ON THE SOLUTION OF CERTAIN DIFFERENTIAL EQUATIONS BY CHARACTERISTIC FUNCTION EXPANSIONS*

BY

TSE-SUN CHOW

Research Staff, General Motors Corporation, Warren, Michigan

1. In this article we seek for the solution of the differential equation

$$p_0 \frac{\partial^2 u}{\partial x^2} + p_1 \frac{\partial u}{\partial x} + p_2 u = \frac{\partial u}{\partial t}, \tag{1.1}$$

where p_0, p_1, p_2 are functions of x, (x real) with the initial condition

$$u(x, t) = u_0(t), \qquad t = 0, \tag{1.2}$$

and the following boundary conditions

$$\alpha_1 u(a, t) + \alpha_2 u(b, t) + \alpha_3 u_x(a, t) + \alpha_4 u_x(b, t) = f(t), \tag{1.3}$$

$$\beta_1 u(a, t) + \beta_2 u(b, t) + \beta_3 u_x(a, t) + \beta_4 u_x(b, t) = g(t), \tag{1.4}$$

where α_i, β_i are constants[1], and $u_x = \partial u/\partial x$. Equation (1.1) is a differential equation of the second order of the parabolic type. Special cases of problems of this kind occur in heat conduction and diffusion, usually with simpler types of boundary conditions. Since the boundary conditions (1.3) and (1.4) are non-homogeneous and time-dependent, the method of separation of variables cannot be used directly. We shall first use a transformation[2] to remove the non-homogeneous boundary conditions and then separate the variables. This results in the well known Sturm-Liouville system and the solution of (1.1) $\cdots$ (1.4) will be sought as expansions of the characteristic functions of this system. With arbitrary values of α_i, β_i the resulting Sturm-Liouville system is in general not self-adjoint, and the characteristic functions are not orthogonal. Yet, it is known in the theory of differential equations that if we introduce the adjoint system, the characteristic functions of the two systems will be bi-orthogonal, i.e., the characteristic function of one system for one particular characteristic number will be orthogonal to all the characteristic functions of the other system with the exception of one of the same characteristic number. In carrying out the expansion procedure to find the solution of (1.1) $\cdots$ (1.4) we shall make use of this bi-orthogonality relationship and shall show in the final solution how the time-dependent functions, $f(t)$ and $g(t)$ are related to the boundary forms complementary to those of the adjoint system. These are given by (4.3), (4.4) and (4.5)[3].

*Received Feb. 1, 1957; revised manuscript received June 10, 1957.

The author wishes to thank Professor R. C. F. Bartels of the University of Michigan for comments. He is also indebted to the referee for pointing out a somewhat misleading statement in the manuscript.

[1]We assume that σ_{12}, σ_{13}, σ_{24}, $\sigma_{34} \neq 0$, where $\sigma_{12} = \alpha_1\beta_2 - \alpha_2\beta_1$ etc.

[2]A homogeneous differential equation with non-homogeneous boundary conditions is equivalent to a non-homogeneous differential equation with homogeneous boundary conditions [1].

[3]The theory of non-self-adjoint boundary value problems and the associated expansions of functions in terms of bi-orthogonal systems of characteristic functions does not appear nearly as well known as the theory of self-adjoint systems. Readers are referred to Coddington and Levinsen [2] for information on this subject.

2. To remove the non-homogeneous boundary conditions we make the substitution

$$u(x, t) = \zeta(x, t) + \sum_{i=1}^{2} X_i T_i$$

in Eqs. (1.1) $\cdots$ (1.4) where X_i and T_i are functions of x and t respectively. We get

$$p_0 \frac{\partial^2 \zeta}{\partial x^2} + p_1 \frac{\partial \zeta}{\partial x} + p_2 \zeta - \frac{\partial \zeta}{\partial t} = \sum_{i=1}^{2} \{X_i T_i' - T_i L(X_i)\}, \tag{2.1}$$

$$\zeta(x, 0) = u_0(x) - \sum_{i=1}^{2} X_i T_i(0), \tag{2.2}$$

$$\begin{aligned} &\alpha_1 \zeta(a, t) + \alpha_2 \zeta(b, t) + \alpha_3 \zeta_x(a, t) + \alpha_4 \zeta_x(b, t) \\ &\quad + \sum_{i=1}^{2} \{\alpha_1 X_i(a) T_i(t) + \alpha_2 X_i(b) T_i(t) + \alpha_3 X_i'(a) T_i(t) + \alpha_4 X_i'(b) T_i(t)\} = f(t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\beta_1 \zeta(a, t) + \beta_2 \zeta(b, t) + \beta_3 \zeta_x(a, t) + \beta_4 \zeta_x(b, t) \\ &\quad + \sum_{i=1}^{2} \{\beta_1 X_i(a) T_i(t) + \beta_2 X_i(b) T_i(t) + \beta_3 X_i'(a) T_i(t) + \beta_4 X_i'(b) T_i(t)\} = g(t). \end{aligned} \tag{2.4}$$

