# On the solution of the laminar boundary layer equations

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## Introduction

For some years after its suggestion an approximate method of solution of the boundary layer equations due to Kármán and Pohlhausen was thought to be reasonably accurate. The present writer (1934) recommended it for general use because it agreed with experiment as far as the point of separation for the flow past a circular cylinder (when the observed pressure distribution was used in the theoretical solution). There seems to be little doubt that this method gives a reasonably accurate solution in a region of accelerated flow, but more recently its adequacy in a region of retarded flow has been questioned. The flow past a circular cylinder is not an exhaustive test for a retarded region because the pressure rises very rapidly from its minimum value leaving little doubt as to the position of separation.

Schubauer (1935) has measured the pressure distribution around an elliptic cylinder of fineness ratio 2·96 : 1 and also observed, by introducing smoke just beyond the separation point, the actual position of separation. On applying Pohlhausen's method to his observed pressure distribution Schubauer fails to find any separation at all. By measurements of the velocity distribution in the boundary layer he finds that Pohlhausen's method agrees reasonably with the observed one up to a point about five-sevenths of the way between the pressure minimum and the observed point of separation; the calculated distribution then diverges from the observed one.

Fairly recently Kármán and Millikan (1934) put forward another approximate means of solution of the boundary layer equations. Millikan (1936) has applied their method to Schubauer's pressure distribution; he finds separation about six-sevenths of the distance from the pressure minimum to the observed point of separation. More precisely, if $x$ is the ratio of the distance measured along the surface from the forward stagnation point to the length of the minor axis, the pressure minimum occurs when $x = 1 \cdot 30$, the observed point of separation is $x = 1{\cdot}99$ and Millikan finds $x = 1{\cdot}88$ for the position of separation.

A new approximate method of solution of the boundary layer equations is suggested below. This method, when applied to Schubauer's observed pressure distribution, gives separation when $x = 1{\cdot}925$ compared with the observed value 1·99. Reasons are given below (see pp. 575 and 576) for believing that the Reynolds number of Schubauer's experiments is scarcely high enough for the boundary layer equations to be valid. There is, in fact, an appreciable pressure drop across the boundary layer. It is difficult to estimate how important this pressure drop is, but the agreement between the theoretically obtained position of separation and the observed one is probably as good as could be expected in the circumstances; the agreement is somewhat better than that obtained by Kármán and Millikan's method.

This new method of solution is also compared with the exact solution given by Falkner and Skan (1930) when the velocity distribution at the edge of the boundary layer is of the form $x^{-m}$; fairly good agreement is obtained.

The method suggested below developed from a comparison between the results obtained by Kármán and Millikan's method, Kármán and Pohlhausen's method and an accurate solution of a particular problem.

Kármán and Millikan applied their method, when it was first introduced, to the problem of the flow along a flat plate placed edgewise to an incident stream when a retarding pressure gradient varying linearly with the distance from the leading edge is superposed. We may write the velocity distribution at the edge of the boundary layer as

$$U = b_0 - b_1 x,$$

so that the pressure gradient is $\rho b_1(b_0 - b_1 x)$; Kármán and Millikan find that separation occurs when $x^*(=b_1 x/b_0)$ is equal to 0·102, whereas Pohlhausen's method does not give separation until $x^* = 0{\cdot}156$. In order to determine which, if either, of the methods is reasonably accurate it was decided to attempt to solve the boundary layer equations accurately for this velocity distribution. This problem is discussed at length below in Part I; it admits of a solution in series of the type

$$u = \frac{b_0}{2}\{f_0'(\eta) - 8x^* f_1'(\eta) + (8x^*)^2 f_2'(\eta) - \dots\},$$

where $\eta$ and some of the coefficients $f_0, f_1, \dots$ are defined below in equations (2), (5)–(13). Separation occurs when

$$f_0''(0) - 8x^* f_1''(0) + (8x^*)^2 f_2''(0) - \dots = 0.$$

The coefficients in this series have been determined up to and including $f_8''(0)$. They are sufficient to show that unless subsequent coefficients increase enormously—and there seems to be no reason to believe they do for, apart from $f_0$, $f_1$ and $f_2$ which are exceptional, they decrease steadily over the range covered—the value $x^* = 0{\cdot}102$ for separation is much too small. Furthermore, $x^* = 0{\cdot}156$ is very much too large.

Unfortunately the series converges very slowly in the neighbourhood of $8x^* = 1$ and sufficient terms have not been obtained to give the point of separation. However, if we assume that the higher terms in the series continue to alternate in sign and decrease (or remain constant in absolute magnitude as the earlier ones $f_3, \ldots f_8$ do) it is easy to show that the point of separation lies between $x^* = 0{\cdot}119$ and $0{\cdot}129$. An approximate method of determining the error obtained by retaining only the terms calculated gives $x^* = 0{\cdot}120$ as the point of separation.

In Part II below a method of solution of the boundary layer equations is suggested utilizing the solution of the problem of Part I by replacing the velocity distribution at the edge of the boundary layer by a polygon of infinitesimally small sides and joining on the solution in adjacent sides by making the momentum integral

$$\theta = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{u}{U}\, dy$$

continuous at the vertex. Up to the pressure minimum Pohlhausen's solution may be used. Alternatively, if the number of terms computed in the solution in series starting from the forward stagnation point (Howarth 1934) is sufficient to carry the solution as far as the pressure minimum this method is to be preferred in the accelerated region. If this solution does not extend quite far enough it may be extended by the solution corresponding to the one given below for the retarded region; this extension is limited, because the series does not converge rapidly enough beyond $x^* = 0{\cdot}10$.

A solution in series can also be obtained when the velocity distribution at the edge of the boundary layer is of the form

$$U = b_0 - b_1 x - b_2 x^2,$$

so that, if necessary, the method outlined above could be made more accurate by replacing the actual velocity distribution at the edge of the boundary layer by a series of parabolas.

## PART I

### The solution of the boundary layer equations for a particular pressure distribution

We consider, first of all, the solution in series for a velocity distribution at the edge of the boundary layer of the form

$$U = b_0 - b_1 x,$$

where $b_0$ and $b_1$ are positive constants.

We assume an expansion of the form

$$\psi = b_0^{\frac{1}{2}} x^{\frac{1}{2}} \nu^{\frac{1}{2}} \{f_0(\eta) - 8x^* f_1(\eta) + (8x^*)^2 f_2(\eta) - (8x^*)^3 f_3(\eta) + (8x^*)^4 f_4(\eta) - \dots\} \tag{1}$$

for the stream function $\psi$, where

$$\eta = \tfrac{1}{2} y x^{-\frac{1}{2}} \nu^{-\frac{1}{2}} b_0^{\frac{1}{2}}, \quad \text{and} \quad x^* = b_1 x / b_0. \tag{2}$$

Substituting this form in the boundary layer equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\nu \partial^2 u}{\partial y^2}, \tag{3}$$

where

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = -b_1 (b_0 - b_1 x), \tag{4}$$

and equating coefficients of the various powers of $x^*$, we find that

$$f_0''' + f_0 f_0'' = 0, \tag{5}$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + 3f_0'' f_1 = -1, \tag{6}$$

$$f_2''' + f_0 f_2'' - 4f_0' f_2' + 5f_0'' f_2 = -1/8 + (2f_1'^2 - 3f_1 f_1''), \tag{7}$$

$$f_3''' + f_0 f_3'' - 6f_0' f_3' + 7f_0'' f_3 = (6f_1' f_2' - 3f_1 f_2'' - 5f_1'' f_2), \tag{8}$$

$$f_4''' + f_0 f_4'' - 8f_0' f_4' + 9f_0'' f_4 = (4f_2'^2 - 5f_2 f_2'') + (8f_1' f_3' - 3f_1 f_3'' - 7f_1'' f_3), \tag{9}$$

$$f_5''' + f_0 f_5'' - 10f_0' f_5' + 11f_0'' f_5 = (10f_1' f_4' - 3f_1 f_4'' - 9f_1'' f_4) + (10f_2' f_3' - 5f_2 f_3'' - 7f_2'' f_3), \tag{10}$$

$$f_6''' + f_0 f_6'' - 12f_0' f_6' + 13f_0'' f_6 = (12f_1' f_5' - 3f_1 f_5'' - 11f_1'' f_5) + (12f_2' f_4' - 5f_2 f_4'' - 9f_2'' f_4) + (6f_3'^2 - 7f_3 f_3''), \tag{11}$$

$$f_7''' + f_0 f_7'' - 14f_0' f_7' + 15f_0'' f_7 = (14f_1' f_6' - 3f_1 f_6'' - 13f_1'' f_6) + (14f_2' f_5' - 5f_2 f_5'' - 11f_2'' f_5) + (14f_3' f_4' - 7f_3 f_4'' - 9f_3'' f_4), \tag{12}$$

$$f_8''' + f_0 f_8'' - 16 f_0' f_8' + 17 f_0'' f_8 = (16 f_1' f_7' - 3 f_1 f_7'' - 15 f_1'' f_7)$$
$$+ (16 f_2' f_6' - 5 f_2 f_6'' - 13 f_2'' f_6) + (16 f_3' f_5' - 7 f_3 f_5'' - 11 f_3'' f_5)$$
$$+ (8 f_4'^2 - 9 f_4 f_4''), \quad (13)$$

