On the Solution of Fredholm Integral Equations of the First Kind

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Abstract: - As it is well known the problem of solving the Fredholm integral equation of the first kind belongs to the class of ill-posed problems. The Tikhonov regularization method is well known. This method is usually applied to an integral equation and a system of linear algebraic equations. The authors firstly propose to reduce the integral equation of the first kind to a system of linear algebraic equations. This system is usually extremely ill-posed. Therefore, it is necessary to carry out the Tikhonov regularization for the system of equations. In this paper, to form a system of linear algebraic equations, local polynomial and non-polynomial spline approximations of the second order of approximation are used. The results of numerical experiments are presented.

Key-Words: - Fredholm integral equation of the first kind, polynomial spline, non-polynomial spline, Tikhonov regularization.

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1 Introduction

As is well known, the problem of solving the Fredholm integral equation of the first kind

$$\int_a^b K(x,s)y(s)ds = f(x)$$

belongs to the class of ill-posed problems. This is manifested in the fact that the equation cannot be solved for an arbitrary right-hand side $f(x)$. If the equation has a solution, then this solution may not be the only one. In addition, small errors introduced in the right side of the equation can lead to large errors in the solution.

The Tikhonov regularization method for such problems is well known (see [1]). As a result a system of linear algebraic equations is obtained. The approximate values of the required function are found at the grid nodes after solving the system of the linear algebraic equations. Richard Bellman said that the theory of matrices can be regarded as the arithmetic of higher mathematics and that solving a linear system is the most fundamental problem in analysis, to which researchers attempt to reduce many other problems and, to their horror, are sometimes successful.

Suppose an integral equation has a unique solution. The authors propose to first reduce the integral equation of the first kind to a system of linear algebraic equations. This system is usually extremely ill-posed. Therefore, it is necessary to carry out the Tikhonov regularization for the system of equations ([2], [3]).

There are many methods for reducing an integral equation to a system of linear algebraic equations [4]. Good results are obtained by using various quadrature formulas (middle rectangles, trapezoids, Simpson's or more exact quadrature formulas). The method of moments (Galerkin's method) gives a good result. At present, spline approximations are used to solve various problems [3], [5]. Of particular interest are the approximations using integro-differential splines [5]. To solve the Fredholm integral equation of the second kind, they gave a lesser error than when using traditional methods [4]. Note that recently many authors have investigated the feasibility and effectiveness of using splines and wavelets to construct a numerical solution of integral equations.

A new collocation technique for the numerical solution of the Fredholm, Volterra and mixed Volterra-Fredholm integral equations of the second kind was introduced and also developed a numerical integration formula on the basis of the linear Legendre multi-wavelets in paper [6].

In [7], the tension spline approximation to obtain the numerical solution of the Volterra–Fredholm integral equation is developed.

In paper [8], a general spline maximum entropy method for the approximation of solutions for solving the Fredholm integral equations is described.

In paper [9], the quadratic rule for the numerical solution of the linear and nonlinear two-dimensional Fredholm integral equations based on spline quasi-interpolant is suggested.

In paper [10], the solution of the Fredholm integro-differential equations of the second kind is approximate by using an exponential spline function.

In paper [11], an iterative numerical method for approximating the solution of fuzzy functional integral equations of the Fredholm type is proposed.

In paper [12], a numerical scheme based on the modified moving least-square (MMLS) method for solving the Fredholm–Hammerstein integral equations on 2D irregular domains is described.

The approximations with splines on the irregular set of nodes are of particular interest [13].

The application of the generalized Haar spaces is very useful [14].

In this paper, we compare the results of applying various methods for solving the Fredholm integral equation of the first kind under the assumption that the solution is unique.

2 Problem Formulation

The integral equation

$$Az \equiv \int_0^1 K(x,s)y(s)ds = f(x), x \in [0,1] \quad (1)$$

is considered.

Here the kernel $K(x,s)$ is a given continuous function, $y(s)$ is an unknown continuous function, and the right-side $f(x)$ belongs to space $L_2[0,1]$ or $C[0,1]$. The problem of solving the equation

$$Az = f$$

belongs to the class of ill-posed problems (see [1]). Suppose the operator A^{-1} exists but is not limited. In this case, small errors introduced in the right side of the equation can lead to large errors in the solution.

