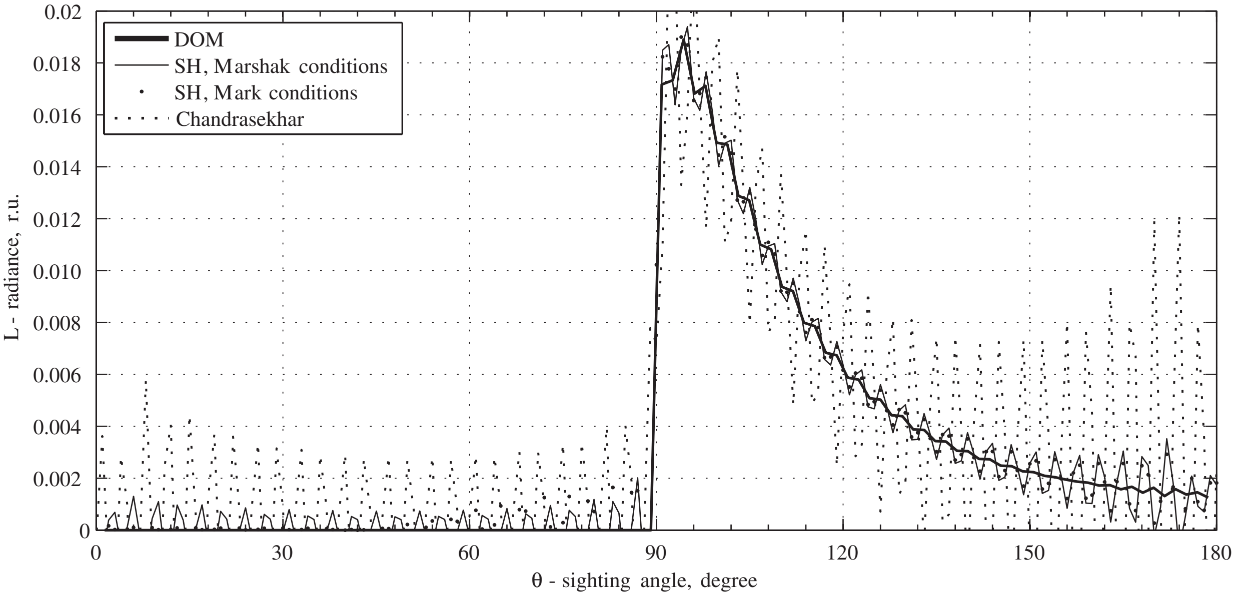

Fig. 1. Determination of the solution smooth part by the different numerical methods: $\tau_0 = 5.0$, $\theta_0 = 40^{\circ}$, $\Lambda = 0.8$, $g = 0.97$, $N = 101$, $M = 8$.

4. Discrete ordinates method

Let us present the scattering phase function by (4), and the desired function by Fourier series on an azimuth

$$\tilde{L}(\tau, \hat{\mathbf{l}}) = \sum_{m=-\infty}^{\infty} C^m(\tau, \mu) e^{im\varphi}, \tag{13}$$

that will reduce after substitution in RTE (5), taking into account (7) and the orthogonality of an azimuth harmonics to the connected set of equations

$$\mu \frac{dC^m(\tau, \mu)}{d\tau} = -C^m(\tau, \mu) + \sum_{k=0}^{\infty} \frac{2k+1}{4\pi} F_k^m(\tau) Q_k^m(\mu) + \frac{\Lambda}{2} \sum_{k=m}^{N} (2k+1) x_k Q_k^m(\mu) \int_{-1}^{1} Q_k^m(\mu') C^m(\tau, \mu') d\mu'. \tag{14}$$

Let us replace the integral in Eq. (14) by Gaussian quadrature

$$\int_{-1}^{1} Q_k^m(\mu') C^m(\tau, \mu') d\mu' \approx \sum_{j=1}^{N} w_j C_j^m(\tau) Q_k^m(\mu_j), \tag{15}$$

where $C_j^m(\tau) \equiv C^m(\tau, \mu_j)$, w_j are the weight coefficients of Gaussian quadrature, μ_j are the roots of the polynomial $P_{N+1}(\mu)$.

In this case, the set (14) can be replaced with the set of the N ordinary differential equations

$$\overleftrightarrow{M} \frac{d}{d\tau} \vec{C}^m(\tau) = -\vec{C}^m(\tau) + \overleftrightarrow{S} \vec{C}^m(\tau) + \vec{F}^m(\tau), \tag{16}$$

where

$$\overleftrightarrow{M} = \mathrm{diag}(\mu_i),\ \vec{F}^m = \left[\sum_{k=m}^{\infty} \frac{2k+1}{4\pi} F_k^m(\tau) Q_k^m(\mu_i)\right],\ \overleftrightarrow{S} = \left[\frac{w_j}{2} \sum_{k=m}^{N} (2k+1) x_k Q_k^m(\mu_i) Q_k^m(\mu_j)\right],\ \vec{C}^m = \left[C_i^m\right].$$

The further solution corresponds completely to the SH method, and the boundary conditions are similar to Mark's boundary conditions. The results of calculations for the medium parameters similar to those that were used earlier are given in Fig. 1. It is evident that DOM has the best convergence. Moreover, $N = 101$ is taken for the identity of the comparison parameters, although it is possible to use $N = 51$ without noticeable deterioration of precision. But the DOM key feature is that all expressions obtain the simple physical sense of the space selected "rays".

5. Usage of the finite element method

The representation (2) really makes the rest $\tilde{L}(\tau, \hat{\mathbf{l}})$ a smooth, slowly varying function in space of arguments. Our analysis has shown that practically for any case the zenith directions amount corresponds to $N<51$, and the azimuth harmonics amount to $M<8$. According to a finite element method [9], it allows to construct in the space the tetrahedrons or hexahedrons mesh to store in every vertex the discrete ordinate of the decomposition coefficients in the series of the azimuth angular distribution. The values between the knots are calculated using one of the approximation schemes [9]. The suggested method of RTE solution allows solving problems in the transport theory in media with an arbitrary 3D geometry. However, the total number of equations in the gained set can be as large as hundreds thousands.

