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On the Solutions of Generalized Fractional Kinetic Equations

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Abstract

In this paper, we derive the solution of generalized fractional kinetic equation involving the generalized M-series. The result obtained here is quite general in nature and capable of yielding a very large number of results (new and known) hitherto scattered in the literature. Special cases, involving the generalized Mittag-Leffler function and generalized Gauss hypergeometric function are also considered. The obtained results imply more precisely the known results.

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1 Introduction

The great importance of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of physics and astrophysics. A spherically symmetric non-rotating, self-gravitating model of star like the Sun is assumed to be in thermal equilibrium and hydrostatic equilibrium. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of state, nuclear energy generation rate and the opacity. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star (Kourganoff [11], Perdang [17] and Clayton [2]). Energy in such stellar structures is being produced by the process of chemical reactions (thermonuclear reactions). Computation of such chemical reactions is of the prime importance as it plays the central role in the evolution of such stellar structures. The two most important nuclear reactions (cycles) in stars, during their evolution, are pp chain (proton-proton chain) and CNO cycle (involves nuclei of carbon, nitrogen and oxygen). The total energy production and luminosity of the star is based on the pp chain and the composition of stellar plasma described by CNO cycle. The production and destruction of nuclei in such chemical reactions can be described by the reaction-type (kinetic) equations. Solutions of such reaction-type (linear/nonlinear) equations determine distribution functions of the dynamical states of a single particle. The linear reaction-type equation, $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}$ can be used to describe the fundamental principles of standard Boltzmann-Gibbs statistical mechanics. The nonlinear generalization of the reaction-type equation $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{y}^{\mathrm{q}}$, leads to new insights into generalized Boltzmann-Gibbs statistical mechanics which is also called nonextensive statistical mechanics. Recently, Ferro et al. [6] studied that a very small deviation from the Maxwell-Boltzman particle distribution and the use of nonextensive statistical mechanics can be applied to describe the modified nuclear reaction rates in stellar plasmas which is consistent with the need of the modification of the nuclear reaction rates of stellar plasma and their chemical composition.

Consider an arbitrary reaction characterized by a time dependent quantity $\mathrm{N} = \mathrm{N(t)}$. It is possible to calculate rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, that is $\mathrm{dN/dt} = -\mathrm{d}+\mathrm{p}$. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself: $\mathrm{d} = \mathrm{d(N)}$ or $\mathrm{p} = \mathrm{p(N)}$. This dependence is complicated since the destruction or production at time t depends not only on N(t) but also on the past history $\mathrm{N}(\tau)$, $\tau < \mathrm{t}$, of the variable N. This may be formally represented by (Haubold and Mathai [7])

$$\frac{\mathrm{dN}}{\mathrm{dt}} = -\,\mathrm{d}\,(\mathrm{N_t}) + \mathrm{p}(\mathrm{N_t}), \tag{1}$$

where N_t denotes the function defined by $N_t(\mathrm{t}^*) = \mathrm{N}(\mathrm{t} - \mathrm{t}^*)$, $\mathrm{t}^* > 0$.

Haubold and Mathai [7] studied a special case of this equation, when spatial fluctuations or inhomogenities in quantity N(t) are neglected, is given by the equation

$$\frac{\mathrm{d\,N_i}}{\mathrm{dt}} = -\,\mathrm{c_i\,N_i(t)} \tag{2}$$

with the initial condition that N_i (t = 0) = N_0 is the number density of species i at time t = 0; constant $c_i > 0$, known as standard kinetic equation.
The solution of the equation (2) is given by

$$\mathrm{N_i(t)} = \mathrm{N_0\, e^{-c_i t}}. \tag{3}$$

An alternative form of the same equation can be obtained on integration:

$$\mathrm{N(t)} - \mathrm{N_0} = \mathrm{c}\ {}_0\mathrm{D_t^{-1}}\,\mathrm{N(t)}, \tag{4}$$

where ${}_0\mathrm{D_t^{-1}}$is the standard integral operator. Haubold and Mathai [7] have given the fractional generalization of the standard kinetic equation (2) as

$$\mathrm{N(t)} - \mathrm{N_0} = \mathrm{c^v}\ {}_0\mathrm{D_t^{-\nu}}\,\mathrm{N(t)}, \tag{5}$$

where ${}_0\mathrm{D_t^{-\nu}}$is the well known Riemann-Liouville fractional integral operator (Oldham and Spanier 16]; Samko et al. [19]; Miller and Ross [13]) defined by

$${}_0\mathrm{D_t^{-\nu}} = \frac{1}{\Gamma(\nu)} \int_0^t (\mathrm{t-u})^{\nu-1}\,\mathrm{f(u)\,du}, \quad \mathrm{R}(\nu) > 0, \tag{6}$$