Let us choose some quadrature formula of the form

$$\int_0^1 g(s)ds \approx \sum_{k=1}^{n} A_k g(s_k)$$

and apply it to calculate the integral in (1), as a result of which we arrive at the approximate equation

$$\sum_{k=1}^{n} A_k K(x,s_k) y(s_k) = f(x). \quad (2)$$

Let us substitute $x = s_j$, $j = 1,\dots,n$. As a result we obtain a system of linear algebraic equations (SLAE)

$$\sum_{k=1}^{n} A_k K_{jk} z_k = f_j, \quad j = 1,\dots,n.$$

Here $z_k = y(s_k)$. Let us write it in the matrix form:

$$CZ = F. \quad (3)$$

This system is ill-posed, i.e. its condition number is large, and to solve it, it is necessary to apply regularization methods. In standard form, the Tikhonov regularization method [1] leads to the Euler equation

$$(C^*C + \alpha E)Z = C^*F,$$

whose solution, for some choice $\alpha \downarrow 0$, tends to the normal solution of the system $CZ = F$ (see [1]). Let us call it the framework of an approximate solution to equation (1) by analogy with the general scheme of approximate methods [4]. In [2], methods of constructing a SLAE of the Euler equation with the smallest condition numbers are described.

The described scheme fits the following approach: we will obtain an approximate solution of equation (1) in the form of an expansion

$$z_n(s) = \sum_{k=1}^{n} c_k w_k(s)$$

in some system of functions $\{\omega_k\}$. After substituting this expansion into the left-hand side of equation (1), we calculate the integrals

$$\int_0^1 K(x,s)w_k(s)ds,$$

and substitute $x_j, j = 1, \dots, n$, instead of x. Now we arrive at an equation of the form (3), and then proceed as described above. As $\{w_k\}$ we can select polynomial or non-polynomial spline functions (see [3], [5]).

Book [1] describes the following regularization method for the original equation (1).

Suppose that there is a unique solution to the original equation (1), which belongs to some compact subset of the space $C[0,1]$. To define the compact set, a functional

$$\Omega(z) = \int_0^1 (z^2(s) + p(z')^2)ds, p > 0$$

is used. The functional is nonnegative in the space $L_2(0,1)$.

It has the property that the set of all elements z satisfying the inequality $\Omega(z) \leq d$ is compact in $C[0,1]$ for any number $d > 0$. Then the problem is posed to minimize the functional

$$M_\alpha(A,f,z) = ||Az - f||_{L_2}^2 + \alpha\Omega(z)$$

for a fixed positive value of the regularization parameter α. It leads to the Euler equation

$$(A^*A + \alpha L)z = A^*f,\ Lz = z(s) - pz''(s),\ p > 0,$$

with some boundary conditions [1].

Solutions of the Euler equation, for a certain choice $\alpha \downarrow 0$, converge to the desired solution. Of course, to solve this integro-differential equation, the discretization of this integro-differential equation is required. The discretization leads to a SLAE.

The Galerkin method for solving our problem is effective. We obtain an approximate solution in the form

$$z_n(s) = \sum_{k=1}^n c_k w_k(s).$$

The coefficients c_k are determined from the condition of equality of the moments

$$(Az_n, w_j) = (f, w_j), j = 1, \dots, n,$$

which leads to the SLAE

$$\sum_{k=1}^n c_k(Aw_k, w_j) = (f, w_j), j = 1, \dots, n.$$

This SLAE is ill-posed, and the techniques described above should be used to regularize it. Note the rather high complexity of the method and its high efficiency.

3 Numerical experiments

The polynomial basis splines can be taken in the form:

$$\omega_j(s) = \frac{s - x_{j+1}}{x_j - x_{j+1}}, \qquad s \in [x_j, x_{j+1}],$$
$$\omega_{j+1}(s) = \frac{s - x_j}{x_{j+1} - x_j}, \qquad s \in [x_j, x_{j+1}].$$

The function $u(s)$ can be approximated by the polynomial spline (see [15])

$$U(s) = u(x_j)\omega_j(s) + u(x_{j+1})\omega_{j+1}(s), \quad s \in [x_j, x_{j+1}], \tag{4}$$

llowing theorem was proved in [15].

Let us denote

$$\| u'' \|_{[x_j,x_{j+1}]} = \max_{[x_j,x_{j+1}]} |u''(x)|, h_j = x_{j+1} - x_j.$$

Theorem 1. Let $u \in C^2[a,b]$. To approximate the function $u(x)$, $x \in [x_j, x_{j+1}]$, by spline (4), the following inequality is valid:

$$| u(x) - U(x)| \leq K{h_j}^2 \| u'' \|_{[x_j,x_{j+1}]}, K = 1/8.$$

Proof. It is easy to notice that U is an interpolation polynomial of the first degree, and x_j, x_{j+1} are the interpolation nodes, $u(x_j) = U(x_j)$, $u(x_{j+1}) = U(x_{j+1})$. Using the remainder term we get

$$u(x) - U(x) = \frac{u''(\tau)}{2!}(x - x_j)(x - x_{j+1}).$$

Thus,

$$\max_{[x_j,x_{j+1}]} |u(x) - U(x)| \leq \frac{1}{8}{h_j}^2 \max_{[x_j,x_{j+1}]} |u''(x)|.$$

The details can be seen in paper [15]. The proof is complete.