The best technique for the solution of such connected equation set is the method of iterations. This could be done by employing an integral transfer equation obtained from the boundary-value problem by the formal solution in the assumption of a known right-hand member. In this case the integral equation set is

$$\tilde{L}(\tau, \hat{\mathbf{l}}) = \begin{cases} \Delta(\tau, \hat{\mathbf{l}}) + \frac{\Lambda}{4\pi\mu} \int_0^{\tau} e^{-(\tau-t)/\mu} \oint x(\hat{\mathbf{l}}, \hat{\mathbf{l}}') \tilde{L}(t, \hat{\mathbf{l}}') d\hat{\mathbf{l}}' dt,\ \mu \geqslant 0, \\ \Delta(\tau, \hat{\mathbf{l}}) + \frac{\Lambda}{4\pi|\mu|} \int_{\tau}^{\tau_0} e^{-(\tau-t)/\mu} \oint x(\hat{\mathbf{l}}, \hat{\mathbf{l}}') \tilde{L}(t, \hat{\mathbf{l}}') d\hat{\mathbf{l}}' dt,\ \mu < 0, \end{cases} \tag{17}$$

where the source function has the form $\Delta(\tau,\hat{\mathbf{l}}) = \sum_{k=0}^{\infty}((2k+1)/4\pi)\,\Delta_k(\tau,\mu)\,P_k(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}_0)$:

$$\Delta_k(\tau,\mu) = \begin{cases} \frac{\mu_0 \Lambda x_k}{\mu_0 - b_k\mu}\left\{\exp\left[(\mu_0 - b_k\mu)\,\frac{\tau}{\mu\mu_0}\right] - 1\right\}\mathrm{e}^{-\tau/\mu} - \exp\left(-b_k\,\frac{\tau}{\mu_0}\right) + \exp(-\tau/\mu_0),\ \mu\geqslant 0, \\ \left(\frac{\mu_0 \Lambda x_k}{\mu_0 - b_k\mu}\left\{1 - \exp\left[(\mu_0 - b_k\mu)\,\frac{\tau_0-\tau}{\mu\mu_0}\right]\right\} - 1\right)\,\exp\left(-b_k\,\frac{\tau}{\mu_0}\right) + \exp(-\tau/\mu_0),\ \mu<0. \end{cases} \tag{18}$$

Let us expand the scattering phase function into a series (4), and the regular solution part $\tilde{L}(\tau,\hat{\mathbf{l}})$ and the source function into an azimuth series (13). Transferring to DOM, we will present a space integral by the Gaussian quadrature that reduces the set (17) to the expressions

$$C^m(\tau,\mu_i) = \begin{cases} \Delta^m(\tau,\mu_i) + \sum_{j=1}^{N} S_{ij}\int_0^{\tau}\exp(-(\tau-t)/\mu_i)\,C^m(t,\mu_j)\,\mathrm{d}t,\ \mu_i\geqslant 0, \\ \Delta^m(\tau,\mu_i) + \sum_{j=1}^{N} S_{ij}\int_{\tau}^{\tau_0}\exp(-(\tau-t)/\mu_i)\,C^m(t,\mu_j)\,\mathrm{d}t,\ \mu_i< 0, \end{cases} \tag{19}$$

where

$$S_{ij} \equiv \frac{\Lambda w_j}{2|\mu_i|}\sum_{k=0}^{\infty}(2k+1)x_k\,Q_k^m(\mu_i)\,Q_k^m(\mu_j),\ \Delta^m(\tau,\mu) = \sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\,\Delta_k(\tau,\mu)\,Q_k^m(\mu_0)\,Q_k^m(\mu).$$

The obtained set (19) is easily solved by a method of iterations. At the calculation of a spatial integral, we will use a grid of values $C_l^m(\tau,\mu_i)$ by the depth with the intermediate magnitude approximation by a cubic spline. In Fig. 2, the calculation of the radiance angular distribution of the radiation reflected by the slab is shown depending on the number of iterations. Let us note that the convergence for the downward radiation is essentially better and the amount of iterations does not exceed 2–4.

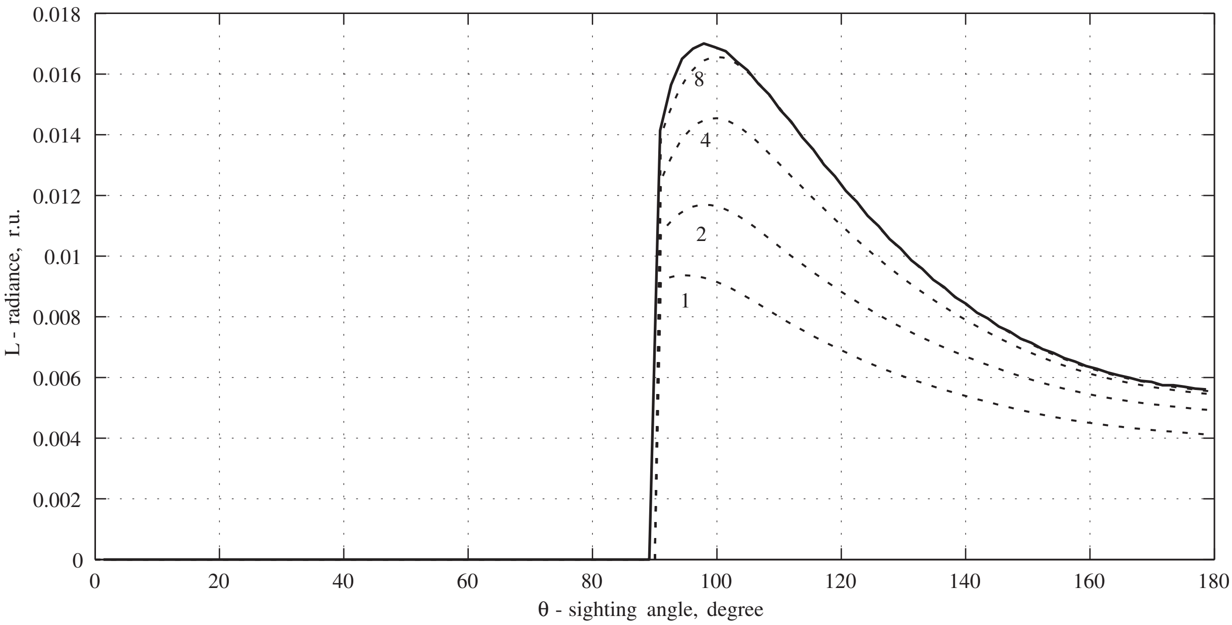

Fig. 2. Convergence of the method of iteration for the reflected radiation: $\tau_0 = 5.0$, $\theta_0 = 10^\circ$, $\Lambda = 0.8$, $g = 0.9$. The solid curve is the exact solution. Numbers near the dotted curves are the number of iteration.

6. Small-angle modification of the SH method (MSH)

The main difficulty to expand the present method on other geometries is that SA in the form Goudsmit and Saunderson [10,11] is possible only for the flat geometry at the almost normal incidence of the radiation on the medium.

From the analysis of the angular spectrum of the radiance distribution, it is possible to formulate the approximate equation describing an anisotropic part of the solution—small-angle MSH [11]: in the neighborhood of the singularity, the spectrum slowly decreases with the number of harmonics. This approach makes it possible to select analytically the singularities from an RTE solution and to state the boundary-value problem for the regular part of the solution. The analytic form of MSH as a radiance decomposition on the surface harmonic essentially simplifies the calculation of the source function in the equation for the regular part.