In these equations $L \equiv p_0\, d^2/dx^2 + p_1\, d/dx + p_2$, $\zeta x = \partial \zeta / \partial x$ and all primes, the corresponding derivatives. We next choose X_i such that

$$\begin{cases} X_1(a) = 1, & X_1(b) = X_1'(a) = X_1'(b) = 0, \\ X_2(b) = 1, & X_2(a) = X_2'(a) = X_2'(b) = 0. \end{cases} \tag{2.5}$$

and furthermore

$$\begin{cases} T_1(t) = \{\beta_2 f(t) - \alpha_2 g(t)\}/\sigma_{12}, \\ T_2(t) = \{\alpha_1 g(t) - \beta_1 f(t)\}/\sigma_{12}. \end{cases} \tag{2.6}$$

Then the boundary conditions to be satisfied by $\zeta(x, t)$ are

$$\begin{cases} \alpha_1 \zeta(a, t) + \alpha_2 \zeta(b, t) + \alpha_3 \zeta_x(a, t) + \alpha_4 \zeta_x(b, t) = 0, \\ \beta_1 \zeta(a, t) + \beta_2 \zeta(b, t) + \beta_3 \zeta_x(a, t) + \beta_4 \zeta_x(b, t) = 0. \end{cases} \tag{2.7}$$

In the meantime a particular choice of $X_i(x)$ can be immediately determined by (2.5). Thus

$$X_1(x) = \frac{(x - b)^3}{(a - b)^3} - 3\, \frac{(x - b)^2 (x - a)}{(a - b)^3}, \tag{2.8}$$

$$X_2(x) = \frac{(x - a)^3}{(b - a)^3} - 3\, \frac{(x - a)^2 (x - b)}{(b - a)^3}. \tag{2.9}$$

3. With X_i, T_i determined, the right-hand side of (2.1) is known completely, and we are led to consider the following ordinary differential equation associated with (2.1),

$$L_n(\psi_n) \equiv p_0 \psi_n'' + p_1 \psi_n' + (p_2 + \lambda_n)\psi_n = 0, \tag{3.1}$$

with the boundary conditions

$$\begin{cases} U_1\{\psi_n\} = \alpha_1 \psi_n(a) + \alpha_2 \psi_n(b) + \alpha_3 \psi_n'(a) + \alpha_4 \psi_n'(b) = 0, \\ U_2\{\psi_n\} = \beta_1 \psi_n(a) + \beta_2 \psi_n(b) + \beta_3 \psi_n'(a) + \beta_4 \psi_n'(b) = 0, \end{cases} \tag{3.2}$$

where λ_n is the characteristic number. Let Ψ_{n1}, Ψ_{n2} be two fundamental solutions of (3.1), then the characteristic numbers λ_n are the roots of the following determinant:

$$\begin{vmatrix} U_1\{\Psi_{n1}\} & U_1\{\Psi_{n2}\} \\ U_2\{\Psi_{n1}\} & U_2\{\Psi_{n2}\} \end{vmatrix} = 0. \tag{3.3}$$

Now consider the following system which is adjoint to the original system defined by (3.1), (3.2):

$$L_n^*(\chi_n) \equiv L^*(\chi_n) + \lambda_n\chi_n \equiv (p_0\chi_n)'' - (p_1\chi_n)' + (p_2 + \lambda_n)\chi_n = 0, \tag{3.4}$$

$$\begin{cases} V_1\{\chi_n\} = \gamma_1\chi_n(a) + \gamma_2\chi_n(b) + \gamma_3\chi_n'(a) + \gamma_4\chi_n'(b) = 0, \\ V_2\{\chi_n\} = \delta_1\chi_n(a) + \delta_2\chi_n(b) + \delta_3\chi_n'(a) + \delta_4\chi_n'(b) = 0, \end{cases} \tag{3.5}$$

where $\gamma_1, \cdots, \gamma_4, \delta_1, \cdots, \delta_4$ are constants. The characteristic functions of the two systems are $\psi_n(x)$ and $\chi_n(x)$. It is known in the theory of differential equations [2, 3] that $\psi_n(x)$, $\chi_n(x)$ are orthogonal in the interval (a, b). We further write $\int_a^b \psi_n\chi_n dx = C_n$. These properties will be utilized in obtaining the solution of the system (2.1) $\cdots$ (2.4) in terms of expansions of $\psi_n(x)$.