As it is shown in paper [5], if the functions φ_1, φ_2 form a Chebyshev system, then the basis functions ω_k, $k = j, j+1$, can be determined by solving the system of equations

$$\varphi_1(x_j)\omega_j(x) + \varphi_1(x_{j+1})\omega_{j+1}(x) = \varphi_1(x),$$

$$\varphi_2(x_j)\omega_j(x) + \varphi_2(x_{j+1})\omega_{j+1}(x) = \varphi_2(x),$$

$$x \in [x_j, x_{j+1}].$$

The determinant of the system should be not equal zero.

We construct a non-polynomial approximation of function $u(x), u \in C^2[a,b]$, on each grid interval $[x_j, x_{j+1}]$ in the form

$$U(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x), \quad x \in [x_j, x_{j+1}]. \qquad (5)$$

Suppose $\varphi_1(x) = 1, \varphi_2(x) = \varphi(x)$. In this case the non-polynomial basis splines can be taken in the form:

$$\omega_j(x) = \frac{\varphi(x) - \varphi(x_{j+1})}{\varphi(x_j) - \varphi(x_{j+1})},$$

$$\omega_{j+1}(x) = \frac{\varphi(x) - \varphi(x_j)}{\varphi(x_{j+1}) - \varphi(x_j)}.$$

We can take $\varphi(x) = \exp(x)$ or $(x) = \exp(-x)$. The error of approximation for this case can be seen in paper [15]. When $\varphi(x) = \exp(-x)$the basis functions $\omega_j(x)$, $\omega_{j+1}(x)$ have the form:

$$\omega_j(x_j + th) = (\exp(h - th) - 1)/(\exp(h) - 1),$$

$$\omega_{j+1}(x_j + th) = (\exp(h) - \exp(h - th))/(\exp(h) - 1)),$$

The plots of the basis functions when $h = 1$ are given in the Fig.2

When $\varphi(x) = \exp(x)$ the basis functions $\omega_j(x)$, $\omega_{j+1}(x)$ have the form:

$$\omega_j(x_j + th) = (\exp(h) - \exp(th))/(\exp(h) - 1)),$$
$$\omega_{j+1}(x_j + th) = (\exp(th) - 1)/(\exp(h) - 1)).$$

The plots of the basis functions $h = 1$ are given in the Fig.3.

Now let us investigate the approximation of function $u(x)$ when we use the system $\varphi_1(x) = \cos(x), \varphi_2(x) = \sin(x)$. In this case we obtain $\omega_j(x_j + th) = \frac{\sin(h-th)}{\sin(h)}, \omega_{j+1}(x_j + th) = \frac{\cos(th)}{\sin(h)}$ when $x \in [x_j, x_{j+1}]$, $x = x_j + th$, $x_{j+1} = x_j$, $t \in [0, 1]$.

To estimate the approximation error, we apply the method described in paper [5].

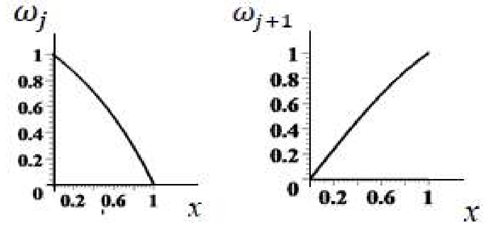

Fig.1. The plots of the basis functions $\omega_j(x_j + th) = \frac{\sin(h-th)}{\sin(h)}, \omega_{j+1}(x_j + th) = \frac{\cos(th)}{\sin(h)}$.

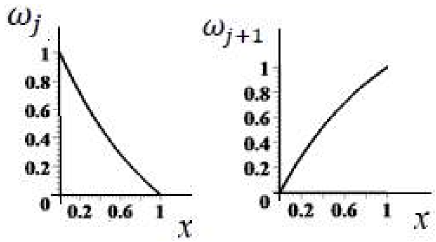

Fig.2. The plots of the basis functions $\omega_j(x_j + th)$, $\omega_{j+1}(x_j + th)$ when $\varphi(x) = \exp(-x)$.