The solution of the fractional kinetic equation (6) is given by (see Haubold and Mathai [7])

$$\mathrm{N(t)} = \mathrm{N_0} \sum_{\mathrm{k=0}}^{\infty} \frac{(-1)^{\mathrm{k}}}{\Gamma(\nu\mathrm{k}+1)}\,(\mathrm{ct})^{\nu\mathrm{k}}. \tag{7}$$

Further Saxena, Mathai and Haubold [20] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which extended the work of Haubold and Mathai [7]. In an another paper Saxena, Mathai and Haubold [21] developed the solutions for fractional kinetic equations associated with the generalized Mittag-Leffler function and R-function.

In the present article we introduce and investigate the further computable extensions of the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the generalized M-series, are written in compact and easily computable form.

2 The generalized M-series and its relationship with some other functions

The generalized M-series is introduced by Sharma and Jain [22], defined as

$$ {}_{p}\overset{\alpha,\beta}{M}{}_{q}(t) = {}_{p}\overset{\alpha,\beta}{M}{}_{q}(a_1, ..., a_p;\ b_1, ..., b_q;\ t) $$

$$ = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{t^k}{\Gamma(\alpha k + \beta)},\ t,\alpha,\beta \in C,\ \mathrm{Re}\,(\alpha) > 0. \qquad (8) $$

Here $(a_j)_k$, $(b_j)_k$are known Poachammer symbols. The series (8) is defined when non of the parameters b_js, j = 1,. . . ,q is a negative inter or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in it. The series in (8) is convergent for t if $p \leq q$, if it is convergent for $|t| < \delta = \alpha^{\alpha}$ if $p \leq q+1$and divergent, if $p > q+1$. When p = q+1 and $|t| = \delta$, the series can converge on conditions depending on the parameters. The generalized M-series yields the following relationship with various classical special functions:
M-series (Sharma [23])

$$ {}_{p}\overset{\alpha,1}{M}{}_{q}(a_1, ..., a_p;\ b_1, ..., b_q;\ t) = {}_{p}\overset{\alpha}{M}{}_{q}(a_1, ..., a_p;\ b_1, ..., b_q;\ t) $$

$$ = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{t^k}{\Gamma(\alpha k + 1)}. \qquad (9) $$

Mittag-Leffler function (Mittag-Leffler ([14], [15]))

$$ {}_{0}\overset{\alpha,1}{M}{}_{0}(-\,;\,-\,;\ t) = E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}. \qquad (10) $$

Generalized Mittag-Leffler function (Wiman ([26], [27]))

$$ {}_{0}\overset{\alpha,\beta}{M}{}_{0}(-\,;\,-\,;\ t) = E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \qquad (11) $$

New generalized Mittag-Leffler function (Prabhakar [18])

$$ {}_{1}\overset{\alpha,\beta}{M}{}_{1}(\gamma\,;\,1\,;\ t) = E^{\gamma}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{t^k}{k\,!}. \qquad (12) $$

Generalized Gauss hypergeometric function (Sharma and Jain [22])

$$ {}_{p}\overset{1,1}{M}{}_{q}(a_1, ..., a_p;\ b_1, ..., b_q;\ t) = {}_{p}F_{q}(a_1, ..., a_p;\ b_1, ..., b_q;\ t). \qquad (13) $$