We now expand the right-hand side of (2.1) into a series of $\psi_n(x)$. Let

$$X_iT_i' = T_i' \sum_{n=1}^{\infty} a_{ni}\psi_n(x), \tag{3.6}$$

$$T_iL(X_i) = T_i \sum_{n=1}^{\infty} b_{ni}\psi_n(x), \tag{3.7}$$

where

$$a_{ni} = \int_a^b X_i\chi_n\, dx/C_n \quad \text{etc.;}$$

assuming further that

$$\zeta(x, t) = \sum_{n=1}^{\infty} F_n(t)\psi_n(x), \tag{3.8}$$

and substituting (3.6), (3.7), (3.8) into (2.1), and collecting coefficients of $\psi_n(x)$ we have

$$-\lambda_nF_n(t) - F_n'(t) = \sum_{i=1}^{2} \{a_{ni}T_i' - b_{ni}T_i\}, \tag{3.9}$$

where use has been made of $L(\psi_n) = -\lambda_n\psi_n$, by (3.1). Upon integration of (3.9) we have immediately

$$F_n(t) = F_n(0) \exp\{-\lambda_n t\} - \int_0^t \left\{\sum_{i=1}^{2} a_{ni}T_i'(\tau) - \sum_{i=1}^{2} b_{ni}T_i(\tau)\right\} \exp\{-\lambda_n(t - \tau)\}\, d\tau. \tag{3.10}$$

The coefficients $F_n(0)$ are to be determined by the initial condition (2.2); thus

$$\sum_{n=1}^{\infty} F_n(0)\psi_n(x) = u_0(x) - \sum_{i=1}^{2} X_i(x)T_i(0),$$

and

$$F_n(0) = \int_a^b \left\{u_0(x) - \sum_{i=1}^{2} X_i(x)T_i(0)\right\}\chi_n(x)\, dx/C_n\, . \tag{3.11}$$

Integrating $\int_0^t \sum_{i=1}^2 a_{ni}T_i'(\tau) \exp\{-\lambda_n(t-\tau)\}\, d\tau$ in (3.10) by parts and remembering that

$$\zeta(x, t) = \sum_{n=1}^{\infty} F_n(t)\psi_n(x) \quad \text{and} \quad u(x, t) = \zeta(x, t) + \sum_{i=1}^{2} X_iT_i\, ,$$

we obtain the formal solution of the given system (1.1) $\cdots$ (1.4) in the following form:

$$\begin{aligned} u(x, t) = &\sum_{n=1}^{\infty} \frac{\psi_n(x)}{C_n} \exp\{-\lambda_n t\} \int_a^b u_0(x)\chi_n(x)\, dx \\ &+ \sum_{i=1}^{2}\sum_{n=1}^{\infty} \frac{\psi_n(x)}{C_n} \int_a^b \chi_n(x)L_n(X_i)\, dx \int_0^t T_i(\tau) \exp\{-\lambda_n(t-\tau)\}\, d\tau, \end{aligned} \tag{3.12}$$

where X_i and T_i are given by (2.8), (2.9) and (2.6).

4. As written in (3.12) the solution contains the functions X_i which are rather arbitrary: they have only to satisfy (2.5). These functions have already been determined; however, it is possible to eliminate them in the final solution. To this end we make use of the Green's formula relating the two systems (3.1), (3.2) and (3.4), (3.5)

$$\begin{aligned} &\int_a^b \{\chi_n L_n(X_i) - X_i L_n^*(\chi_n)\}\, dx \\ &= p_0(b)\chi_n(b)X_i'(b) - p_0(b)\chi_n'(b)X_i(b) - p_0'(b)\chi_n(b)X_i(b) + p_1(b)\chi_n(b)X_i(b) \\ &\quad - p_0(a)\chi_n(a)X_i'(a) + p_0(a)\chi_n'(a)X_i(a) + p_0'(a)\chi_n(a)X_i(a) - p_1(a)\chi_n(a)X_i(a) \\ &= U_1\{X_i\}V_4\{\chi_n\} + U_2\{X_i\}V_3\{\chi_n\} + U_3\{X_i\}V_2\{\chi_n\} + U_4\{X_i\}V_1\{\chi_n\}, \end{aligned} \tag{4.1}$$