Let us consider MSH on the basis of the slab irradiation by a plane unidirectional source under an arbitrary incidence angle and pass from RTE to the infinite set of ordinary differential equations of the SH method. However, in this case we will not truncate the numbers of the series terms and assume the following:

- a continuous dependence of the series coefficients $C^m(k,\tau)$ on the index k, which in integer points coincides with values of the coefficients $C_k^m(\tau)$;
- at a strong anisotropy of the angular distribution its spectrum $C_k^m(\tau)$ is a slowly monotonically decreasing function of the index k that allows to expand it in a Taylor series preserving two or three first terms

$$C^m(\tau, k \pm 1) \approx C^m(\tau, k) \pm \frac{\partial C^m(\tau,k)}{\partial k} + \frac{1}{2}\frac{\partial^2 C^m(\tau,k)}{\partial k^2}; \tag{20}$$

- owing to the anisotropy of the radiance angular distribution the basic contribution to the solution is given by the terms with $k \gg 1$ and its anisotropy is much greater than its asymmetry $k \gg m$.

These assumptions reduce the infinite set of the SH method (9) to one partial equation

$$\mu_0 \frac{\partial C^m}{\partial \tau} + \frac{\sqrt{1-\mu_0^2}}{2}\frac{\partial}{\partial \tau}\left[\frac{\partial C^{m+1}}{\partial \kappa} + \frac{\partial C^{m-1}}{\partial \kappa} + \frac{1}{\kappa}((m+1)C^m - (m-1)C^m)\right] = -(1 - \Lambda x_k)\, C^m(\tau, \kappa),$$

permitting the approximate analytical solution [11]

$$C_k^m(\tau) = \frac{\mathrm{e}^{-\tau/\mu_0}}{2\pi}\int_0^{2\pi} \cos m\varphi \,\exp\left[\frac{\Lambda\tau}{\mu_0}\int_0^{\infty} x(\rho)\,\mathrm{e}^{-\zeta}\,\mathrm{d}\zeta\right]\mathrm{d}\varphi, \tag{21}$$

where $\rho = \sqrt{\kappa^2 + a^2\zeta^2 - 2\kappa a\zeta \cos\varphi}$, $\kappa = \sqrt{k(k+1)}$, $a = \tan\theta_0$.

The obtained solution passes to Goudsmit–Saunderson approximation at the small incidence angles, but, in contrast to it, this solution describes the rotation of the maximum of the radiance angular distribution from an incidence direction on the upper slab border to a vertical one in the medium depth. The analytical form of expression (21) is more complex, and its usage in the approach (2) is possible with difficulties, but we succeeded to formulate the analytical approach to a determination of the solution singularities for an arbitrary medium geometry.

7. MSH in arbitrary medium geometry

Applying the formulated approach, we succeeded in the solutions of RTE for all the fundamental sources. Let us consider for example the boundary-value problem of the point isotropic source located in the origin of coordinates in the infinite turbid medium [12]:

$$\begin{cases} \mu \frac{\partial L(r,\mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial L(r,\mu)}{\partial \mu} = -\varepsilon L(r,\mu) + \frac{\Lambda\varepsilon}{4\pi} \oint L(r,\mu') x(\hat{\mathbf{l}},\hat{\mathbf{l}}') \, d\hat{\mathbf{l}}', \\ L(r,\mu)\big|_{r\to 0,\mu>0} = \frac{1}{4\pi r^2} \delta(1-\mu), \; L(r,\mu)\big|_{r\to\infty} = 0, \end{cases} \tag{22}$$

where, in consequence of the spherical symmetry of the problem, the solution $L(r,\hat{\mathbf{l}}) = L(r,\mu), \; \mu = (r,\hat{\mathbf{l}})/r$.

We present the angular dependences of all the functions in RTE as the decomposition on the Legendre polynomials

$$L(r,\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi r^2} C_k(r) P_k(\mu) \tag{23}$$

that according to the SH method gives us the infinite set of the ordinary differential equations

$$\frac{d}{dr}[(k+1) C_{k+1}(r) + k C_{k-1}(r)] + \frac{k(k+1)}{r} (C_{k+1}(r) - C_{k-1}(r)) = -(2k+1)\varepsilon(1 - \Lambda x_k) C_k(r). \tag{24}$$

Now we determine the continuous dependence of the series coefficients $C(k,r)$ on the index k and expand it in a Taylor series preserving two first terms (20). In this case, Eq. (24) takes the following form:

$$\frac{\partial C}{\partial k} + \frac{2k(k+1)}{2k+1} \frac{1}{r} \frac{\partial C}{\partial k} + \varepsilon(1 - \Lambda x(k)) C(k,r) = 0, \tag{25}$$

where $x(k)$ is the continuous representation of the angular spectrum of the phase function x_k.

Let us make the transformation of variables in Eq. (25):

$$(r,k) \to (r, \xi = r/\kappa), \; \kappa = \sqrt{k(k+1)} \tag{26}$$

that reduces it to the ordinary differential equation

$$\frac{dC}{dr} + \varepsilon[1 - \Lambda x(r/\xi)] C(r/\xi, r) = 0, \tag{27}$$

which is explicitly solvable:

$$Y(k,r) = f\left(r\Big/\sqrt{k(k+1)}\right) \exp\left\{-\varepsilon\left(1 - \frac{\Lambda}{r\sqrt{k(k+1)}} \int_0^{\sqrt{k(k+1)}} x(\xi) \, d\xi\right) r\right\}, \tag{28}$$

where $f(\cdot)$ is an arbitrary smooth function defined by boundary conditions.

Neglecting the backscattering in SA, the boundary conditions take the form

$$\forall k \in \overline{1,\infty}: \begin{cases} C_k(r) \to 1, r \to 0, \\ C_k(r) \to 0, r \to \infty, \end{cases} \tag{29}$$

which gives us the final solution for the radiance angular distribution in the field of the point isotropic source

$$L(r,\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi r^2} \exp\left\{-\varepsilon\left(1 - \frac{\Lambda}{r\sqrt{k(k+1)}} \int_0^{\sqrt{k(k+1)}} x(\xi) \, d\xi\right) r\right\} P_k(\mu). \tag{30}$$

In this case, the solution has the singularities not only in the direct radiation but also in the first two orders of scattering. Let us analyze the singularities of MSH in this case. We expand the expression for MSH in Taylor series on the single scattering albedo. It is equivalent to the solution representation by the scattering orders. We take into account the first three terms:

$$L_0(r,\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi r^2} e^{-\varepsilon r} P_k(\mu) = \frac{e^{-\varepsilon r}}{2\pi r^2} \delta(1-\mu), \tag{31}$$

$$L_1(r,\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi r^2} \mathrm{e}^{-\varepsilon r} \left(\frac{\Lambda \varepsilon r}{\sqrt{k(k+1)}} \int_0^{\sqrt{k(k+1)}} x(\xi)\,\mathrm{d}\xi \right) P_k(\mu), \tag{32}$$

$$L_2(r,\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi r^2} \mathrm{e}^{-\varepsilon r} \frac{1}{2} \left(\frac{\Lambda \varepsilon r}{\sqrt{k(k+1)}} \int_0^{\sqrt{k(k+1)}} x(\xi)\,\mathrm{d}\xi \right)^2 P_k(\mu). \tag{33}$$

As can be seen from (31), zero scattering order includes the δ-singularity of the exact solution. Since we are interested in the behavior of (32) and (33) about the singular point $\mu = 1$, the greatest contribution will be given by the series terms with $k \gg 1$. According to Mie theory, it is possible to note for an arbitrary phase function

$$\int_0^{\sqrt{k(k+1)}} x(\zeta)\,\mathrm{d}\zeta \underset{k\to\infty}{\longrightarrow} C_0, \tag{34}$$

where C_0 is an arbitrary constant.