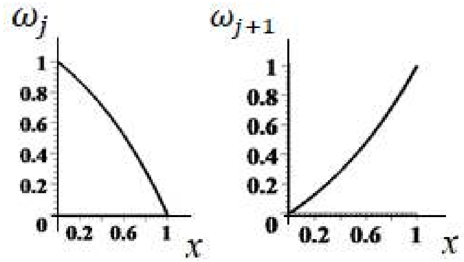

Fig.3. The plots of the basis functions $\omega_j(x_j + th)$, $\omega_{j+1}(x_j + th)$ when $\varphi(x) = \exp(x)$.

Note that this formula for function interpolation can also be applied on a uniform grid of nodes.

Table 1 shows the actual errors of approximation of some functions $u(x)$ obtained with the use of the polynomial and non-polynomial splines when $h = 0.1$, $[a, b] = [-1,1]$.

Table 2 shows the theoretical errors of approximation of some functions obtained with the use of the polynomial splines when $h = 0.1$, $[a, b] = [-1,1]$.

Table 1. The actual errors of approximation of some functions obtained with the use of the polynomial and non-polynomial splines

$u(x)$	$\varphi_1(x) = 1$, $\varphi_2(x) = x$.	$\varphi_1(x) = \cos(x)$, $\varphi_2(x) = \sin(x)$	$\varphi_1(x) = 1$, $\varphi_2(x) = \exp(-x)$
$\exp(x)$	0.00323	0.00647	0.00646
$\sin(x)$	0.00102	0.0	0.00177
x^2	0.00250	0.00363	0.00487
$\exp(-x)$	0.00323	0.00647	0.0
$\sin(2x)$	0.00498	0.00374	0.00558

Table 2. The theoretical errors of approximation of some functions with the polynomial splines

$u(x)$	$\max_{[-1,1]} \|u - U\|$
$\exp(x)$	0.00340
$\sin(x)$	0.00105
x^2	0.00250

Let ordered distinct nodes $\{x_j\}$ be such that $x_{j+1} - x_j = h$.

Theorem 2. Let function $u(x)$ be such that $u \in C^2([a,b])$. Suppose the basis splines $\omega_j(x), \omega_{j+1}(x)$ are constructed when

$$U(x) = u(x), u(x) = \varphi_1(x), \varphi_2(x),$$
$$\text{for } x \in [x_i, x_{i+1}],$$
$$\varphi_1(x) = \cos(x)\,, \varphi_2(x) = \sin(x).$$

Then, $x \in [x_i, x_{i+1}]$ we have

$$|u(x) - U(x)| \ \le K_2 h^2 \|Lu\|,$$
$$x \in [x_j, x_{j+1}],\ K_2 > 0.$$

Here $Lu = u''(x) + u(x)$.

Proof. In the case of the non-polynomial splines as it was shown in paper [5] we construct a homogeneous equation, which has a fundamental system of solutions $\varphi_1(x) = \cos(x)\,, \varphi_2(x) = \sin(x)$.

$$Lu = \begin{vmatrix} \cos(x) & \sin(x) & u(x) \\ -\sin(x) & \cos(x) & u'(x) \\ -\cos(x) & -\sin(x) & u''(x) \end{vmatrix} = 0.$$

It is easy to see, that the Wronskian $W(x) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$ does not equal zero. Now we can construct a general solution of the nonhomogeneous equation $Lu = F$ by the method of variation of the constants.

Expanding the determinant according to the elements of the last column and dividing all terms of the equation by $W(x)$ we obtain the equation $Lu = 0$ in the form. $u'' + qu' + pu = 0$. Here q and p are some coefficients.

After we have constructed a general solution of nonhomogeneous equation $Lu = F$ by the method of variation of the constants we obtain the function $u(x), x \in [x_j, x_{j+1}]$.

$$u(x) = \int_{x_j}^{x} \sin(t)\,(u''(t) + u(t))dt + c_1\cos(x) + c_2\sin(x).$$

Here c_i, $i = 1, 2,$ are some arbitrary constants,$x \in [x_i, x_{i+1}]$. We construct the approximation of $u(x)$ in the form:

$$U(x) = u(x_j)\omega_j(x) + u(x_{j+1})\omega_{j+1}(x),$$
$$x \in [x_j, x_{j+1}].$$

Thus, using the results from paper [5], we get

$$|u(x) - U(x)| \ \le K_2 h^2 \|Lu\|, x \in [x_j, x_{j+1}],$$

The proof is complete.