where $U_3\ \{X_i\}$, $U_4\ \{X_i\}$ are linear combinations of $X_i(a)$, $X_i(b)$, $X_i'(a)$, $X_i'(b)$ and $V_3\ \{\chi_n\}$, $V_4\ \{\chi_n\}$ are linear combinations of $\chi_n(a)$, $\chi_n(b)$, $\chi_n'(a)$, $\chi_n'(b)$. U_1, U_2 and V_1, V_2 have been defined by (3.2) and (3.5) respectively. Noting that

$$L_n^*(\chi_n) = 0, \; V_1\{\chi_n\} = V_2\{\chi_n\} = 0,$$

and remembering $X_i(x)$ has to satisfy (2.5), we have

$$\begin{aligned} \int_a^b \chi_n L_n(X_i)\, dx = &\ \delta_{i1}\{p_0(a)\chi_n'(a) + p_0'(a)\chi_n(a) - p_1(a)\chi_n(a)\} \\ &+ \delta_{i2}\{-p_0(b)\chi_n'(b) - p_0'(b)\chi_n(b) + p_1(b)\chi_n(b)\} = \alpha_i V_4\{\chi_n\} + \beta_i V_3\{\chi_n\}, \end{aligned} \tag{4.2}$$

where δ_{i1}, δ_{i2} are the Kronecker deltas. Substituting the expressions for T_i as given by (2.6) and the result (4.2) just obtained into (3.12) we obtain the solution in the following form:

$$\begin{aligned} u(x, t) = &\sum_{n=1}^{\infty} \frac{\psi_n(x)}{C_n} \exp\{-\lambda_n t\} \int_a^b u_0(x)\chi_n(x)\, dx \\ &+ \sum_{n=1}^{\infty} \frac{\psi_n(x)V_4\{\chi_n\}}{C_n} \int_0^t f(\tau) \exp\{-\lambda_n(t-\tau)\}\, d\tau \\ &+ \sum_{n=1}^{\infty} \frac{\psi_n(x)V_3\{\chi_n\}}{C_n} \int_0^t g(\tau) \exp\{-\lambda_n(t-\tau)\}\, d\tau, \end{aligned} \tag{4.3}$$

where

$$V_3\{\chi_n\} = -\frac{1}{\sigma_{12}}\{\alpha_1[(p_0'(b) - p_1(b))\chi_n(b) + p_0(b)\chi_n'(b)] + \alpha_2[(p_0'(a) - p_1(a))\chi_n(a) + p_0(a)\chi_n'(a)]\}, \tag{4.4}$$

and

$$V_4\{\chi_n\} = \frac{1}{\sigma_{12}}\{\beta_1[(p_0'(b) - p_1(b))\chi_n(b) + p_0(b)\chi_n'(b)] + \beta_2[(p_0'(a) - p_1(a))\chi_n(a) + p_0(a)\chi_n'(a)]\}. \tag{4.5}$$

5. As an example of the previous discussions, consider the diffusion equation for the axi-symmetric case, $a < r < b$:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t}, \tag{5.1}$$

with the initial condition

$$u(r, t) = u_0(t), \qquad t = 0, \tag{5.2}$$

and the boundary conditions

$$\begin{cases}\alpha_1 u(a, t) + \alpha_2 u(b, t) + \alpha_3 u_r(a, t) + \alpha_4 u_r(b, t) = f(t), \\ \beta_1 u(a, t) + \beta_2 u(b, t) + \beta_3 u_r(a, t) + \beta_4 u_r(b, t) = g(t).\end{cases} \tag{5.3}$$

By following the same procedure as outlined in the previous paragraphs we are led to consider the system:

$$L_n\{\psi_n\} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + \lambda_n\right)\psi_n = 0, \tag{5.4}$$

$$\begin{cases}U_1\{\psi_n\} = \alpha_1\psi_n(a) + \alpha_2\psi_n(b) + \alpha_3\psi_n'(a) + \alpha_4\psi_n'(b) = 0, \\ U_2\{\psi_n\} = \beta_1\psi_n(a) + \beta_2\psi_n(b) + \beta_3\psi_n'(a) + \beta_4\psi_n'(b) = 0\end{cases} \tag{5.5}$$

and the adjoint system:

$$L_n^*\{\chi_n\} \equiv \left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + \lambda_n + \frac{1}{r^2}\right)\chi_n = 0, \tag{5.6}$$

$$\begin{cases}V_1\{\chi_n\} = \gamma_1\chi_n(a) + \gamma_2\chi_n(b) + \gamma_3\chi_n'(a) + \gamma_4\chi_n'(b) = 0, \\ V_2\{\chi_n\} = \delta_1\chi_n(a) + \delta_2\chi_n(b) + \delta_3\chi_n'(a) + \delta_4\chi_n^2(b) = 0,\end{cases} \tag{5.7}$$