Therefore, expressions (32) and (33) about the point $\mu \cong 1$ become

$$L_1(r,\mu) = \frac{\Lambda \varepsilon C_0}{2\pi r} \mathrm{e}^{-\varepsilon r} \sum_{k=0}^{\infty} P_k(\mu), \; L_2(r,\mu) = \frac{(\Lambda\varepsilon)^2 C_0^2}{8\pi} \mathrm{e}^{-\varepsilon r} \sum_{k=0}^{\infty} \left(\frac{1}{k+1} + \frac{1}{k} \right) P_k(\mu), \tag{35}$$

where we used the approximate equality $2\sqrt{k(k+1)} \approx 2k+1$ for the case $k \gg 1$.

On the basis of the course-of-value function properties for Legendre's polynomials, it is possible to sum the last series, and finally we have

$$L_1(r,\mu) = \frac{\Lambda \varepsilon C_0}{2\pi r} \mathrm{e}^{-\varepsilon r} \frac{1}{\sqrt{2(1-\mu)}}, \quad L_2(r,\mu) = \frac{(\Lambda\varepsilon)^2 C_0^2}{8\pi} \mathrm{e}^{-\varepsilon r} (-\ln(1-\mu)). \tag{36}$$

Thus, MSH contains all the singularities of the exact solution of RTE [3].

Now let us consider the simplest 3D boundary-value problem of the point isotropic source in the turbid medium slab. Using expression (30) and the method that is similar to the one using Section 5 with the coaxial cylindrical mesh, we calculated the radiance angular distribution on one slab bound from a point isotropic

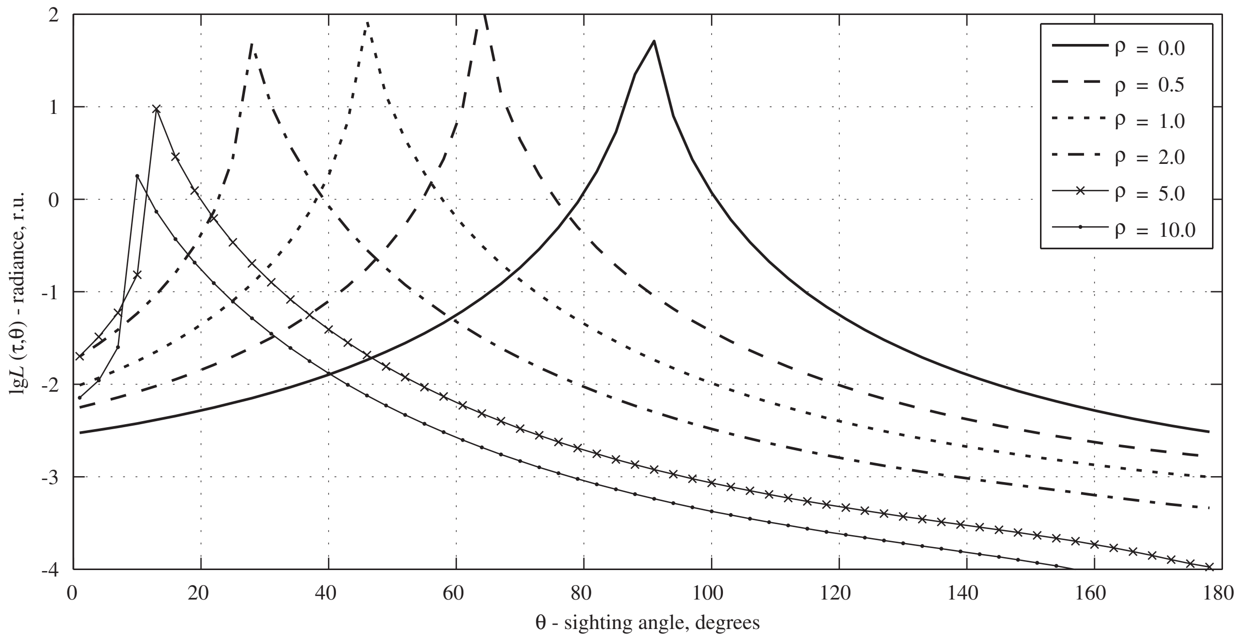

Fig. 3. The radiance angle distribution of the light field of the point isotropic source on the other slab bound. ρ is the distance to the point from the origin in the XOY plane.

source allocated on the other bound in the origin. The position of all the points in the space is specified by (τ, ρ), where ρ is the distance to the point from the origin in the XOY plane. The results are given in Fig. 3 for the slab with optical parameters: $\Lambda = 0.8$, $\tau = 1.0$ and $g = 0.9$.

Similarly, it is possible to find the RTE solution for the point unidirectional source located in the origin of coordinates in the infinite medium [13]:

$$L(r,\eta,\mu,\varphi) = \frac{1}{2\pi}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{2k+1}{2}\frac{2l+1}{2}\sqrt{\frac{(k-m)!}{(k+m)!}}\sqrt{\frac{(l-m)!}{(l+m)!}}C_{kl}^{m}(r)\,P_k^m(\mu)\,P_l^m(\eta)\,\mathrm{e}^{\mathrm{i}m\varphi}, \tag{37}$$

where

$$C_{kl}^{m}(r) = \frac{\mathrm{e}^{-\varepsilon r}}{2\pi r^2}\int_0^{2\pi}\exp\left\{-\mathrm{i}m\psi + \Lambda\varepsilon r\int_0^1 x(\rho(\zeta))\,\mathrm{d}\zeta\right\}\mathrm{d}\psi, \quad \rho(\zeta) = \sqrt{\lambda^2(1-\zeta^2)+\kappa^2\zeta^2 - 2\zeta(1-\zeta)\lambda\kappa\cos\psi},$$

$\lambda = \sqrt{l(l+1)}$, $\mu = (\hat{\mathbf{l}},\mathbf{r})/r$, $\eta = (\hat{\mathbf{q}},\mathbf{r})/r$, $\hat{\mathbf{q}}$ is the radiation direction of the point unidirectional source, φ is the dihedral angle between the $(\hat{\mathbf{l}}\times\mathbf{r})$ plane and $(\hat{\mathbf{q}}\times\mathbf{r})$ plane.