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(where dashes denote differentiations with regard to $\eta$) together with the boundary conditions:

$$\left.\begin{aligned} & f_r = f_r' = 0 \quad \text{when } \eta = 0 \text{ for all values of } r, f_0' = 2, \\ & f_1' = \tfrac{1}{4}, \quad f_2' = f_3' = f_4' = \ldots = 0 \quad \text{when } \eta = \infty. \end{aligned}\right\} \quad (14)$$

Blasius (1908), Toepfer (1912) and Goldstein (1930, p. 15) have given the solution of (5). Indeed the problem considered is a particular case of a more general problem discussed by Goldstein (1930, p. 15), though no more of the coefficients $f_r$ are given in that paper. The method of solution for the remaining equations, which are all linear, is exactly the same as that described by the present writer in a previous paper (1934). The following results have been obtained:

$f_0''(0) = 1{\cdot}328242$, $f_1''(0) = 1{\cdot}02054$, $f_2''(0) = -0{\cdot}06926$,

$f_3''(0) = 0{\cdot}0560$, $f_4''(0) = -0{\cdot}0372$, $f_5''(0) = 0{\cdot}0272$,

$f_6''(0) = -0{\cdot}0212$, $f_7''(0) = 0{\cdot}0174$; $f_8''(0) = -0{\cdot}0147$.

The velocity distribution is given by

$$u = \frac{b_0}{2} \{ f_0'(\eta) - (8x^*) f_1'(\eta) + (8x^*)^2 f_2'(\eta) - \ldots \}. \quad (15)$$

The functions $f_0 f_0' f_0'' \ldots f_6 f_6' f_6''$ are tabulated in Table I.

The condition for separation $(\partial u / \partial y)_0 = 0$ leads to

$$f_0''(0) - (8x^*) f_1''(0) + (8x^*)^2 f_2''(0) - \ldots = 0. \quad (16)$$

From the terms calculated it will be seen that the series converges quite slowly in the neighbourhood of $8x^* = 1$; probably at least eight more terms would be required in order to determine the value of the series in (16) to three places of decimals. We can, however, set fairly close limits to the values of $x^*$ for separation (henceforward we shall refer to this number as $x_s^*$) if we assume that for $r \geq 9$ the values of $f_r''(0)$ do not increase in absolute

TABLE I

| $\eta$ | $f_0$ | $f_0'$ | $f_0''$ | $f_1$ | $f_1$ | $f_1''$ |
|---|---|---|---|---|---|---|
| 0·0 | 0·00000 | 0·00000 | 1·32824 | 0·00000 | 0·00000 | 1·02054 |
| 0·1 | 0·00664 | 0·13282 | 1·32795 | 0·00494 | 0·09705 | 0·92054 |
| 0·2 | 0·02656 | 0·26553 | 1·32589 | 0·01908 | 0·18411 | 0·82054 |
| 0·3 | 0·05974 | 0·39788 | 1·32033 | 0·04142 | 0·26116 | 0·72057 |
| 0·4 | 0·10611 | 0·52942 | 1·30957 | 0·07098 | 0·32823 | 0·62073 |
| 0·5 | 0·16557 | 0·65957 | 1·29204 | 0·10674 | 0·38532 | 0·52125 |
| 0·6 | 0·23795 | 0·78756 | 1·26637 | 0·14771 | 0·43250 | 0·42258 |
| 0·7 | 0·32298 | 0·91253 | 1·23147 | 0·19291 | 0·46989 | 0·32545 |
| 0·8 | 0·42032 | 1·03352 | 1·18666 | 0·24137 | 0·49768 | 0·23094 |
| 0·9 | 0·52952 | 1·14953 | 1·13173 | 0·29214 | 0·51621 | 0·14045 |
| 1·0 | 0·65003 | 1·25954 | 1·06701 | 0·34432 | 0·52596 | 0·05565 |
| 1·1 | 0·78120 | 1·36263 | 0·99341 | 0·39706 | 0·52759 | −0·02161 |
| 1·2 | 0·92230 | 1·45798 | 0·91237 | 0·44959 | 0·52195 | −0·08948 |
| 1·3 | 1·07252 | 1·54492 | 0·82582 | 0·50124 | 0·51006 | −0·14633 |
| 1·4 | 1·23099 | 1·62303 | 0·73603 | 0·55143 | 0·49309 | −0·19093 |
| 1·5 | 1·39682 | 1·69210 | 0·64544 | 0·59973 | 0·47231 | −0·22262 |
| 1·6 | 1·56911 | 1·75218 | 0·55651 | 0·64581 | 0·44900 | −0·24145 |
| 1·7 | 1·74696 | 1·80354 | 0·47151 | 0·68949 | 0·42442 | −0·24817 |
| 1·8 | 1·92954 | 1·84666 | 0·39234 | 0·73069 | 0·39972 | −0·24418 |
| 1·9 | 2·11605 | 1·88224 | 0·32050 | 0·76946 | 0·37588 | −0·23138 |
| 2·0 | 2·30576 | 1·91104 | 0·25694 | 0·80592 | 0·35367 | −0·21198 |
| 2·1 | 2·49806 | 1·93392 | 0·20208 | 0·84027 | 0·33363 | −0·18827 |
| 2·2 | 2·69238 | 1·95174 | 0·15589 | 0·87273 | 0·31608 | −0·16240 |
| 2·3 | 2·88826 | 1·96537 | 0·11793 | 0·90357 | 0·30116 | −0·13625 |
| 2·4 | 3·08534 | 1·97558 | 0·08748 | 0·93305 | 0·28879 | −0·11130 |
| 2·5 | 3·28329 | 1·98309 | 0·06363 | 0·96141 | 0·27882 | −0·08861 |
| 2·6 | 3·48189 | 1·98849 | 0·04537 | 0·98889 | 0·27098 | −0·06879 |
| 2·7 | 3·68094 | 1·99231 | 0·03171 | 1·01566 | 0·26496 | −0·05212 |
| 2·8 | 3·88031 | 1·99496 | 0·02173 | 1·04192 | 0·26045 | −0·03855 |
| 2·9 | 4·07990 | 1·99675 | 0·01459 | 1·06779 | 0·25715 | −0·02785 |
| 3·0 | 4·27964 | 1·99795 | 0·00961 | 1·09338 | 0·25480 | −0·01965 |
| 3·1 | 4·47948 | 1·99873 | 0·00620 | 1·11878 | 0·25315 | −0·01356 |
| 3·2 | 4·67938 | 1·99922 | 0·00392 | 1·14403 | 0·25203 | −0·00914 |
| 3·3 | 4·87931 | 1·99954 | 0·00243 | 1·16919 | 0·25128 | −0·00603 |
| 3·4 | 5·07928 | 1·99973 | 0·00148 | 1·19430 | 0·25079 | −0·00389 |
| 3·5 | 5·27926 | 1·99984 | 0·00088 | 1·21936 | 0·25048 | −0·00245 |
| 3·6 | 5·47925 | 1·99991 | 0·00051 | 1·24440 | 0·25029 | −0·00151 |
| 3·7 | 5·67924 | 1·99995 | 0·00029 | 1·26942 | 0·25017 | −0·00091 |
| 3·8 | 5·87924 | 1·99997 | 0·00017 | 1·29443 | 0·25010 | −0·00054 |
| 3·9 | 6·07923 | 1·99999 | 0·00009 | 1·31944 | 0·25005 | −0·00031 |
| 4·0 | 6·27923 | 1·99999 | 0·00005 | 1·34444 | 0·25003 | −0·00018 |
| 4·1 | 6·47923 | 2·00000 | 0·00003 | 1·36944 | 0·25002 | −0·00010 |
| 4·2 | 6·67923 | 2·00000 | 0·00001 | 1·39444 | 0·25001 | −0·00005 |
| 4·3 | 6·87923 | 2·00000 | 0·00001 | 1·41944 | 0·25000 | −0·00003 |
| 4·4 | 7·07923 | 2·00000 | 0·00000 | 1·44445 | 0·25000 | −0·00002 |