where $\gamma_1, \cdots, \delta_4$ are to be determined. Now the fundamental solutions of (5.4) are $J_0(\lambda_n^{1/2}r)$, $Y_0(\lambda_n^{1/2}r)$, being Bessel functions of the first and the second kind of the zero order. The characteristic function of the system (5.4), (5.5) is therefore

$$\psi_n(r) = J_0(\lambda_n^{1/2}r) - EY_0(\lambda_n^{1/2}r), \tag{5.8}$$

where E is a constant and is determined by $U_1\{\psi_n\} = U_2\{\psi_n\} = 0$. Similarly the fundamental solutions of (5.6) are $rJ_0(\lambda_n^{1/2}r)$ and $rY_0(\lambda_n^{1/2}r)$, and the characteristic function of the system (5.6), (5.7) is

$$\chi_n(r) = rJ_0(\lambda_n^{1/2}r) - E'rY_0(\lambda_n^{1/2}r), \tag{5.9}$$

E' to be determined by $V_1\{\chi_n\} = V_2\{\chi_n\} = 0$. The Green's formula connecting the two systems is:

$$\int_a^b \{\chi_n L_n(\psi_n) - \psi_n L_n^*(\chi_n)\}\, dr$$

$$= \psi_n'(b)\chi_n(b) - \psi_n(b)\chi_n'(b) + \frac{1}{b}\psi_n(b)\chi_n(b) \tag{5.10}$$

$$- \psi_n'(a)\chi_n(a) + \psi_n(a)\chi_n'(a) - \frac{1}{a}\psi_n(a)\chi_n(a)$$

$$= U_1\{\psi_n\}V_4\{\chi_n\} + U_2\{\psi_n\}V_3\{\chi_n\} + U_3\{\psi_n\}V_2\{\chi_n\} + U_4\{\psi_n\}V_1\{\chi_n\}.$$

Here $U_1\{\psi_n\}$, $U_2\{\psi_n\}$ have already been defined, as by (5.5); if we take[4]

$$U_3\{\psi_n\} = \frac{1}{\sigma_{13}\sigma_{24}}\left\{\alpha_4\frac{\sigma_{13}}{\sigma_{24}}\psi_n(a) + \alpha_3\psi_n(b)\right\}, \tag{5.11}$$

$$U_4\{\psi_n\} = -\frac{1}{\sigma_{13}\sigma_{24}}\left\{\beta_4\frac{\sigma_{13}}{\sigma_{24}}\psi_n(a) + \beta_3\psi_n(b)\right\}, \tag{5.12}$$

we can find $V_1\{\chi_n\}, \cdots, V_4\{\chi_n\}$ by comparing the coefficients of $\psi_n(a)$, $\psi_n(b)$, $\psi_n'(a)$, $\psi_n'(b)$ in (5.10). This results[5]

$$V_1\{\chi_n\} = \alpha_3\sigma_{24}\left\{\left(\frac{1}{a} - \frac{\alpha_1}{\alpha_3}\right)\chi_n(a) - \chi_n'(a)\right\} + \alpha_4\sigma_{13}\left\{\left(\frac{1}{b} - \frac{\alpha_2}{\alpha_4}\right)\chi_n(b) - \chi_n'(b)\right\}, \tag{5.13}$$

$$V_2\{\chi_n\} = \beta_3\sigma_{24}\left\{\left(\frac{1}{a} - \frac{\beta_1}{\beta_3}\right)\chi_n(a) - \chi_n'(a)\right\} + \beta_4\sigma_{13}\left\{\left(\frac{1}{b} - \frac{\beta_2}{\beta_4}\right)\chi_n(b) - \chi_n'(b)\right\}, \tag{5.14}$$

$$V_3\{\chi_n\} = \frac{1}{\sigma_{34}}\{\alpha_4\chi_n(a) + \alpha_3\chi_n(b)\}, \tag{5.15}$$

$$V_4\{\chi_n\} = -\frac{1}{\sigma_{34}}\{\beta_4\chi_n(a) + \beta_3\chi_n(b)\}. \tag{5.16}$$