In Table 3 we compare the actual errors of approximation of some functions with the polynomial splines and trigonometrical splines when $h = 0.1$.

Table 3. The actual errors of approximation of some functions with the polynomial splines and trigonometrical splines

$u(x)$	$\max_{[-1,1]} \|u - U\|$ Polynomial splines	$\max_{[-1,1]} \|u - U\|$ Trigonometric splines
$1/(1 + 25x^2)$	0.0418	0.0407
$\sin(5x)$	0.0311	0.0298
$\sin(2x)$	0.00498	0.00374

The plots of the errors of approximation of the Runge function $u(x) = 1/(1 + 25x^2)$, $u(x) = \sin(5x)$, $u = \sin(2x)$, with the polynomial splines when $h = 0.1$ is shown in Figs.4, 7, 9. The plots of the errors of approximation of the Runge function $1/(1 + 25x^2)$with the trigonometric splines when $h = 0.1$ is shown in Fig.5, 6, 8.

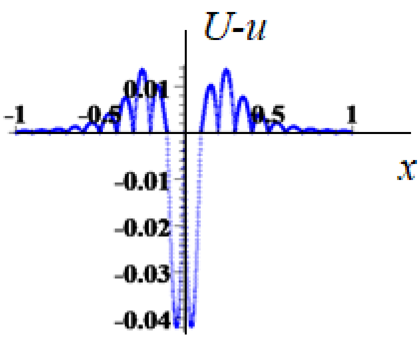

Fig.4.The plot of the error of approximation of the Runge function $1/(1+25x^2)$ with the polynomial splines.

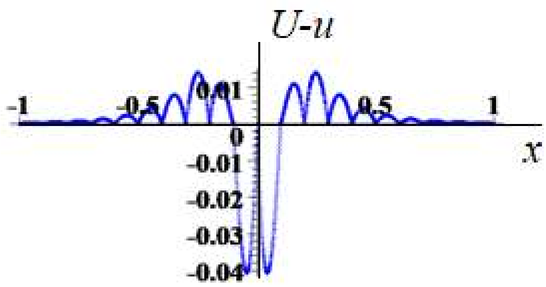

Fig.5.The plot of the error of approximation of the Runge function $1/(1+25x^2)$ with the trigonometric splines.

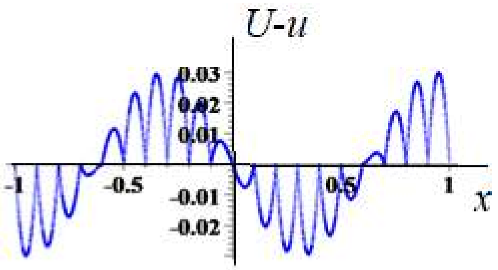

Fig.6.The plot of the error of approximation of function $\sin(5x)$, with the trigonometric splines.

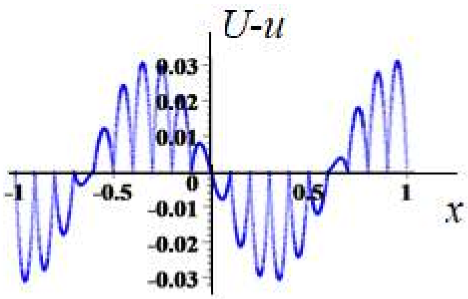

Fig.7.The plot of the error of approximation of function $\sin(5x)$ with the polynomial splines.

Having analyzed the results of the presented numerical experiments, we can draw the following conclusions: A decrease in the approximation error is possible both by increasing the number of nodes and by choosing a different type of approximation for a given approximation order.

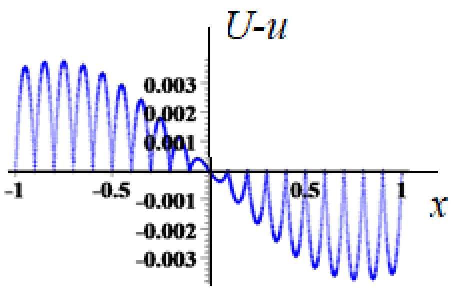

Fig.8.The plot of the error of approximation of function $\sin(2x)$ with the trigonometric splines.

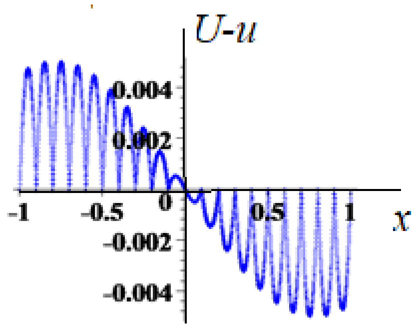

Fig.9.The plot of the error of approximation of function $\sin(2x)$ with the polynomial splines.