Table I (*continued*)

| $\eta$ | $f_2$ | $f_2'$ | $f_2''$ | $f_3$ | $f_3'$ | $f_3''$ |
|---|---|---|---|---|---|---|
| 0·0 | 0·0000 | 0·0000 | −0·0693 | 0·0000 | 0·0000 | 0·0560 |
| 0·1 | −0·0004 | −0·0075 | −0·0816 | 0·0003 | 0·0056 | 0·0560 |
| 0·2 | −0·0015 | −0·0163 | −0·0932 | 0·0011 | 0·0112 | 0·0560 |
| 0·3 | −0·0037 | −0·0261 | −0·1031 | 0·0025 | 0·0168 | 0·0559 |
| 0·4 | −0·0068 | −0·0368 | −0·1108 | 0·0045 | 0·0224 | 0·0556 |
| 0·5 | −0·0111 | −0·0482 | −0·1156 | 0·0070 | 0·0279 | 0·0551 |
| 0·6 | −0·0165 | −0·0599 | −0·1170 | 0·0101 | 0·0334 | 0·0541 |
| 0·7 | −0·0230 | −0·0715 | −0·1144 | 0·0137 | 0·0387 | 0·0524 |
| 0·8 | −0·0307 | −0·0826 | −0·1077 | 0·0178 | 0·0438 | 0·0499 |
| 0·9 | −0·0395 | −0·0928 | −0·0965 | 0·0224 | 0·0486 | 0·0463 |
| 1·0 | −0·0493 | −0·1018 | −0·0812 | 0·0275 | 0·0530 | 0·0414 |
| 1·1 | −0·0598 | −0·1089 | −0·0620 | 0·0330 | 0·0569 | 0·0352 |
| 1·2 | −0·0710 | −0·1141 | −0·0397 | 0·0388 | 0·0600 | 0·0275 |
| 1·3 | −0·0825 | −0·1168 | −0·0152 | 0·0450 | 0·0623 | 0·0185 |
| 1·4 | −0·0943 | −0·1171 | 0·0100 | 0·0513 | 0·0637 | 0·0082 |
| 1·5 | −0·1059 | −0·1148 | 0·0346 | 0·0577 | 0·0639 | −0·0028 |
| 1·6 | −0·1171 | −0·1102 | 0·0571 | 0·0640 | 0·0631 | −0·0142 |
| 1·7 | −0·1278 | −0·1035 | 0·0762 | 0·0702 | 0·0611 | −0·0252 |
| 1·8 | −0·1378 | −0·0951 | 0·0908 | 0·0762 | 0·0581 | −0·0352 |
| 1·9 | −0·1468 | −0·0855 | 0·1005 | 0·0818 | 0·0541 | −0·0437 |
| 2·0 | −0·1549 | −0·0752 | 0·1049 | 0·0870 | 0·0494 | −0·0500 |
| 2·1 | −0·1619 | −0·0647 | 0·1046 | 0·0917 | 0·0442 | −0·0540 |
| 2·2 | −0·1678 | −0·0544 | 0·1000 | 0·0958 | 0·0387 | −0·0555 |
| 2·3 | −0·1728 | −0·0448 | 0·0922 | 0·0994 | 0·0332 | −0·0547 |
| 2·4 | −0·1768 | −0·0361 | 0·0822 | 0·1025 | 0·0278 | −0·0519 |
| 2·5 | −0·1800 | −0·0284 | 0·0710 | 0·1050 | 0·0228 | −0·0476 |
| 2·6 | −0·1825 | −0·0219 | 0·0595 | 0·1071 | 0·0183 | −0·0422 |
| 2·7 | −0·1844 | −0·0165 | 0·0484 | 0·1087 | 0·0144 | −0·0363 |
| 2·8 | −0·1859 | −0·0122 | 0·0384 | 0·1100 | 0·0111 | −0·0304 |
| 2·9 | −0·1869 | −0·0088 | 0·0296 | 0·1109 | 0·0083 | −0·0247 |
| 3·0 | −0·1876 | −0·0062 | 0·0222 | 0·1117 | 0·0061 | −0·0195 |
| 3·1 | −0·1882 | −0·0043 | 0·0163 | 0·1122 | 0·0044 | −0·0150 |
| 3·2 | −0·1885 | −0·0029 | 0·0116 | 0·1126 | 0·0031 | −0·0112 |
| 3·3 | −0·1888 | −0·0019 | 0·0081 | 0·1128 | 0·0021 | −0·0082 |
| 3·4 | −0·1889 | −0·0012 | 0·0055 | 0·1130 | 0·0014 | −0·0059 |
| 3·5 | −0·1890 | −0·0008 | 0·0037 | 0·1131 | 0·0010 | −0·0041 |
| 3·6 | −0·1891 | −0·0005 | 0·0024 | 0·1132 | 0·0006 | −0·0028 |
| 3·7 | −0·1891 | −0·0003 | 0·0015 | 0·1132 | 0·0004 | −0·0018 |
| 3·8 | −0·1891 | −0·0002 | 0·0009 | 0·1133 | 0·0002 | −0·0012 |
| 3·9 | −0·1891 | −0·0001 | 0·0006 | 0·1133 | 0·0001 | −0·0007 |
| 4·0 | −0·1892 | −0·0001 | 0·0003 | 0·1133 | 0·0001 | −0·0005 |
| 4·1 | −0·1892 | −0·0000 | 0·0002 | 0·1133 | 0·0001 | −0·0003 |
| 4·2 | −0·1892 | −0·0000 | 0·0001 | 0·1133 | 0·0000 | −0·0002 |
| 4·3 | −0·1892 | −0·0000 | 0·0001 | 0·1133 | 0·0000 | −0·0001 |
| 4·4 | −0·1892 | −0·0000 | 0·0000 | 0·1133 | 0·0000 | −0·0001 |

Table I (*continued*)