Now we introduce

$$\Omega_{ij} = J_i(\lambda_n^{1/2}a)Y_j(\lambda_n^{1/2}b) - J_j(\lambda_n^{1/2}b)Y_i(\lambda_n^{1/2}a), \tag{5.17}$$

$$\begin{cases} P(\alpha) = -\alpha_1\Omega_{01} + \alpha_3\lambda_n^{1/2}\Omega_{11}, \\ Q(\alpha) = -\alpha_1\Omega_{00} + \alpha_3\lambda_n^{1/2}\Omega_{10}, \\ R(\alpha) = \alpha_2\Omega_{00} - \alpha_4\lambda_n^{1/2}\Omega_{01}, \\ S(\alpha) = \alpha_2\Omega_{10} - \alpha_4\lambda_n^{1/2}\Omega_{11}, \end{cases} \tag{5.18}$$

$$\begin{cases} K(\alpha) = \alpha_1 J_0(\lambda_n^{1/2}a) - \alpha_3\lambda_n^{1/2}J_1(\lambda_n^{1/2}a), \\ L(\alpha) = \alpha_2 J_0(\lambda_n^{1/2}b) - \alpha_4\lambda_n^{1/2}J_1(\lambda_n^{1/2}b), \\ M(\alpha) = \alpha_1 Y_0(\lambda_n^{1/2}a) - \alpha_3\lambda_n^{1/2}Y_1(\lambda_n^{1/2}a), \\ N(\alpha) = \alpha_2 Y_0(\lambda_n^{1/2}b) - \alpha_4\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b), \end{cases} \tag{5.19}$$

[4] Of course, other forms of $U_3\{\psi_n\}$ and $U_4\{\psi_n\}$ may be chosen as long as $U_1\{\psi_n\}, - - -, U_4\{\psi_n\}$ form an independent set in the quantities $\psi_n(a)$, $\psi_n(b)$, $\psi_n'(a)$, $\psi_n'(b)$.

[5] Equations (5.15) and (5.16) are equivalent to (4.4) and (4.5) if we put in the latter $p_0 = 1$, $p_1(a) = 1/a$, etc., and make use of $V_1\{\chi_n\} = V_2\{\chi_n\} = 0$.

and similar expressions for $P(\beta), \cdots, N(\beta)$. Also let $\sigma = b\sigma_{13}/a\sigma_{24}$, then, using the fact that $J_0'(\lambda_n^{1/2}r) = -\lambda_n^{1/2} J_1(\lambda_n^{1/2}r)$, $Y_0'(\lambda_n^{1/2}r) = -\lambda_n Y_1(\lambda_n^{1/2}r)$, where the prime denotes the derivative with respect to r and that $J_1(\lambda_n^{1/2}r)Y_0(\lambda_n^{1/2}r) - Y_1(\lambda_n^{1/2}r)J_0(\lambda_n^{1/2}r) = 2/\pi\lambda_n^{1/2}r$ we obtain, after some algebraic details,

$$E = \frac{K(\alpha) + L(\alpha)}{M(\alpha) + N(\alpha)} = \frac{K(\beta) + L(\beta)}{M(\beta) + N(\beta)}, \tag{5.20}$$

$$\psi_n(a) = \frac{R(\alpha) + 2\alpha_3/\pi a}{M(\alpha) + N(\alpha)} = \frac{R(\beta) + 2\beta_3/\pi a}{M(\beta) + N(\beta)}, \tag{5.21}$$

$$\psi_n(b) = \frac{Q(\alpha) + 2\alpha_4/\pi b}{M(\alpha) + N(\alpha)} = \frac{Q(\beta) + 2\beta_4/\pi b}{M(\beta) + N(\beta)}, \tag{5.22}$$

$$E' = \frac{K(\alpha) + \sigma L(\alpha)}{M(\alpha) + \sigma N(\alpha)} = \frac{K(\beta) + \sigma L(\beta)}{M(\beta) + \sigma N(\beta)}, \tag{5.23}$$

$$\chi_n(a) = a\frac{\sigma R(\alpha) + 2\alpha_3/\pi a}{M(\alpha) + \sigma N(\alpha)} = a\frac{\sigma R(\beta) + 2\beta_3/\pi a}{M(\beta) + \sigma N(\beta)}, \tag{5.24}$$

$$\chi_n(b) = b\frac{Q(\alpha) + 2\alpha_4\sigma/\pi b}{M(\alpha) + \sigma N(\alpha)} = b\frac{Q(\beta) + 2\beta_4\sigma/\pi b}{M(\beta) + \sigma N(\beta)}. \tag{5.25}$$