3 Generalized Fractional Kinetic Equations

Theorem 1. If $\nu > 0$, $c > 0$,$d > 0$, $\mu > 0$, $\mathrm{Re}(s) > |d|^{\nu/\alpha}$, $c \neq d$ then for the solution of the generalized fractional kinetic equation

$$N(t) \;-\; N_0\, t^{\mu-1}\, {}_p\overset{\nu,\mu}{M}{}_q\,(a_1,...,a_p;\, b_1,...,b_q;\, -d^\nu\, t^\nu) \;=\; -\, c^\nu\, {}_0D_t^{-\nu}\, N(t), \quad (14)$$

there holds the formula

$$N(t) \;=\; N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-1)^r\, (ct)^{r\nu}\; {}_p\overset{\nu,\mu+r\nu}{M}{}_q\,(a_1,...,a_p;\, b_1,...,b_q$$

$$;\, -d^\nu t^\nu)\,. \quad (15)$$

Proof. Applying the Laplace transform both the sides of equation (14), we get

$$\bar{N}(s) \;-\; N_0 \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k}\, \frac{(-d^\nu)^k}{s^{\nu k+\mu}} \;=\; -\, c^\nu\, s^{-\nu}\, \bar{N}(s). \quad (16)$$

Solving for $\bar{N}(s)$, it gives

$$\bar{N}(s) \;=\; \frac{N_0}{(1 + c^\nu s^{-\nu})} \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k}\, \frac{(-d^\nu)^k}{s^{\nu k+\mu}}\,. \quad (17)$$

Now, taking inverse Laplace transform both the sides of (17), we obtain the desired result (15).

4 Special Cases

When $p = 1 = q$, $b_1 = 1$ and $c = d$, then we arrive at the following result recently obtained by Saxena, Mathai and Haubold [21]

Corollary 1. If $c > 0$, $\nu > 0$, $\mu > 0$, then the solution of the equation

$$N(t) \;-\; N_0\, t^{\mu-1}\, E^{a_1}_{\nu,\mu}[-c^\nu t^\nu] \;=\; -\, c^\nu\, {}_0D_t^{-\nu}\, N(t), \quad (18)$$

there holds the formula

$$N(t) \;=\; N_0\, t^{\mu-1}\, E^{a_1+1}_{\nu,\mu}(-c^\nu t^v). \quad (19)$$

If we set $p = 0 = q$, then we get the following result obtained by Saxena, Mathai and Haubold [20].

Corollary 2. If $\nu > 0$, $c > 0$, $d > 0$, $c \neq d$, then for the solution of

$$N(t) \;-\; N_0\, t^{\mu-1}\, E_{\nu,\mu}[-d^\nu t^\nu] \;=\; -\, c^\nu\, {}_0D_t^{-\nu}\, N(t), \quad (20)$$

the following result holds

$$N(t) = N_0 \frac{t^{\mu-\nu-1}}{c^\nu - d^\nu} [E_{\nu,\mu-1}(-d^\nu t^\nu) - E_{\nu,\mu-\nu}(-c^\nu t^\nu)]. \tag{21}$$

If we take c = d, p = 0 = q, then we arrive at the following result given by Saxena, Mathai and Haubold [20]

Corollary 3. If $c > 0$, $\nu > 0$, $\mu > 0$, then for the solution of the equation

$$N(t) - N_0\, t^{\mu-1} E_{\nu,\mu}(-c^\nu t^\nu) = -c^\nu\, {}_0D_t^{-\nu} N(t), \tag{22}$$

the following result holds

$$N(t) = \frac{N_0}{\nu} t^{\mu-1} [E_{\nu,\mu-1}(-c^\nu t^\nu) + (1+\nu-\mu) E_{\nu,\mu}(-c^\nu t^\nu)]. \tag{23}$$

Finally, for $\mu = 1 = v$, we arrive at the following interesting result.

Corollary 4. If $c > 0$, $d > 0$, $c \neq d$, then for the solution of the equation

$$N(t) - N_0\, {}_pF_q(a_1, ..., a_p; b_1, ..., b_q; -dt) = -c\, {}_0D_t^{-1} N(t), \tag{24}$$

there holds the formula

$$N(t) = N_0 \sum_{r=0}^{\infty} (-1)^r (ct)^r {}_pF_{q+1}(a_1, ..., a_p; b_1, ..., b_q, r; -dt). \tag{25}$$

5. Conclusion

In this paper we have introduced an extended fractional generalization of the standard kinetic equation and established solution for the same. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation discussed in this article, involving generalized M-series contains a number of known (may be new also) fractional kinetic equations involving various other special functions (the M-series, the generalized Mittag-Leffler function, Mittag-Leffler function etc.). The results obtained in the present paper provide an extension of the results given by Haubold and Mathai [7] and Saxena, Mathai and Haubold ([20] and [21]).

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