| $\eta$ | $f_4$ | $f_4'$ | $f_4''$ | $f_5$ | $f_5'$ | $f_5''$ |
|---|---|---|---|---|---|---|
| 0·0 | 0·0000 | 0·0000 | −0·0372 | 0·0000 | 0·0000 | 0·0272 |
| 0·1 | −0·0002 | −0·0037 | −0·0372 | 0·0001 | 0·0027 | 0·0272 |
| 0·2 | −0·0007 | −0·0074 | −0·0371 | 0·0005 | 0·0054 | 0·0272 |
| 0·3 | −0·0017 | −0·0111 | −0·0369 | 0·0012 | 0·0082 | 0·0270 |
| 0·4 | −0·0030 | −0·0148 | −0·0365 | 0·0022 | 0·0109 | 0·0267 |
| 0·5 | −0·0046 | −0·0184 | −0·0358 | 0·0034 | 0·0135 | 0·0262 |
| 0·6 | −0·0067 | −0·0220 | −0·0348 | 0·0049 | 0·0162 | 0·0254 |
| 0·7 | −0·0090 | −0·0254 | −0·0334 | 0·0067 | 0·0187 | 0·0243 |
| 0·8 | −0·0117 | −0·0286 | −0·0314 | 0·0087 | 0·0210 | 0·0228 |
| 0·9 | −0·0147 | −0·0316 | −0·0289 | 0·0109 | 0·0232 | 0·0210 |
| 1·0 | −0·0180 | −0·0344 | −0·0258 | 0·0133 | 0·0251 | 0·0186 |
| 1·1 | −0·0216 | −0·0368 | −0·0220 | 0·0160 | 0·0268 | 0·0159 |
| 1·2 | −0·0254 | −0·0388 | −0·0175 | 0·0187 | 0·0282 | 0·0127 |
| 1·3 | −0·0293 | −0·0403 | −0·0124 | 0·0216 | 0·0293 | 0·0090 |
| 1·4 | −0·0334 | −0·0412 | −0·0066 | 0·0245 | 0·0300 | 0·0049 |
| 1·5 | −0·0376 | −0·0416 | −0·0004 | 0·0276 | 0·0303 | 0·0006 |
| 1·6 | −0·0417 | −0·0413 | 0·0060 | 0·0307 | 0·0301 | −0·0038 |
| 1·7 | −0·0458 | −0·0404 | 0·0125 | 0·0336 | 0·0295 | −0·0083 |
| 1·8 | −0·0498 | −0·0388 | 0·0186 | 0·0365 | 0·0285 | −0·0125 |
| 1·9 | −0·0535 | −0·0367 | 0·0241 | 0·0394 | 0·0271 | −0·0163 |
| 2·0 | −0·0571 | −0·0340 | 0·0286 | 0·0420 | 0·0253 | −0·0195 |
| 2·1 | −0·0603 | −0·0310 | 0·0319 | 0·0445 | 0·0232 | −0·0221 |
| 2·2 | −0·0633 | −0·0277 | 0·0339 | 0·0466 | 0·0209 | −0·0237 |
| 2·3 | −0·0659 | −0·0243 | 0·0346 | 0·0485 | 0·0185 | −0·0244 |
| 2·4 | −0·0681 | −0·0208 | 0·0339 | 0·0503 | 0·0161 | −0·0243 |
| 2·5 | −0·0700 | −0·0175 | 0·0321 | 0·0518 | 0·0138 | −0·0233 |
| 2·6 | −0·0716 | −0·0144 | 0·0294 | 0·0531 | 0·0115 | −0·0217 |
| 2·7 | −0·0729 | −0·0116 | 0·0262 | 0·0541 | 0·0094 | −0·0197 |
| 2·8 | −0·0740 | −0·0092 | 0·0226 | 0·0549 | 0·0075 | −0·0171 |
| 2·9 | −0·0748 | −0·0071 | 0·0190 | 0·0555 | 0·0059 | −0·0147 |
| 3·0 | −0·0754 | −0·0054 | 0·0155 | 0·0560 | 0·0046 | −0·0123 |
| 3·1 | −0·0759 | −0·0040 | 0·0124 | 0·0564 | 0·0034 | −0·0101 |
| 3·2 | −0·0762 | −0·0029 | 0·0096 | 0·0567 | 0·0025 | −0·0080 |
| 3·3 | −0·0765 | −0·0021 | 0·0073 | 0·0569 | 0·0019 | −0·0061 |
| 3·4 | −0·0766 | −0·0014 | 0·0054 | 0·0571 | 0·0013 | −0·0046 |
| 3·5 | −0·0767 | −0·0010 | 0·0039 | 0·0572 | 0·0009 | −0·0034 |
| 3·6 | −0·0768 | −0·0006 | 0·0027 | 0·0573 | 0·0006 | −0·0025 |
| 3·7 | −0·0769 | −0·0004 | 0·0019 | 0·0573 | 0·0004 | −0·0018 |
| 3·8 | −0·0769 | −0·0003 | 0·0011 | 0·0573 | 0·0003 | −0·0012 |
| 3·9 | −0·0769 | −0·0002 | 0·0008 | 0·0573 | 0·0002 | −0·0008 |
| 4·0 | −0·0770 | −0·0001 | 0·0005 | 0·0573 | 0·0001 | −0·0005 |
| 4·1 | −0·0770 | −0·0001 | 0·0003 | | | |
| 4·2 | −0·0770 | −0·0000 | 0·0002 | | | |
| 4·3 | −0·0770 | −0·0000 | 0·0001 | | | |
| 4·4 | −0·0770 | −0·0000 | 0·0001 | | | |

TABLE I (*continued*)

| $\eta$ | $f_6$ | $f'_6$ | $f''_6$ | $\eta$ | $f_6$ | $f'_6$ | $f''_6$ |
|---|---|---|---|---|---|---|---|
| 0·0 | 0·0000 | 0·0000 | −0·0212 | 2·1 | −0·0347 | −0·0184 | 0·0165 |
| 0·1 | −0·0001 | −0·0021 | −0·0212 | 2·2 | −0·0364 | −0·0167 | 0·0179 |
| 0·2 | −0·0004 | −0·0042 | −0·0212 | 2·3 | −0·0380 | −0·0148 | 0·0186 |
| 0·3 | −0·0010 | −0·0064 | −0·0211 | 2·4 | −0·0394 | −0·0129 | 0·0186 |
| 0·4 | −0·0017 | −0·0085 | −0·0208 | 2·5 | −0·0405 | −0·0111 | 0·0181 |
| 0·5 | −0·0027 | −0·0105 | −0·0204 | 2·6 | −0·0416 | −0·0094 | 0·0171 |
| 0·6 | −0·0038 | −0·0125 | −0·0197 | 2·7 | −0·0424 | −0·0077 | 0·0157 |
| 0·7 | −0·0052 | −0·0145 | −0·0188 | 2·8 | −0·0431 | −0·0062 | 0·0140 |
| 0·8 | −0·0067 | −0·0163 | −0·0177 | 2·9 | −0·0437 | −0·0049 | 0·0124 |
| 0·9 | −0·0084 | −0·0180 | −0·0162 | 3·0 | −0·0440 | −0·0038 | 0·0102 |
| 1·0 | −0·0103 | −0·0197 | −0·0145 | 3·1 | −0·0442 | −0·0028 | 0·0084 |
| 1·1 | −0·0123 | −0·0209 | −0·0123 | 3·2 | −0·0444 | −0·0021 | 0·0066 |
| 1·2 | −0·0145 | −0·0221 | −0·0099 | 3·3 | −0·0445 | −0·0016 | 0·0051 |
| 1·3 | −0·0168 | −0·0229 | −0·0072 | 3·4 | −0·0446 | −0·0011 | 0·0038 |
| 1·4 | −0·0191 | −0·0234 | −0·0042 | 3·5 | −0·0447 | −0·0007 | 0·0028 |
| 1·5 | −0·0215 | −0·0237 | −0·0009 | 3·6 | −0·0447 | −0·0005 | 0·0021 |
| 1·6 | −0·0239 | −0·0236 | 0·0026 | 3·7 | −0·0447 | −0·0003 | 0·0015 |
| 1·7 | −0·0263 | −0·0231 | 0·0061 | 3·8 | −0·0447 | −0·0002 | 0·0010 |
| 1·8 | −0·0285 | −0·0223 | 0·0093 | 3·9 | −0·0447 | −0·0002 | 0·0007 |
| 1·9 | −0·0307 | −0·0212 | 0·0122 | 4·0 | −0·0447 | −0·0001 | 0·0004 |
| 2·0 | −0·0328 | −0·0199 | 0·0146 | | | | |

magnitude as $r$ increases and continue to alternate in sign.† (Each term in (16) after the first is then negative.)

We can then obtain limits for $x_s^*$ by putting first $f''_r(0) = 0$ for $r \geqslant 9$ and then by putting $|f''_r(0)| = |f''_8(0)|$ for $r \geqslant 9$. In the first case we find $x_s^* = 0{\cdot}129$ and in the second $x_s^* = 0{\cdot}119$. It is, therefore, fairly safe to assume that $x_s^*$ lies within the interval 0·119 to 0·129.

We can obtain an approximate answer in the following way. We may regard the terms already obtained up to and including $x^6$, say, as an approximate solution.‡ It appears that each of the terms $f'_5$ and $f'_6$, and as far as they have been obtained, $f'_7$ and $f'_8$ are expressible with reasonable accuracy in the form $K_r \eta e^{-\alpha\eta^3}$, where the $K_r$ are constants (different for different $f_r$).§ In fig. 1 $f'_6$ is compared with $K_6 \eta e^{-\alpha\eta^3}$ with an appropriate value −0·0221 for $K_6$ and $\alpha = 0{\cdot}1$. It is evident by trial that $\alpha$ retains the same value for

† It will be seen from the equations (5)–(13) that apart from $f_0, f_1$ and $f_2$, which have exceptional right-hand sides, the equations defining the $f_r$ may be considered to be of the same form; it seems reasonable therefore to assume that they continue to alternate in sign and decrease in absolute magnitude beyond $r = 8$.

‡ The terms in $x^7$ and $x^8$ were not obtained sufficiently accurately in the outer part of the boundary layer to be used here; it is, in fact, necessary to obtain $f''_r(0)$ to an accuracy considerably greater than is required for $f''_r$ in the outer part of the boundary layer.