With a small number of grid nodes, the choice of trigonometric splines can significantly reduce the approximation error.

Now we will consider in detail the construction of computational schemes for solving the Volterra equation of the second kind. In the construction of the computational schemes, we will use the approximation formulas with polynomial and non-polynomial splines of the second order of approximation.

Transforming the integral $\int_{x_j}^{x_{j+1}} K(x,s)u(s)ds$ using formula (4) or (5), we obtain

$$\int_{x_j}^{x_{j+1}} K(x,s)u(s)ds$$

$$= u(x_j)\int_{x_j}^{x_{j+1}} K(x,s)\omega_j(s)ds +$$

$$u(x_{j+1})\int_{x_j}^{x_{j+1}} K(x,s)\omega_{j+1}(s)ds + O(h^3).$$

To construct a numerical method, we discard the error and denote $\tilde{u}(x_j) \approx u(x_j)$.

It is assumed that the integrals can be calculated exactly, or can be applied a quadrature formula with an error not less than $O(h^3)$.

The polynomial and non-polynomial splines are easy to apply on irregular grid of nodes.

Now apply the splines to the solution of the integral equations. Let us we have the equation

$$Az \equiv \int_0^1 K(x,s)y(s)ds = f(x), x \in [0,1].$$

We take $y(s) = 1$ and calculate the corresponding right-hand side $f(x)$ for the given kernel $K(x,s) = \cos(5 + x - s)$.

Next, we will consider the application of three methods for solving the integral equation. The calculations will be done using Maple with $\alpha = 10^{-15}$, Digits=20.

In the interval [0,1] we construct an equidistant grid of nodes $x_j, j = 1, \dots, n.$ Method 1 uses a composite middle rectangle rule. Methods 2 and 3 use piecewise linear polynomial splines. Note that the piecewise splines of the second order of approximation cannot guarantee the reliability of the result at the points x_0, x_n. The application of the composite trapezoidal rule also do not guarantee the reliability of the result at the points x_0, x_n.

Method 1. At first we take the composite middle rectangle rule with 32 equidistant nodes on the interval [0,1]. The plot of the error between the exact solution and the approximate solution obtained with the composite middle rectangle rule before regularization is given in Fig.10. Here we connected the points with straight line segments for clarity of the drawing. The graph shows the nodes of the grid at the interval [0,1] and the values of the errors at these nodes.

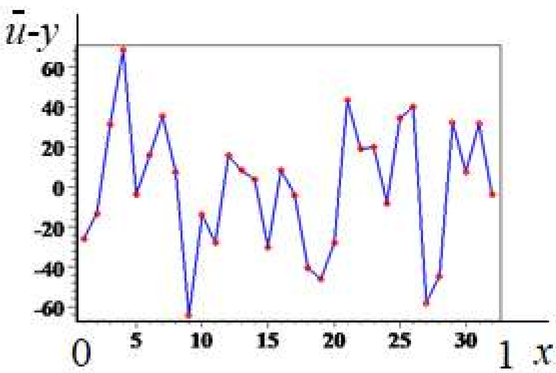

Fig.10. The plot of the error between the exact solution and the approximate solution before the regularization.

The plot of the error between the exact solution and the approximate solution obtained with the rule of composed middle rectangles after regularization is given in Fig.11. The graph shows the nodes of the grid at the interval [0,1] and the values of the errors at these nodes.

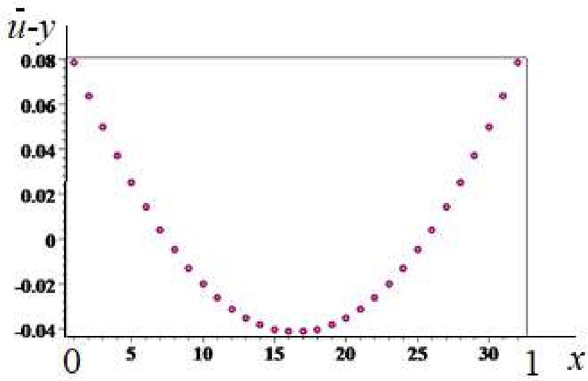

Fig.11.The plot of the error between the exact solution and the approximate solution after the regularization.