The idea of the solution representation as sum (2) was proposed on the basis of Goudsmit–Saunderson approximation in articles [14,15] that restricted this approach only to frameworks of the plane-parallel geometry. Expressions (30) and (37) allow to select the solution singularities of the RTE and to state the equation for the regular part of the solution in the arbitrary medium geometry.

It is possible to show that MSH is the most general form of SA and all the rest forms follow from it under the requirements of a normal incidence, small sighting angle and strong scattering anisotropy. MSH neglects only the variance of the trajectory length of the scattered rays and their backscattering. MSH contains all the singularities of the exact solution of RTE. The prevalent form of the SA as a Fourier transform of the radiance distribution on the boundary [9,10] does not allow selecting similarly the solution singular part, since it gives an expression for the source function, which considerably complicates the initial equation.

8. Generalization of MSH for the vectorial case of polarized radiation

However, in the vectorial case, there is still another problem: the reference planes of the incident and scattering rays as well as the scattering plane do not coincide [16]. Therefore, it is necessary to apply the rotator $\overleftrightarrow{R}(\chi)$ that disturbs the transformation symmetry of the different Stokes parameters and makes impossible using the addition theorem for the surface harmonics. In paper [16], the determination of the polarization on the basis of the circular polarization (CP-presentation) is presented, which is connected with Stokes polarization (SP-presentation) by the linear relation

$$\mathbf{L}_{\mathrm{CP}} = \begin{bmatrix} L_{+2} \\ L_{+0} \\ L_{-0} \\ L_{-2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} Q-\mathrm{i}U \\ I-V \\ I+V \\ Q+\mathrm{i}U \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 & -\mathrm{i} & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & \mathrm{i} & 0 \end{bmatrix}\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} \equiv \overleftrightarrow{T}_{\mathrm{CS}}\mathbf{L}_{\mathrm{SP}}, \quad \mathbf{L}_{\mathrm{SP}} = \overleftrightarrow{T}_{\mathrm{CS}}^{-1}\mathbf{L}_{\mathrm{CP}} \equiv \overleftrightarrow{T}_{\mathrm{SC}}\mathbf{L}_{\mathrm{CP}}, \tag{38}$$

where $\mathbf{L} = \{I,\ Q,\ U,\ V\}$ is the Stokes vector parameter.

In this case, the rotator takes a simpler form

$$\overleftrightarrow{R}_{\mathrm{CP}}(\chi) = \overleftrightarrow{T}_{\mathrm{CS}}\,\overleftrightarrow{R}(\chi)\overleftrightarrow{T}_{\mathrm{SC}} = \mathrm{diag}(\mathrm{e}^{+\mathrm{i}2\chi},1,1,\mathrm{e}^{-\mathrm{i}2\chi}), \tag{39}$$

which makes it possible using for every component $\mathbf{L}_{\mathrm{CP}}$ a special type of the generalized spherical harmonic $P_{mn}^{l}(x)$, for which its own form of the addition theorem [16] is correct:

$$\mathrm{e}^{-\mathrm{i}m\chi}\,P_{mn}^{l}(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}')\,\mathrm{e}^{-\mathrm{i}n\chi'} = \sum_{q=-l}^{l}(-1)^q\,P_{mq}^{l}(\hat{\mathbf{l}}\cdot\hat{z})\,P_{qn}^{l}(\hat{z}\cdot\hat{\mathbf{l}}')\,\mathrm{e}^{\mathrm{i}q(\varphi-\varphi')}, \tag{40}$$

where χ (χ') is a dihedral angle between the $\hat{\mathbf{l}}\times\hat{\mathbf{z}}$ plane ($\hat{\mathbf{l}}'\times\hat{\mathbf{z}}$ plane) and the $\hat{\mathbf{l}}\times\hat{\mathbf{l}}'$ plane.

Unfortunately, in CP presentation, all the coefficients in the vectorial RTE (VRTE) become complex, which makes it difficult to use the effective numerical methods of the VRTE solution that is based on the sorting algorithm. Therefore, to solve problem (1) by taking into account the radiation polarization, we at first convert the equation to CP presentation, then subtract SA and get the equation for the smooth part $\vec{\mathbf{L}}(\tau,\hat{\mathbf{l}})$, and finally return to SP presentation and solve the obtained equation. Let us present the angular distribution of all the functions in VRTE as decomposition by the generalized SH

$$[\overleftrightarrow{x}_{\mathrm{CP}}(\cos\gamma)]_{rs} = \sum_{k=0}^{\infty}(2k+1)x_{rs}^{k}P_{r,s}^{k}(\cos\gamma),\quad \mathbf{L}(\tau,v,\varphi) = \sum_{m=-\infty}^{+\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\mathrm{e}^{\mathrm{i}m\varphi}\overleftrightarrow{Y}_{k}^{m}(v)\,\mathbf{f}_{k}^{m}(\tau). \tag{41}$$

Here $\overleftrightarrow{Y}_{k}^{m}(\mu) = \mathrm{diag}\{P_{m,+2}^{k}(\mu), P_{m,+0}^{k}(\mu), P_{m,-0}^{k}(\mu), P_{m,-2}^{k}(\mu)\}$; each of the indices r, s runs through $+2,\ +0,\ -0,\ -2$; $v = (\hat{\mathbf{l}},\hat{\mathbf{l}}_0)$ is a cosine of the zenith angle and φ is the azimuth angle $\hat{\mathbf{l}}$ with reference to the direction of the radiation incidence; $\mu = v\mu_0 + \sqrt{1-\mu_0^2}\sqrt{1-v^2}\cos\varphi$; $(\underset{k}{\overleftrightarrow{x}})_{rs} \equiv x_{rs}^{k}$.