§ The error incurred by expressing $f'_4$ in this form is not great.

the different functions $f'_r$. If we assume that all the terms after $x^6$ are of the form $K_r \eta e^{-0\cdot 1\eta^3}$ the velocity $u$ at any point may be written $(u_0+u_1)$, where

$$u_0 = \frac{b_0}{2}\{f'_0(\eta)-(8x^*)f'_1(\eta)+\ldots+(8x^*)^6 f'_6(\eta)\} \tag{17}$$

and

$$u_1 = \frac{b_0}{2} F(x^*)\,\eta e^{-0\cdot 1\eta^3}, \tag{18}$$

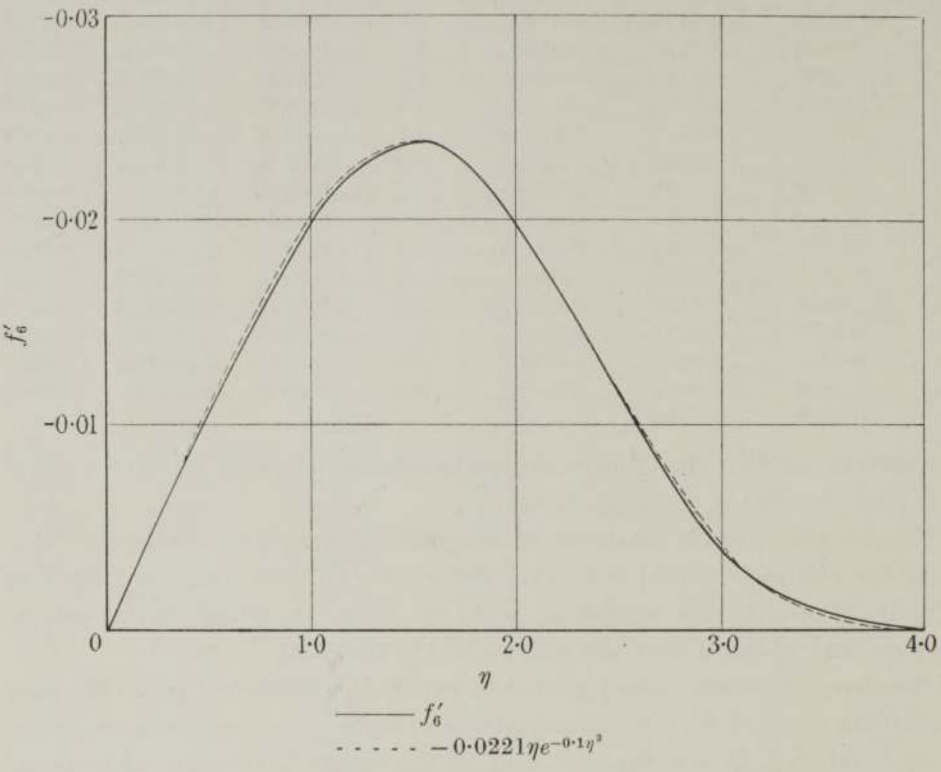


Fig. 1

where $F(x^*)$ is a function to be determined. It will be noticed that the function $\eta e^{-0\cdot 1\eta^3}$ satisfies the conditions

$$u = \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \ldots = 0 \quad \text{when } \eta = \infty,$$

$$u = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = 0 \quad \text{when } \eta = 0,$$

a considerably greater number of conditions than the polynomial used in the ordinary Pohlhausen method. Moreover, the choice of $\alpha$ $(=0\cdot 1)$ so as to give agreement with the forms of $f'_5$ and $f'_6$ presumably corresponds to satisfying an additional condition, though it is not obvious what this condition is.

We may write the boundary layer momentum integral in the form

$$\int_0^\infty \frac{\partial}{\partial x}(u^2 - U^2)\,dy - U\int_0^\infty \frac{\partial}{\partial x}(u-U)\,dy = -\nu\left(\frac{\partial u}{\partial y}\right)_0, \tag{19}$$

i.e.

$$\int_0^\infty \frac{\partial}{\partial x}(u_0^2 - U^2)\,dy - U\int_0^\infty \frac{\partial}{\partial x}(u_0 - U)\,dy + 2\int_0^\infty \frac{\partial}{\partial x}(u_0 u_1)\,dy$$
$$+\int_0^\infty \frac{\partial}{\partial x}(u_1^2)\,dy - U\int_0^\infty \frac{\partial u_1}{\partial x}dy = -\nu\left(\frac{\partial u_0}{\partial y}\right)_0 - \nu\left(\frac{\partial u_1}{\partial y}\right)_0. \tag{20}$$

Let us write
$$u_0 = \frac{b_0}{2}\{f_0' - 8x^* f_1' + \ldots + (8x^*)^6 f_6'\} = \frac{b_0}{2}v, \tag{21}$$

$$\frac{\partial u_0}{\partial x} = \frac{b_0}{4x}\{-\eta f_0'' + 8x^*(\eta f_1'' - 2f_1') - \ldots - (8x^*)^6(\eta f_6'' - 12 f_6')\} = \frac{b_0 v'}{4x},^\dagger \tag{22}$$

$$\frac{\partial u_0}{\partial y} = \frac{b_0^{\frac{3}{2}}}{4x^{\frac{1}{2}}\nu^{\frac{1}{2}}}\{f_0'' - 8x^* f_1'' + \ldots + (8x^*)^6 f_6''\} = \frac{b_0^{\frac{3}{2}}}{4x^{\frac{1}{2}}\nu^{\frac{1}{2}}}V'.^\dagger \tag{23}$$

Equation (20) may then be written

$$\int_0^\infty \{vv' + 8x^*(1-x^*)\}\,d\eta - (1-x^*)\Big[(-\eta f_0' + f_0) + 8x^*\left(\eta f_1' - 3f_1 + \frac{\eta}{2}\right)$$
$$-(8x^*)^2(\eta f_2' - 5f_2) + \ldots - (8x^*)^6(\eta f_6' - 13 f_6)\Big]$$
$$+F(x^*)\left[-\int_0^\infty v(\eta e^{-0\cdot 1\eta^3} - 0\cdot 3\eta^4 e^{-0\cdot 1\eta^3}) + \int_0^\infty v'\eta e^{-0\cdot 1\eta^3}\,d\eta - (1-x^*)\int_0^\infty \eta e^{-0\cdot 1\eta^3}\,d\eta\right]$$
$$+2F'(x^*)\,x^*\left[\int_0^\infty v\eta e^{-0\cdot 1\eta^3}\,d\eta - (1-x^*)\int_0^\infty \eta e^{-0\cdot 1\eta^3}\,d\eta\right]$$
$$+2F(x^*)\,F'(x^*)\,x^*\int_0^\infty \eta^2 e^{-0\cdot 2\eta^3}\,d\eta - [F(x^*)]^2\int_0^\infty \eta^2 e^{-0\cdot 2\eta^3}(1 - 0\cdot 3\eta^3)\,d\eta$$
$$= -\tfrac{1}{2}V' - \tfrac{1}{2}F(x^*). \tag{24}$$

Starting from the value of $F(x^*)$ at $x^* = 0\cdot 0875$ given by the terms in $x^{*7}$ and $x^{*8}$, equation (24) may be integrated graphically for $F(x^*)$. The condition for separation is, of course,

$$F(x^*) = -V'. \tag{25}$$

† The dashes on $v$ and $V$ do not denote differentiations.

It will be seen from equation (24) that $F'(x^*)$ becomes infinite when

$$F(x^*)\int_0^\infty \eta^2 e^{-0\cdot 2\eta^3}\,d\eta + \int_0^\infty v\eta e^{-0\cdot 1\eta^3}\,d\eta - (1-x^*)\int_0^\infty \eta e^{-0\cdot 1\eta^3}\,d\eta = 0. \quad (26)$$

Once the integral curve reaches this curve it becomes imaginary. The integral curve together with the curves given by (25) and (26) is shown in fig. 2. It will be noticed that the integral reaches the curve given by (26) when $x^* = 0\cdot 120$ and that it has not then reached the curve given by (25) by a quantity of the order of $2 \times 10^{-2}$.