Method 2. Now again we take 32 equidistant nodes on the interval [0,1]. We calculate

$$\int_{x_j}^{x_{j+1}} K(x,s)y(s)ds$$

using the polynomial spline.

$$\int_{x_j}^{x_{j+1}} K(x,s)y(s)ds$$

$$= y(x_j)\int_{x_j}^{x_{j+1}} K(x,s)\omega_j(s)ds +$$

$$y(x_{j+1})\int_{x_j}^{x_{j+1}} K(x,s)\omega_{j+1}(s)ds + O(h^3).$$

To construct a numerical method, we discard the error and denote $\tilde{u}(x_j) \approx y(x_j)$. Let us introduce the notation

$$W_j(x) = \tilde{u}(x_j)\int_{x_j}^{x_{j+1}} K(x,s)\omega_j(s)ds +$$

$$\tilde{u}(x_{j+1})\int_{x_j}^{x_{j+1}} K(x,s)\omega_{j+1}(s)ds.$$

Next, we substitute x with x_k $(x = x_k)$ and solve the system of equations

$\sum_{j=1}^{n} W_j(x_k) = f(x_k), k = 1, \dots n.$ In this case, we first have to regularize this system according to Tikhonov method. The plot of the error between the exact solution and the approximate solution obtained with the polynomial splines before regularization is given in Fig.12. The graph shows the nodes of the grid at the interval [0,1] and the values of the errors at these nodes.

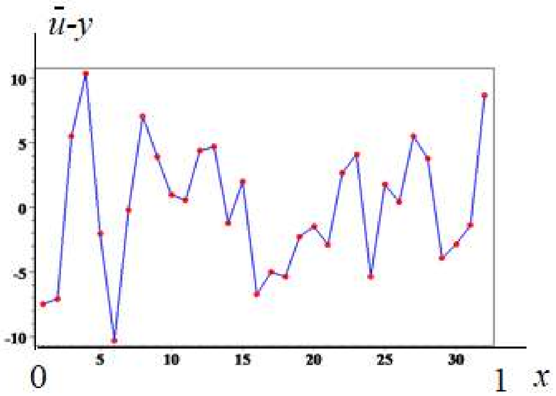

Fig.12.The plot of the error between the exact solution and the approximate solution before the regularization.

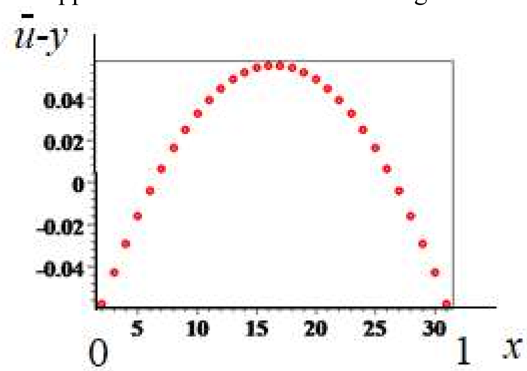

Fig.13. The plot of the error in absolute value between the exact solution and the approximate solution after the regularization.

The plot of the error between the exact solution and the approximate solution obtained with the polynomial splines after regularization is given in Fig.13. The graph shows the results of calculations at $j = 2, \dots, n-1$.

Method 3. Now we obtain an approximate solution using Galerkin's method. The plot of the error between the exact solution and the approximate solution obtained with the polynomial splines before regularization is given in Fig.14. The plot of the error between the exact solution and the approximate solution obtained with the polynomial splines after regularization is given in Fig.15. The graph shows the results of calculations at $j = 2, \dots, n-1$.

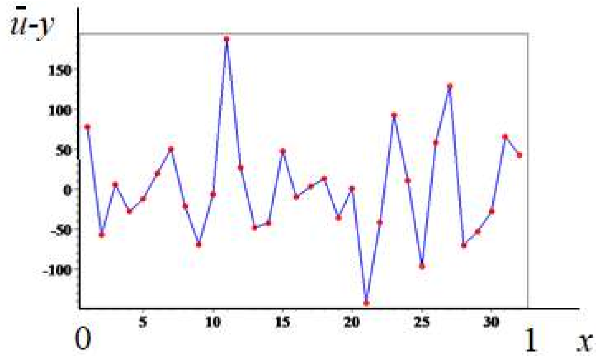

Fig.14.The plot of the error between the exact solution and the approximate solution before the regularization.