After the substitution in VRTE, taking into account the orthogonality of the general SH will result in the combined set of the ordinary differential equation

$$\begin{aligned}\frac{1}{2k+1}\frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg\{&\mu_0(\underset{k}{\overleftrightarrow{A}}\mathbf{f}_{k-1}^{m} + \underset{k+1}{\overleftrightarrow{A}}\mathbf{f}_{k+1}^{m} + m\underset{k}{\overleftrightarrow{B}}\mathbf{f}_{k}^{m}) + \frac{i}{2}\sqrt{1-\mu_0^2}\Big[d_{+(m-1)}\underset{k}{\overleftrightarrow{C}}\mathbf{f}_{k-1}^{m-1} - g_{-(m-1)}\underset{k+1}{\overleftrightarrow{C}}\mathbf{f}_{k+1}^{m-1}\\ &+ h_{-(m-1)}\underset{k}{\overleftrightarrow{B}}\mathbf{f}_{k}^{m-1} + d_{-(m-1)}\underset{k}{\overleftrightarrow{C}}\mathbf{f}_{k-1}^{m+1} - g_{+(m+1)}\underset{k+1}{\overleftrightarrow{C}}\mathbf{f}_{k+1}^{m+1} + h_{-(m+1)}\underset{k}{\overleftrightarrow{B}}\mathbf{f}_{k}^{m+1}\Big]\Bigg\} + (\overleftrightarrow{1} - \Lambda\underset{k}{\overleftrightarrow{x}})\mathbf{f}_{k}^{m} = 0.\end{aligned} \tag{42}$$

Symbols from (42) denote the following:

$$\underset{k}{\overleftrightarrow{A}} = \frac{k^2-s^2}{k}\delta_{rs},\ \underset{k}{\overleftrightarrow{B}} = \frac{(2k+1)s}{k(k+1)}\delta_{rs},\ \underset{k}{\overleftrightarrow{C}} = \frac{\sqrt{k^2-s^2}}{k}\delta_{rs},$$

$$d_{\pm m} = \sqrt{(k\pm m+1)(k\pm m+2)},\ g_{\pm m} = \sqrt{(k\pm m-1)(k\pm m)},\ h_{\pm m} = \sqrt{(k\pm m)(k\mp m+2)}.$$

Let us determine the continuous dependence $\mathbf{f}^{m}(k,\tau)$ of the coefficients $\mathbf{f}_{k}^{m}(\tau)$ on the numbers. In the case of the strong anisotropy, this dependence is a slow monotonic decreasing function that allows expanding it in a Taylor series with the maintenance of the first two terms [17]:

$$\mathbf{f}_{k\pm1}^{m}(\tau) \equiv \mathbf{f}^{m}(k\pm1,\tau) \approx \mathbf{f}^{m}(k,\tau) \pm \frac{\partial}{\partial k}\mathbf{f}^{m}(k,\tau). \tag{43}$$

Assumption (43) transforms the infinite equation set of the SH method (42) to a partial equation, the solution of which has a very complicated analytical form for the numerical calculation. But since it is important to express only the solution singularities, we can simplify it. Let us assume that the incident angle is near 0°, and that it is possible to present that $\sqrt{1-\mu_0^2} \approx 0$. Under these conditions we obtain a very simple set of the ordinary differential equation

$$\mu_0\frac{\mathrm{d}\mathbf{f}_{k}^{m}}{\mathrm{d}\tau} + (\overleftrightarrow{1} - \Lambda\underset{k}{\overleftrightarrow{x}})\mathbf{f}_{k}^{m}(\tau) = 0, \tag{44}$$

the solution of which has the following form:

$$\mathbf{f}_{k}^{m}(\tau) = \exp\left[-\frac{(\overleftrightarrow{1} - \Lambda\underset{k}{\overleftrightarrow{x}})}{\mu_0}\tau\right]\mathbf{f}_{k}^{m}(0) \equiv \overleftrightarrow{\mathbf{I}}_{k}(\tau)\,\mathbf{f}_{k}^{m}(0). \tag{45}$$

For the boundary conditions of the incident radiation with the radiance L, the polarization degree p, the ellipticity q and the polarization azimuth φ_0, we can set the solution in the following form:

$$\mathbf{L}_{\mathrm{MSH}}(\tau,\nu,\varphi)=\sum_{m=-\infty}^{+\infty}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi}\,\mathrm{e}^{\mathrm{i}m\varphi}\,\overset{\leftrightarrow m}{Y}_k(\nu)\,\overset{\leftrightarrow}{\mathbf{I}}_k(\tau)\,\mathbf{f}_k^m(0),\quad \mathbf{f}_k^m(0)=\pi L\begin{bmatrix} p\mathrm{e}^{-\mathrm{i}2\varphi_0}\delta_{m2}\\ (1-q)\delta_{m0}\\ (1+q)\delta_{m0}\\ p\mathrm{e}^{\mathrm{i}2\varphi_0}\delta_{m,-2}\end{bmatrix}. \tag{46}$$

9. Regular part of VRTE solution

Now let us consider the smooth part of Stokes parameters field in the slab irradiated above the PU source (46). In this case we have VRTE

$$\mu\frac{\partial}{\partial\tau}\tilde{\mathbf{L}}(\tau,\hat{\mathbf{l}})+\tilde{\mathbf{L}}(\tau,\hat{\mathbf{l}})=\frac{\Lambda}{4\pi}\oint\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0)\,\overset{\leftrightarrow}{x}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\,\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}_0\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}')\,\tilde{\mathbf{L}}(\tau,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}'+\Delta(\tau,\hat{\mathbf{l}}), \tag{47}$$

where the full solution is presented by the expression

$$\mathbf{L}(\tau,\mu,\varphi)=\tilde{\mathbf{L}}(\tau,\mu,\varphi)+\mathbf{L}_{\mathrm{MSH}}(\tau,\mu,\varphi), \tag{48}$$

and the source function is

$$\Delta(\tau,\hat{\mathbf{l}})=\frac{\Lambda}{4\pi}\oint\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}_0)\,\overset{\leftrightarrow}{x}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\,\overset{\leftrightarrow}{R}(\hat{\mathbf{l}}_0\times\hat{\mathbf{l}}'\to\hat{\mathbf{l}}\times\hat{\mathbf{l}}')\,\mathbf{L}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}'-\mu\frac{\partial}{\partial\tau}\mathbf{L}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}})-\mathbf{L}_{\mathrm{MSH}}(\tau,\hat{\mathbf{l}}). \tag{49}$$

Then we deal with the scattering integral. We convert all the functions under the integral to CP presentation, expand the scattering matrix in series on a generalized SH, use the addition theorem and return to SP presentation. It can be written as follows:

$$\begin{aligned}\mathbf{I}_{\mathrm{S}}&=\overset{\leftrightarrow}{T}_{\mathrm{SC}}\frac{\Lambda}{4\pi}\oint\overset{\leftrightarrow}{T}_{\mathrm{CS}}\,\overset{\leftrightarrow}{R}(\chi)\overset{\leftrightarrow}{T}_{\mathrm{SC}}\overset{\leftrightarrow}{T}_{\mathrm{CS}}\,\overset{\leftrightarrow}{x}(\hat{\mathbf{l}},\hat{\mathbf{l}}')\overset{\leftrightarrow}{T}_{\mathrm{SC}}\overset{\leftrightarrow}{T}_{\mathrm{CS}}\,\overset{\leftrightarrow}{R}(\chi')\overset{\leftrightarrow}{T}_{\mathrm{SC}}\overset{\leftrightarrow}{T}_{\mathrm{CS}}\,L(z,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}'\\ &=\frac{\Lambda}{4\pi}\oint\left(\sum_{l=0}^{\infty}(2l+1)\sum_{n=-l}^{l}\mathrm{e}^{\mathrm{i}n(\varphi-\varphi')}\,\overset{\leftrightarrow l}{P}_n(\mu)\,\overset{\leftrightarrow}{\chi}_l\,\overset{\leftrightarrow l}{P}_n(\mu')\right)L(\tau,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}',\end{aligned} \tag{50}$$