This failure to find separation—although the actual value of the skin friction given by the last real point on the integral curve is small—must be due to the form assumed for the correction term. It seems fairly reasonable to assume that it is only in the neighbourhood of the point where $F'(x^*)$ becomes infinite that the solution is invalidated. If we suppose that the solution given is valid as far as $F(x^*) = -0\cdot 110$, $x^* = 0\cdot 119$ say, we may complete the solution using a result stated by Goldstein (1930, p. 4), viz.

$$\left(\frac{\partial u}{\partial y}\right)_x = \left(\frac{\partial u}{\partial y}\right)_0 + \nu\frac{\left(\dfrac{\partial^4 u}{\partial y^4}\right)_0}{\left(\dfrac{\partial u}{\partial y}\right)_0}x + \nu^2\frac{\left(\dfrac{\partial^4 u}{\partial y^4}\right)^2 + \dfrac{\partial u}{\partial y}\dfrac{\partial^7 u}{\partial y^7}}{2\left(\dfrac{\partial u}{\partial y}\right)^3}x^2 + \dots. \quad (27)$$

Taking the values of $\partial u/\partial y$ and $\partial^4 u/\partial y^4$ obtained by our method of solution we find separation when $x^* = 0\cdot 120$. The values of $\partial^7 u/\partial y^7$ given by the approximate method are sufficient to show that the third term in the series is negligible over the range of values of $x^*$ $(0\cdot 001)$ over which we require the solution.

An alternative method of solution is obtained by using the result given by differentiating the first boundary layer equation twice with regard to $y$; we find

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \nu\frac{\partial^4 u}{\partial y^4}\Big/\frac{\partial u}{\partial y}\dagger \quad (28)$$

along the wall. We may use the skin friction given by the first nine terms of the series as an approximation and again use a term

$$u_1 = \frac{b_0}{2}\phi(x^*)\,\eta e^{-0\cdot 1\eta^3}$$

as a correction. Equation (28) then gives a first order differential equation for $\phi$. Although we are still using the same form for the correction term, yet applying (28) (which gives the correct growth of the skin friction along the

† This result is, of course, contained in (27).

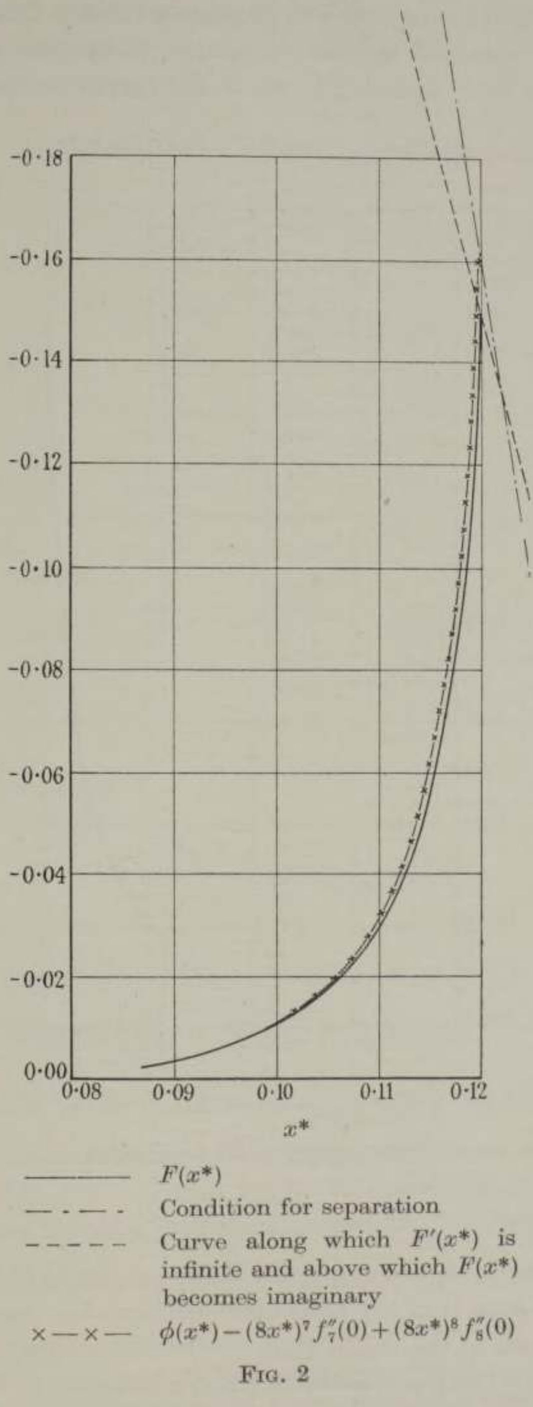
-0·18
-0·16
-0·14
-0·12
-0·10
-0·08
-0·06
-0·04
-0·02
0·00
0·08
0·09
0·10
0·11
0·12
$x^*$
$F(x^*)$
Condition for separation
Curve along which $F'(x^*)$ is infinite and above which $F(x^*)$ becomes imaginary
$\phi(x^*) - (8x^*)^7 f_7''(0) + (8x^*)^8 f_8''(0)$

FIG. 2

wall) instead of (24) (which is the expression of the momentum law throughout the layer) would be expected to produce a totally different result from (24) if the form assumed for the correction term were inadequate. The integral curve for $\phi(x^*)$ is shown in fig. 3; the corresponding value of

$$\phi(x^*)-f_7''(0)\,(8x^*)^7+f_8''(0)\,(8x^*)^8$$

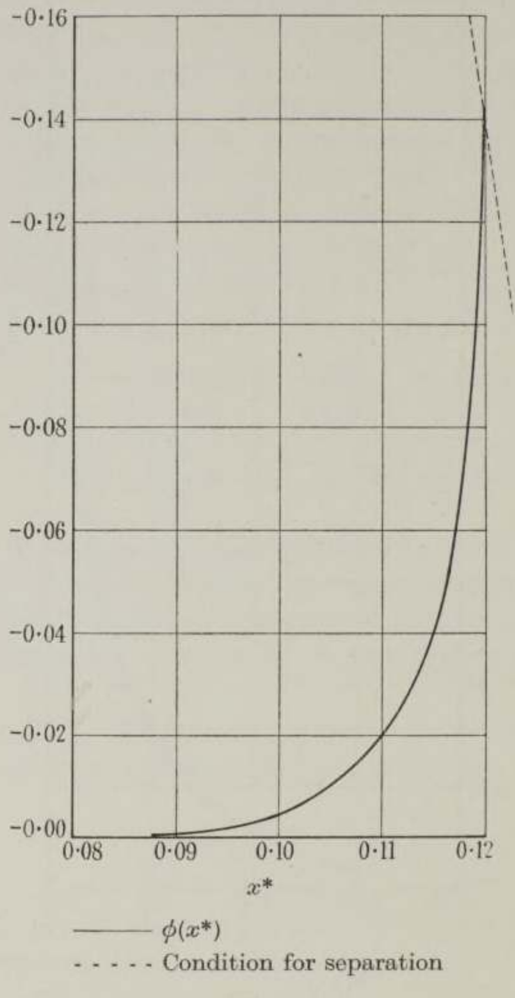


Fig. 3

is shown dotted in fig. 2. This quantity is the one we have to compare with $F(x^*)$. It will be seen that the agreement between the curves is very good and that both lead to the result that separation occurs when $x^* = 0{\cdot}120$. The curve in fig. 3 when $\phi(x^*)$ becomes infinite coincides in this case with the curve of condition for separation. (Of course, the accurate value of $\partial^4 u/\partial y^4$

is zero at the separation, but the present method, being approximate, does not give this result and consequently yields an infinite value for $\phi'(x^*)$ at the point of separation.)