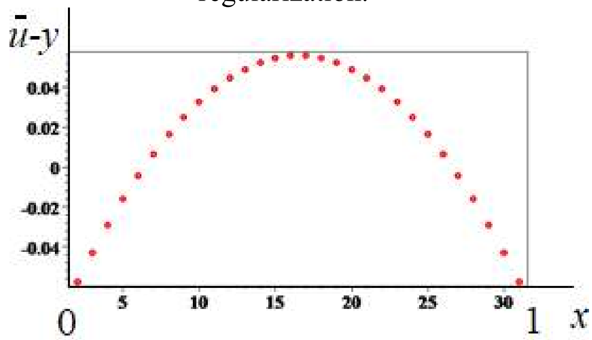

Fig.15. The plot of the error between the exact solution and the approximate solution after the regularization.

Table 4. The maximum of the error in absolute value between the exact solution of the equation and the numerical solutions obtained with methods1,2,3.

n	The error obtained with the use of the composite middle rectangle rule	The error obtained with the use of method 2	The error obtained with the use of method 3
64	$0.822 \cdot 10^{-1}$	$0.721 \cdot 10^{-1}$	$0.721 \cdot 10^{-1}$
32	$0.785 \cdot 10^{-1}$	$0.577 \cdot 10^{-1}$	$0.577 \cdot 10^{-1}$
16	$0.712 \cdot 10^{-1}$	$0.711 \cdot 10^{-1}$	$0.711 \cdot 10^{-1}$
8	$0.578 \cdot 10^{-1}$	0.107	0.107

Table 4 shows the maximum of the error in absolute value between the exact solution of the equation and the numerical solution obtained with the application of the composite middle rectangle rule (column 2). The numerical solution obtained with the application of method 2 with the polynomial splines is presented in column 3. The numerical solution obtained with the application of method 3 with the polynomial splines is given in column 4. The number of nodes (n) is given in the first column of Table 4.

The application of the non-polynomial splines gives us a similar result. Polynomial and nonpolynomial splines can be used at irregular grid nodes. The next question is how to connect the values of solution on the grid points. Using trigonomertic splines for connecting the points can provide a lesser error in solution at the point between two nodes of grid than when using polynomial splines.

This paper discusses the method for solving integral equations of the first kind by reducing it to a system of linear algebraic equations and subsequent regularization of this system of equations. This method is simple to use, but not very accurate. If we apply regularization to the Fredholm integral equation of the first kind, and then use some quadrature formula to obtain a system of linear algebraic equations, then the approximation error turns out to be smaller. This is shown by the following numerical experiment.

*Method 4 (*Tikhonov method*).* We begin with the regularization of the integral equation under consideration by the Tikhonov method, and then we apply the composite middle rectangle rule. Having solved the resulting system of linear algebraic equations, we obtain an approximate solution of the original integral equation. Fig. 16 shows the error of the solution when the number of nodes is 32.

Let us compare the solution errors obtained by both variants of the regularization (they are shown in Fig. 11 and Fig.16). It can be seen that the application of Tikhonov's regularization applied directly to the integral equation gives a smaller error. Note that this approach requires more work and knowledge of theory.

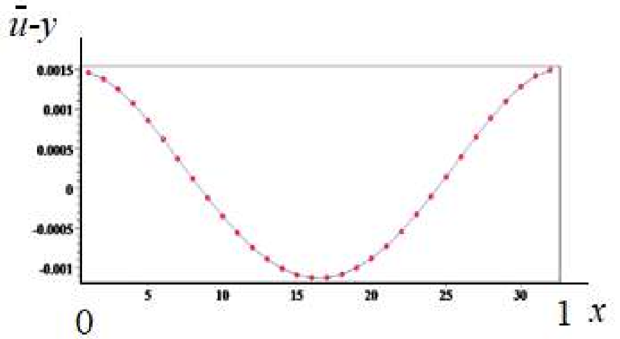

Fig.16. The plot of the error between the exact solution and the approximate solution after the regularization.

4 Conclusion

In this paper, we considered the results of solving the solution of the Fredholm integral equation of the first kind in several ways. Numerical experiments have shown that methods based on the use of splines can give a lesser error than when using traditional methods (for example, the composite middle rectangle rule). In the future, it is proposed to construct numerical methods based on splines of a higher order of approximation. As it was noted, piecewise-linear basic splines do not guarantee the accuracy of the result at the first node x_0 and at the last node x_n of the interval $[0,1]$. The best result is obtained with the number of nodes from 16 to 64. This paper proposes second order trigonometric splines. These splines ensure the coincidence of the approximation and the function being approximated if this function is a sine or cosine. As the results of numerical experiments show, these approximations (in comparison with polynomial approximations) give a smaller approximation error in absolute value, but may required more Digits in the mantissa of numbers for numerical calculations.

To reduce the error, we can apply Tikhonov's regularization directly to the integral equation. It will give a smaller error, but requires more work and knowledge of theory.

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