where $\overset{\leftrightarrow}{x}_k=\overset{\leftrightarrow}{T}_{\mathrm{SC}}\overset{\leftrightarrow}{x}_k\overset{\leftrightarrow}{T}_{\mathrm{CS}}$, $\overset{\leftrightarrow}{x}_k$ is defined by (41):

$$\overset{\leftrightarrow l}{P}_n(\mu)=\begin{bmatrix} Q_l^n(\mu) & 0 & 0 & 0\\ 0 & R_l^n(\mu) & -\mathrm{i}T_l^n(\mu) & 0\\ 0 & \mathrm{i}T_l^n(\mu) & R_l^n(\mu) & 0\\ 0 & 0 & 0 & Q_l^n(\mu)\end{bmatrix}\equiv\overset{\leftrightarrow}{P}_R(\mu)+\mathrm{i}\overset{\leftrightarrow}{P}_I(\mu),$$

$$\frac{1}{2}\left(P^l_{n,+2}(\mu)+P^l_{n,-2}(\mu)\right)\equiv i^nR_l^n(\mu),\qquad \frac{1}{2}\left(P^l_{n,+2}(\mu)-P^l_{n,-2}(\mu)\right)\equiv i^nT_l^n(\mu).$$

It is easy to get convinced by the direct verification that $\overset{\leftrightarrow l}{P}_n(\mu)=\overline{\overset{\leftrightarrow l}{P}_{-n}(\mu')}$, where the line above indicates the complex-conjugate number. It means that the local transformation matrix is a real function. Therefore, there is no necessity to keep all the terms in the azimuth series (50) and to combine the terms of the series with m and $-m$:

$$\mathbf{I}_{\mathrm{S}}=\frac{\Lambda}{4\pi}\oint\left(\sum_{k=0}^{\infty}(2k+1)\sum_{m=0}^{l}(2-\delta_{0,m})\left(\overset{\leftrightarrow m}{C}_k(\mu,\mu')\cos m(\varphi-\varphi)+\overset{\leftrightarrow m}{S}_k(\mu,\mu')\sin m(\varphi-\varphi')\right)\right)\mathbf{L}(\tau,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}', \tag{51}$$

where

$$\overset{\leftrightarrow}{C}{}_k^m(\mu,\mu') = \overset{\leftrightarrow}{P}{}_R^k(\mu)\overset{\leftrightarrow}{\chi}_k\overset{\leftrightarrow}{P}{}_R^k(\mu') - \overset{\leftrightarrow}{P}{}_I^k(\mu)\overset{\leftrightarrow}{\chi}_k\overset{\leftrightarrow}{P}{}_I^k(\mu'), \quad \overset{\leftrightarrow}{S}{}_k^m(\mu,\mu') = \overset{\leftrightarrow}{P}{}_I^k(\mu)\overset{\leftrightarrow}{\chi}_k\overset{\leftrightarrow}{P}{}_R^k(\mu') + \overset{\leftrightarrow}{P}{}_R^k(\mu)\overset{\leftrightarrow}{\chi}_k\overset{\leftrightarrow}{P}{}_I^k(\mu').$$

The obtained expression (51) is similar to the corresponding expression in [18]. Thus, we determine the matrices as [18]

$$\overset{\leftrightarrow}{\phi}_1(\varphi) = \mathrm{diag}\{\cos\varphi, \cos\varphi, \sin\varphi, \sin\varphi\}, \ \overset{\leftrightarrow}{\phi}_2(\varphi) = \mathrm{diag}\{-\sin\varphi, -\sin\varphi, \cos\varphi, \cos\varphi\}, \ \overset{\leftrightarrow}{D}_1 = \mathrm{diag}\{1,1,0,0\},$$
$$\overset{\leftrightarrow}{D}_2 = \mathrm{diag}\{0,0,-1,-1\},$$

$$\overset{\leftrightarrow}{\Pi}{}_k^m(\mu) = \begin{bmatrix} Q_k^m(\mu) & 0 & 0 & 0 \\ 0 & R_k^m(\mu) & -T_k^m(\mu) & 0 \\ 0 & -T_k^m(\mu) & R_k^m(\mu) & 0 \\ 0 & 0 & 0 & Q_k^m(\mu) \end{bmatrix}. \tag{52}$$

This allows transforming (51) into the following form:

$$\mathbf{I}_S = \frac{\Lambda}{4\pi}\oint\left(\sum_{l=0}^{\infty}(2l+1)\sum_{n=0}^{l}(2-\delta_{0,n})(\overset{\leftrightarrow}{\phi}_1(n(\varphi-\varphi'))\overset{\leftrightarrow}{A}\overset{\leftrightarrow}{D}_1 + \overset{\leftrightarrow}{\phi}_2(n(\varphi-\varphi'))\overset{\leftrightarrow}{A}\overset{\leftrightarrow}{D}_2)\right)\mathbf{L}(\tau,\hat{\mathbf{l}}')\,\mathrm{d}\hat{\mathbf{l}}', \tag{53}$$

where $\overset{\leftrightarrow}{A}{}_k^m(\mu,\mu') = \overset{\leftrightarrow}{\Pi}{}_k^m(\mu)\chi_k\overset{\leftrightarrow}{\Pi}{}_k^m(\mu')$.

Let us consider the source function (49) from (47). It is possible to show that, using (49), (53) and (52) after some tedious transformations, we can get the expression for the source function

$$\Delta(\tau,\mu,\varphi) = \sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\frac{2k+1}{2}\left(\overset{\leftrightarrow}{\phi}_1(m\varphi)\overset{\leftrightarrow}{\Pi}{}_m^k(\mu)\overset{\leftrightarrow}{\Phi}_k(\tau)\overset{\leftrightarrow}{\Pi}{}_m^k(\mu_0)\overset{\leftrightarrow}{D}_1 + \overset{\leftrightarrow}{\phi}_2(m\varphi)\overset{\leftrightarrow}{\Pi}{}_m^k(\mu)\overset{\leftrightarrow}{\Phi}_k(\tau)\overset{\leftrightarrow}{\Pi}{}_m^k(\mu_0)\overset{\leftrightarrow}{D}_2\right)\mathbf{f}_k(0), \tag{54}$$