TABLE II

| $x^*$ | $\dfrac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{b_0 b_1^{\frac{1}{2}}}$ | $\dfrac{b_1^{\frac{1}{2}}\delta_1}{\nu^{\frac{1}{2}}}$ | $\chi = \dfrac{b_1^{\frac{1}{2}}\theta}{\nu^{\frac{1}{2}}}$ | $\dfrac{d\chi}{dx^*}$ | $\chi\Big/\dfrac{d\chi}{dx^*}$ | $\dfrac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{U\left(-\frac{dU}{dx}\right)^{\frac{1}{2}}} = \dfrac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{b_0 b_1^{\frac{1}{2}}(1-x^*)}$ |
|---|---|---|---|---|---|---|
| 0·0000 | ∞ | 0·000 | 0·000 | ∞ | 0·000 | ∞ |
| 0·0125 | 2·739 | 0·199 | 0·076 | 3·17 | 0·024 | 2·773 |
| 0·0250 | 1·772 | 0·292 | 0·110 | 2·39 | 0·046 | 1·817 |
| 0·0375 | 1·309 | 0·371 | 0·137 | 2·08 | 0·066 | 1·360 |
| 0·0500 | 1·011 | 0·447 | 0·162 | 1·93 | 0·084 | 1·064 |
| 0·0625 | 0·790 | 0·523 | 0·186 | 1·85 | 0·100 | 0·843 |
| 0·0750 | 0·613 | 0·603 | 0·209 | 1·82 | 0·115 | 0·663 |
| 0·0875 | 0·459 | 0·691 | 0·231 | 1·81 | 0·128 | 0·503 |
| 0·1000 | 0·315 | 0·794 | 0·254 | 1·84 | 0·138 | 0·345 |
| 0·1125 | 0·163 | 0·931 | 0·276 | 1·88 | 0·147 | 0·184 |
| 0·120 | 0·000 | 1·110 | 0·290 | 1·92 | 0·151 | 0·000 |

In Table II the values of

$$\frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{b_0 b_1^{\frac{1}{2}}}, \quad \frac{b_1^{\frac{1}{2}}\theta}{\nu^{\frac{1}{2}}} \quad \text{and} \quad \frac{\delta_1 b_1^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}$$

are tabulated against $x^*$ over the entire range of the solution, where $\theta$ is the momentum integral $\int_0^\infty \left(1-\frac{u}{U}\right)\frac{u}{U}\,dy$ and $\delta_1$ is the displacement thickness $\int_0^\infty \left(1-\frac{u}{U}\right)dy$. The values of these functions are shown graphically in figs. 4, 5 and 6. The velocity distributions corresponding to $x^* = 0{\cdot}0125$, 0·025, 0·0375, 0·05, 0·0625, 0·075, 0·0875, 0·1, 0·1125 and 0·120 are given in Table III and some of these velocity curves are shown graphically in fig. 7. Values of $\nu^{\frac{1}{2}}\frac{\partial u}{\partial y}\Big/b_0 b_1^{\frac{1}{2}}$ are tabulated in Table IV for $x^* = 0{\cdot}025$, 0·05, 0·075, 0·1 and 0·120.

An additional check on the methods of solution used to complete the solution in series may be obtained in the following way. If we denote by

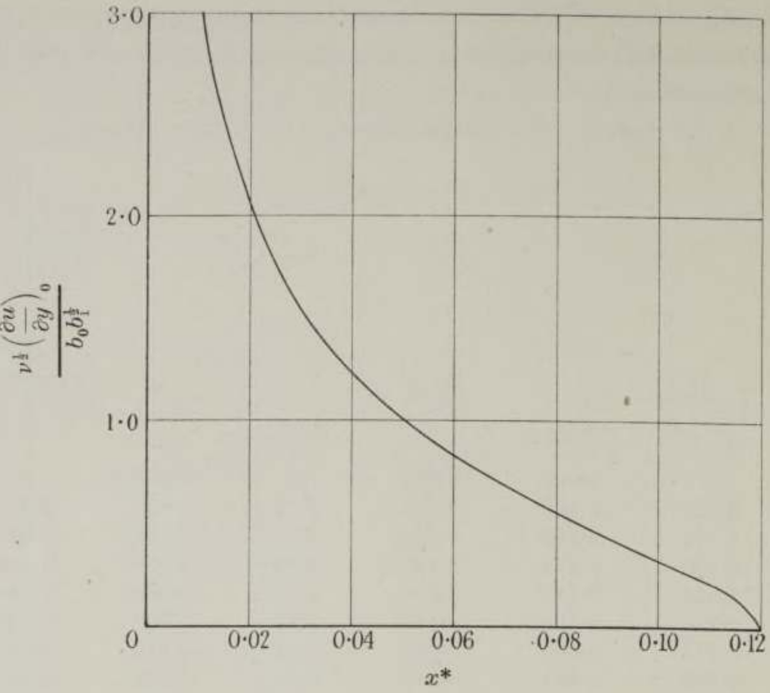
3·0
2·0
1·0
0
0·02
0·04
0·06
0·08
0·10
0·12
$x^*$

FIG. 4

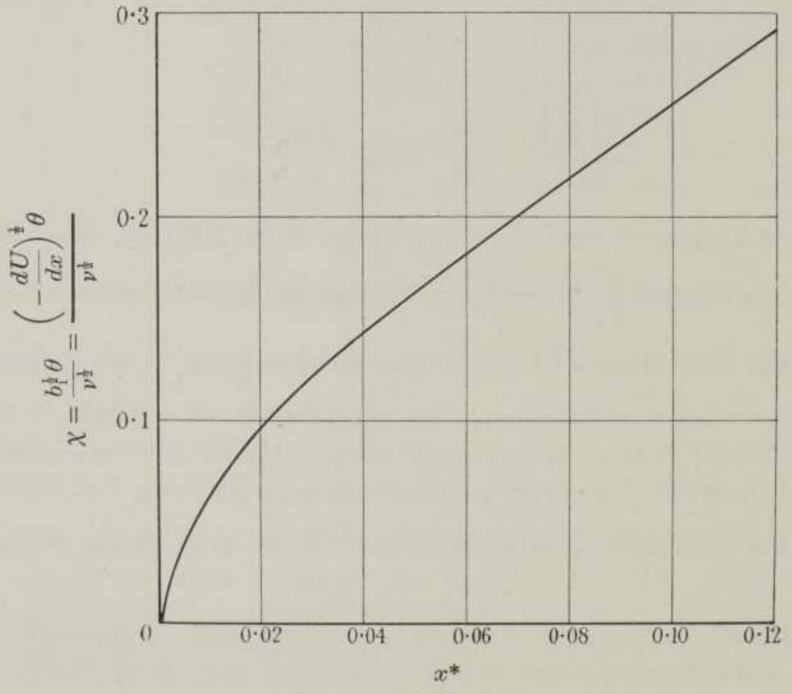
0·3
0·2
0·1
0
0·02
0·04
0·06
0·08
0·10
0·12
$x^*$

FIG. 5

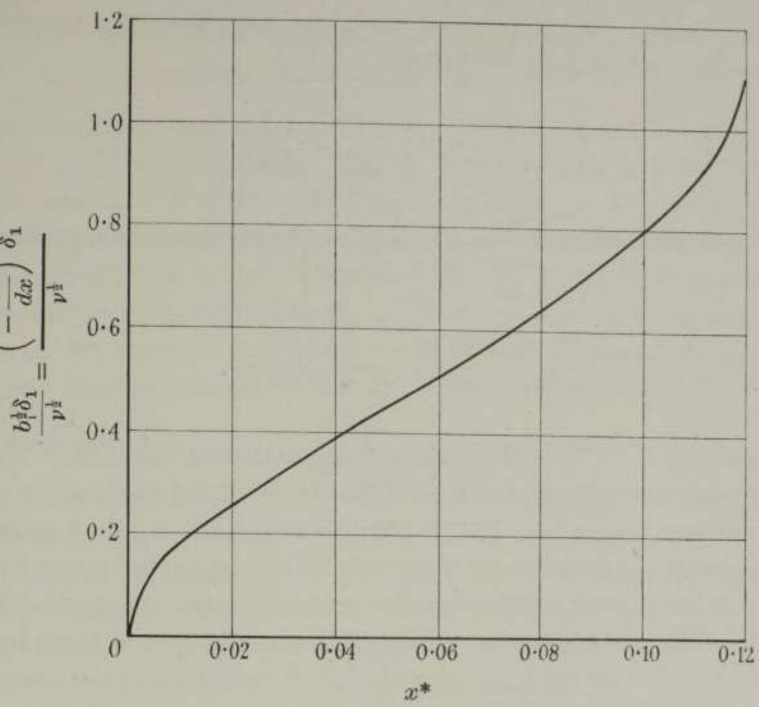
1·2
1·0
0·8
0·6
0·4
0·2
0
0·02
0·04
0·06
0·08
0·10
0·12
x*

FIG. 6

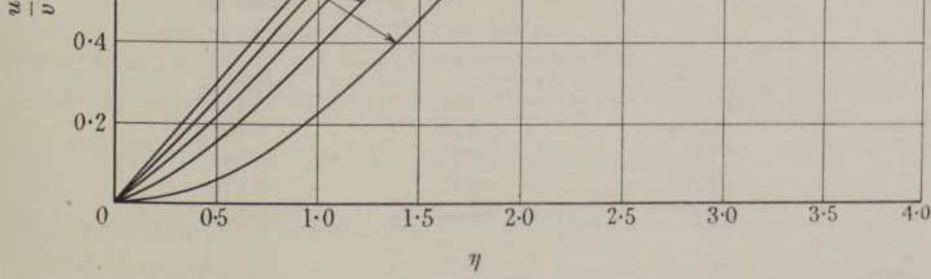
1·0
0·8
0·6
0·4
0·2
0
x*=0·025
=0·050
=0·075
=0·100
=0·120
0·5
1·0
1·5
2·0
2·5
3·0
3·5
4·0
η