Furthermore, since $\psi_m(r)$ and $\chi_n(r)$ are orthogonal in the interval (a, b), we have

$$\int_a^b \psi_m(r)\chi_n(r)\,dr = 0, \qquad m \neq n,$$

and

$$\begin{aligned}\int_a^b &\psi_n(r)\chi_n(r)\,dr \\ &= \frac{b^2}{2}\frac{\{P(\alpha) + 2\alpha_2/\pi\lambda^{1/2}b\}\{P(\alpha) + 2\alpha_2\sigma/\pi\lambda_n^{1/2}b\} + \{Q(\alpha) + 2\alpha_4/\pi b\}\{Q(\alpha) + 2\alpha_4\sigma/\pi b\}}{\{M(\alpha) + N(\alpha)\}\{M(\alpha) + \sigma N(\alpha)\}} \\ &- \frac{a^2}{2}\frac{\{R(\alpha) + 2\alpha_3/\pi a\}\{\sigma R(\alpha) + 2\alpha_3/\pi a\} + \{S(\alpha) + 2\alpha_1/\pi\lambda_n^{1/2}a\}\{\sigma S(\alpha) + 2\alpha_1/\pi\lambda_n^{1/2}a\}}{\{M(\alpha) + N(\alpha)\}\{M(\alpha) + \sigma N(\alpha)\}} \\ &= C_n\,. \end{aligned} \tag{5.26}$$

In the last expression we can replace all the $P(\alpha), \cdots, N(\alpha)$ by the corresponding $P(\beta), \cdots, N(\beta)$. The characteristic numbers λ_n are the roots of the equation

$$\sigma_{34}\Omega_{11}\lambda_n - (\sigma_{14}\Omega_{01} + \sigma_{32}\Omega_{10})\lambda_n^{1/2} + \sigma_{12}\Omega_{00} + \frac{2}{\pi}\left(\frac{\sigma_{13}}{a} + \frac{\sigma_{24}}{b}\right) = 0. \tag{5.27}$$

With these preliminaries the solution of the given problem can be written down according to (4.3). We note in particular that the system (5.4), (5.5) becomes self-adjoint if $a\sigma_{24} = b\sigma_{13}$, i.e., if $\sigma = 1$, in which case $E = E'$ and $\chi_n(r) = r\psi_n(r)$.

6. We now consider a special case of the previous example. Let

$$\begin{cases}\alpha_1 = -1, & \alpha_2 = 0, \quad \alpha_3 = \alpha, \quad \alpha_4 = 0, \\ \beta_1 = 0, & \beta_2 = 1, \quad \beta_3 = 0, \quad \beta_4 = \beta,\end{cases} \tag{6.1}$$

then the boundary conditions are reduced to

$$\begin{cases} -u(a, t) + \alpha u_r(a, t) = f(t), \\ u(b, t) + \beta u_r(b, t) = g(t). \end{cases} \tag{6.2}$$

This problem arises in the evaluation of transient temperature distribution in a homogeneous hollow circular cylinder, $a \leqq r \leqq b$, when the gas temperatures inside and outside the cylinder are functions of time, being $f(t)$ and $g(t)$ respectively, and the initial temperature distribution in the cylinder is $u_0(r)$. The constants α and β are associated with the heat transfer coefficient and the thermal conductivity of the cylinder material (here $\alpha, \beta > 0$). Then $a\sigma_{24} = b\sigma_{13} = 0$, and the system is self-adjoint. Here we have

$$E = E' = \frac{J_0(\lambda_n^{1/2}a) + \alpha\lambda_n^{1/2}J_1(\lambda_n^{1/2}a)}{Y_0(\lambda_n^{1/2}a) + \alpha\lambda_n^{1/2}Y_1(\lambda_n^{1/2}a)} = \frac{J_0(\lambda_n^{1/2}b) - \beta\lambda_n^{1/2}J_1(\lambda_n^{1/2}b)}{Y_0(\lambda_n^{1/2}b) - \beta\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b)}, \tag{6.3}$$

$$\psi_n(r) = \frac{1}{r}\chi_n(r) = J_0(\lambda_n^{1/2}r) - EY_0(\lambda_n^{1/2}r) \tag{6.4}$$

$$\psi_n(a) = \frac{1}{a}\chi_n(a) = -\frac{2\alpha/\pi a}{Y_0(\lambda_n^{1/2}a) + \alpha\lambda_n^{1/2}Y_1(\lambda_n^{1/2}a)} = \frac{\Omega_{00} - \beta\lambda_n^{1/2}\Omega_{01}}{Y_0(\lambda_n^{1/2}b) - \beta\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b)} \tag{6.5}$$

$$\psi_n(b) = \frac{1}{b}\chi_n(b) = -\frac{\Omega_{00} + \alpha\lambda_n^{1/2}\Omega_{10}}{Y_0(\lambda_n^{1/2}a) + \alpha\lambda_n^{1/2}Y_1(\lambda_n^{1/2}a)} = \frac{2\beta/\pi b}{Y_0(\lambda_n^{1/2}b) - \beta\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b)} \tag{6.6}$$