where

$$\overset{\leftrightarrow}{\Phi}_k(\tau) = \frac{1}{2k+1}\left[\overset{\leftrightarrow}{A}_{k+1}\overset{\leftrightarrow}{b}_{k+1}\overset{\leftrightarrow}{\Xi}_{k+1}(\tau)\vec{a}_k + 4\frac{(2k+1)}{k(k+1)}\overset{\leftrightarrow}{b}_k\overset{\leftrightarrow}{\Xi}_k(\tau)\overset{\leftrightarrow}{B} + \overset{\leftrightarrow}{A}_k\overset{\leftrightarrow}{b}_{k-1}\overset{\leftrightarrow}{\Xi}_{k-1}(\tau)\vec{a}_k\right] - \underset{k}{\overset{\leftrightarrow}{b}}\overset{\leftrightarrow}{\Xi}_k(\tau),$$

$$\overset{\leftrightarrow}{b}_k = \overset{\leftrightarrow}{1} - \Lambda\overset{\leftrightarrow}{\chi}_k, \ \overset{\leftrightarrow}{\Xi}_k(\tau) = \overset{\leftrightarrow}{T}_{SC}\overset{\leftrightarrow}{I}_k(\tau)\overset{\leftrightarrow}{T}_{CS}, \ \overset{\leftrightarrow}{a}_k = \mathrm{diag}\{k,\kappa,\kappa,k\}, \ \overset{\leftrightarrow}{A}_k = \frac{\overset{\leftrightarrow}{a}_k}{k}, \ \overset{\leftrightarrow}{B} = \mathrm{diag}\{0,1,1,0\}, \ \kappa = \sqrt{k^2-4}.$$

Let us present the smooth part of the solution similar to the diffusion component

$$\tilde{\mathbf{L}}(\tau,\mu,\varphi) = \sum_{m=0}^{\infty}\left[\overset{\leftrightarrow}{\phi}_1(m\varphi)\tilde{\mathbf{L}}_1^m(\tau,\mu) + \overset{\leftrightarrow}{\phi}_2(m\varphi)\tilde{\mathbf{L}}_2^m(\tau,\mu)\right]. \tag{55}$$

that gives us two integral equations ($i = 1, 2$)

$$\mu\frac{\partial}{\partial\tau}\tilde{\mathbf{L}}_i^m(\tau,\mu) + \tilde{\mathbf{L}}_i^m(\tau,\mu) = \frac{\Lambda}{2}\sum_{k=0}^{\infty}(2k+1)\int_{-1}^{1}\overset{\leftrightarrow}{A}{}_k^m(\mu,\mu')\tilde{\mathbf{L}}_i^m(\tau,\mu')\,\mathrm{d}\mu' + \Delta_i(\tau,\mu), \tag{56}$$

with the boundary conditions

$$\tilde{\mathbf{L}}_i^m(0,\mu)\big|_{\mu 0} = 0, \quad \tilde{\mathbf{L}}_i^m(\tau_0,\mu)\big|_{\mu 0} = -\left[\mathbf{L}_{\mathrm{MSH}}^m(\tau_0,\mu)\right]_i, \tag{57}$$

where

$$\Delta_i(\tau,\mu) = \sum_{k=0}^{\infty}\frac{2k+1}{2}\overset{\leftrightarrow}{\Pi}{}_m^k(\mu)\overset{\leftrightarrow}{\Phi}_k(\tau)\overset{\leftrightarrow}{\Pi}{}_m^k(\mu_0)\overset{\leftrightarrow}{D}_i f_k(0).$$

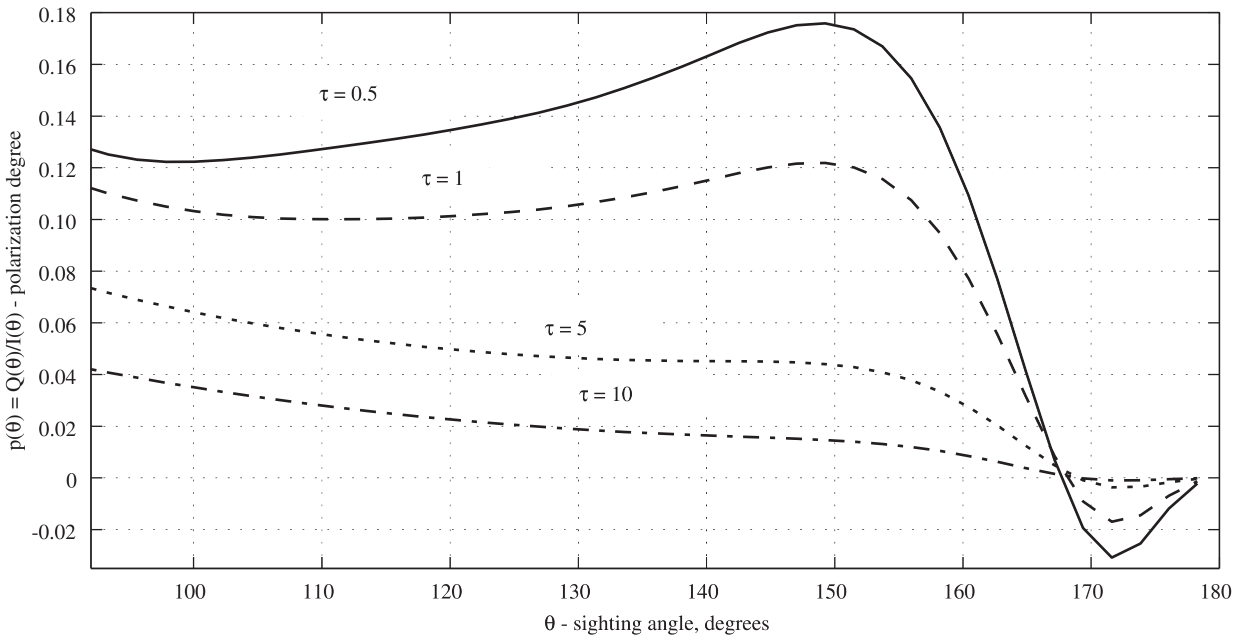

Fig. 4. The polarization degree of the slab reflected radiation depending on the slab thickness τ by the normal incident irradiation.

Thus, Eq. (56) can be already solved by the discrete ordinate method. The polarization degree of the slab reflected radiance is given in Fig. 4 for the scattering matrix of Deirmendjian Water Haze L for wave length equal to 700 nm, depending on the slab thickness by the normal incident irradiation.

10. Conclusion

From the analysis of the angular spectrum of the radiance distribution, it is possible to formulate the approximate equation describing an anisotropic part of the solution as the small-angle modification of the spherical harmonic method (MSH). This approach makes possible to eliminate analytically the singularities from the RTE solution and to state the boundary-value problem for the regular part of the solution. The analytic form of MSH as a decomposition on the surface harmonic essentially simplifies the calculation of the source function in the equation for the regular part. The regular part of the solution is found numerically by the finite element method that enables to extend the solution to the arbitrary medium geometry. The application of the developed method for the solution of the simple problem of the medium slab showed its high performance. However, its expansion to the case of the arbitrary medium geometry demands the development of the mesh-building technique for the numerical determination of the regular part of solution.

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