$$\begin{aligned} C_n &= \int_a^b \psi_n(r)\chi_n(r)\,dr = \int_a^b r\psi_n^2(r)\,dr \\ &= \frac{2}{\pi^2\lambda_n}\left\{\frac{1+\beta^2\lambda_n}{\{Y_0(\lambda_n^{1/2}b) - \beta\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b)\}^2} - \frac{1+\alpha^2\lambda_n}{\{Y_0(\lambda_n^{1/2}a) + \alpha\lambda_n^{1/2}Y_1(\lambda_n^{1/2}b)\}^2}\right\}, \end{aligned} \tag{6.7}$$

and λ_n is a root of

$$\alpha\beta\Omega_{11}\lambda_n - (\alpha\Omega_{10} - \beta\Omega_{01})\lambda_n^{1/2} - \Omega_{00} = 0. \tag{6.8}$$

Furthermore, from (5.15) and (5.16) we have

$$V_3\{\chi_n\} = \chi_n(b)/\beta,$$
$$V_4\{\chi_n\} = -\chi_n(a)/\alpha,$$

so that the solution of this special case can be written down, according to (4.3),

$$\begin{aligned} u(r, t) = \sum_{n=1}^{\infty} \frac{\psi_n(r)}{C_n} \exp(-\lambda_n t) \int_a^b r u_0(r)\psi_n(r)\,dr \\ - \frac{a}{\alpha}\sum_{n=1}^{\infty}\frac{\psi_n(r)\psi_n(a)}{C_n}\int_0^t f(\tau)\exp\{-\lambda_n(t-\tau)\}\,d\tau \\ + \frac{b}{\beta}\sum_{n=1}^{\infty}\frac{\psi_n(r)\psi_m(b)}{C_n}\int_0^t g(\tau)\exp\{-\lambda_n(t-\tau)\}\,d\tau. \end{aligned} \tag{6.9}$$

The solution of this special case for $f(t)$ and $g(t)$ equal to constants has already been given by Carslaw and Jaeger [4], using the method of the Laplace transform. It is easy to verify that the result presented here is the same as that given in [4]. For, if $\varphi(r) =$

$\sum_{n=1}^{\infty} A_n\psi_n(r)$, $a < r < b$, then

$$A_n = \frac{1}{C_n}\int_a^b r\varphi(r)\psi_n(r)\,dr = -\frac{1}{C_n\lambda_n}\int_a^b r\varphi(r)\left(\psi_n'' + \frac{1}{r}\psi_n'\right)dr$$
$$= \frac{1}{C_n\lambda_n}\left\{\frac{b}{\beta}\varphi(b)\psi_n(b) + \frac{a}{\alpha}\varphi(a)\psi_n(a) + \int_a^b r\varphi'(r)\psi_n'(r)\,dr\right\}, \tag{6.10}$$

where use has been made of $-\psi_n(a) + \alpha\psi_n'(a) = \psi_n(b) + \beta\psi_n'(b) = 0$. By putting $\varphi(r) = 1$ and $\log r$ successively in (6.10) we can find the summation of the series $\sum_{n=1}^{\infty} \psi_n(r)\,\psi_n(a)/C_n\lambda_n$ and $\sum_{n=1}^{\infty} \psi_n(r)\,\psi_n(b)/C_n\lambda_n$. Replacing $f(t)$ and $g(t)$ by the constants f and g respectively in (6.9) and using the results just obtained after performing the integration, we get

$$u(r, t) = \sum_{n=1}^{\infty} \frac{\psi_n(r)}{C_n}\exp(-\lambda_n t)\int_a^b ru_0(r)\psi_n(r)\,dr$$
$$- \frac{af\{b\log b/r + \beta\} + bg\{a\log a/r - \alpha\}}{ab\log b/a + a\beta + b\alpha} \tag{6.11}$$
$$+ \sum_{n=1}^{\infty}\frac{\psi_n(r)}{C_n\lambda_n}\left\{\frac{af}{\alpha}\psi_n(a) - \frac{bg}{\beta}\psi_n(b)\right\}\exp(-\lambda_n t),$$

which is the form given by Carslaw and Jaeger. In this expression $\psi_n(r)$, $\psi_n(a)$ and $\psi_n(b)$ are given by (6.4), (6.5) and (6.6) respectively.

References

1. Courant and Hilbert, *Methods mathematical physics*, Vol. 1 (Interscience Publishers, 1953), 277.
2. E. A. Coddington and N. Levinsen, *Theory of ordinary differential equations*, McGraw-Hill, 1955
3. E. L. Ince, *Ordinary differential equations*, Dover, Chapters IX and X
4. H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, Oxford University Press, pp. 278-279, 1950