FIG. 7

$\chi$ the non-dimensional quantity $\theta b_1^{\frac{1}{2}}/\nu^{\frac{1}{2}}$ it is necessary in the following work to determine $d\chi/dx^*$; this may most easily be done from the momentum integral equation, which may be written

$$\frac{d\theta}{dx} + (H+2)\,\theta\,\frac{1}{U}\,\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}, \tag{29}$$

where $H = \delta_1/\theta$. Equation (29) may be written, in the case of our problem, as

$$\frac{d\chi}{dx^*} - \frac{(H+2)\,\chi}{1-x^*} = \frac{\nu^{\frac{1}{2}}\left(\dfrac{\partial u}{\partial y}\right)_0}{(1-x^*)^2\, b_0 b_1^{\frac{1}{2}}}. \tag{30}$$

Thus the value of $d\chi/dx^*$ corresponding to a particular value of $x^*$ may be determined from the values shown in Table II by direct application of the momentum integral equation. If this equation were not satisfied we should expect a discrepancy to arise between the values of $d\chi/dx^*$ given by (30) and those obtained graphically from the curve or from numerical differentiation of the table of values of $\chi$. It was in fact made very evident by trial that an error as small as 0·001 in the value of $x_s^*$ would cause a considerable discrepancy between the value of $d\chi/dx^*$ given by (30) and that obtained from the $\chi(x^*)$ curve. Starting from the value of $\chi$ at $x^* = 0{\cdot}0625$ and by integrating the values of $d\chi/dx^*$ obtained from (30) we obtain the values of $\chi$ given in the second column of the following table. The values of $\chi$ already determined by integration from the velocity curves are given in the third column. The agreement will be seen to be very good.

| $x^*$ | $\chi$ Calculated from the values of $d\chi/dx^*$ given by equation (30) | $\chi$ Calculated by integration from the velocity curves |
|---|---|---|
| 0·0625 | — | 0·186 |
| 0·0750 | 0·209 | 0·209 |
| 0·0875 | 0·232 | 0·231 |
| 0·100 | 0·254 | 0·254 |
| 0·1125 | 0·277 | 0·276 |
| 0·120 | 0·291 | 0·290 |

We may take this agreement as fairly conclusive proof that the value of $x_s^*$ (0·120) determined by either of the approximate methods is the correct one.

## PART II

### New method of solving the boundary layer equations for the general case of retarded flow

For simplicity, we will consider in the first place the method as it was crudely conceived originally. The original idea was to replace the velocity distribution at the edge of the boundary layer, in a retarded region, by a polygon of a finite number of sides. We will suppose that the solution has been carried as far as the retarded region by some other method, because there seems to be little doubt that many of the existing methods (Pohlhausen's for example) are reasonably adequate in an accelerated region. We may suppose, therefore, for our present purpose that the skin friction and momentum integral $\theta = \int_0^\infty \left(1-\frac{u}{U}\right)\frac{u}{U}\,dy$ are known at the commencement of the first side; we may write the velocity distribution corresponding to the first side in the form $U = b_0' - b_1 X$, where $X$ is measured from the first vertex. The essential assumption introduced now is that, once the pressure, the pressure gradient and the skin friction (or the momentum integral) are known at a particular point, the velocity distribution throughout the boundary layer is completely determined and is given by the appropriate one of the singly infinite family of velocity curves given by the problem of Part I. Although not precisely true it is hoped that such an assumption offers a reasonable basis for approximation.

The method consists in determining $b_0$ and $x_0^*$ $(=b_1 x_0/b_0)$ so that when $x = x_0$ in the solution of the problem we have just discussed (with $U = b_0 - b_1 x$) the skin friction is identical with the known skin friction at the beginning of the first side and $U$ is equal to $b_0'$ there. The solution at any point in the first side is then given by putting

$$x^* = x_0^* + \frac{b_1 X}{b_0};$$

the solution as far as the second vertex is therefore known and the process can be repeated. The determination of $b_0$ and $x_0^*$ is simple; we notice that

$$b_0' = b_0 - b_1 x_0 = b_0(1 - x_0^*), \tag{31}$$

and since $(\partial u/\partial y)_{X=0}$ is supposed known we can evaluate

$$\frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{\substack{y=0\\X=0}}}{b_0' b_1^{\frac{1}{2}}}.$$

In virtue of (31)

$$\frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{\substack{y=0\\X=0}}}{b_0' b_1^{\frac{1}{2}}} = \frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{\substack{y=0\\X=0}}}{b_0(1-x^*)\, b_1^{\frac{1}{2}}}. \tag{32}$$

We can plot the right-hand side of (32) once and for all as a function of $x_0^*$ (see Table II and fig. 8) and from the graph we can read off from the given value of

$$\frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{\substack{y=0\\X=0}}}{b_0' b_1^{\frac{1}{2}}}$$

the appropriate value of $x_0^*$. Knowing $x_0^*$, $b_0$ is determined from (31), and at any point of the first side $x^*$ is determined by the expression $x_0^* + (b_1 X/b_0)$. Similarly $x^*$ is determined in subsequent sides.

TABLE III. VALUES OF $u/U$

| $\eta$ \ $x^*$ | 0·0125 | 0·025 | 0·0375 | 0·050 | 0·0625 | 0·075 | 0·0875 | 0·100 | 0·1125 | 0·120 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0·0 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 | 0·000 |
| 0·2 | 0·125 | 0·117 | 0·108 | 0·099 | 0·089 | 0·078 | 0·066 | 0·052 | 0·034 | 0·010 |
| 0·4 | 0·251 | 0·237 | 0·222 | 0·205 | 0·188 | 0·168 | 0·146 | 0·120 | 0·085 | 0·038 |
| 0·6 | 0·377 | 0·358 | 0·338 | 0·317 | 0·293 | 0·267 | 0·237 | 0·202 | 0·152 | 0·085 |
| 0·8 | 0·498 | 0·477 | 0·455 | 0·430 | 0·403 | 0·372 | 0·337 | 0·294 | 0·234 | 0·149 |
| 1·0 | 0·611 | 0·590 | 0·567 | 0·541 | 0·513 | 0·480 | 0·442 | 0·394 | 0·325 | 0·227 |
| 1·2 | 0·711 | 0·692 | 0·670 | 0·645 | 0·617 | 0·585 | 0·546 | 0·498 | 0·426 | 0·318 |
| 1·4 | 0·796 | 0·779 | 0·760 | 0·738 | 0·712 | 0·682 | 0·646 | 0·598 | 0·527 | 0·416 |
| 1·6 | 0·864 | 0·850 | 0·834 | 0·815 | 0·794 | 0·769 | 0·736 | 0·692 | 0·625 | 0·517 |
| 1·8 | 0·914 | 0·904 | 0·891 | 0·877 | 0·860 | 0·839 | 0·812 | 0·776 | 0·716 | 0·616 |
| 2·0 | 0·949 | 0·942 | 0·934 | 0·923 | 0·910 | 0·894 | 0·872 | 0·844 | 0·794 | 0·708 |
| 2·2 | 0·972 | 0·967 | 0·962 | 0·954 | 0·946 | 0·934 | 0·918 | 0·897 | 0·858 | 0·787 |
| 2·4 | 0·985 | 0·983 | 0·979 | 0·975 | 0·969 | 0·961 | 0·951 | 0·936 | 0·908 | 0·853 |
| 2·6 | 0·993 | 0·991 | 0·990 | 0·987 | 0·984 | 0·979 | 0·972 | 0·962 | 0·943 | 0·903 |
| 2·8 | 0·997 | 0·996 | 0·995 | 0·994 | 0·992 | 0·989 | 0·985 | 0·978 | 0·967 | 0·940 |
| 3·0 | 0·998 | 0·998 | 0·998 | 0·997 | 0·996 | 0·995 | 0·992 | 0·989 | 0·982 | 0·965 |
| 3·2 | 0·999 | 0·999 | 0·999 | 0·999 | 0·999 | 0·998 | 0·997 | 0·994 | 0·991 | 0·981 |
| 3·4 | 1·000 | 1·000 | 1·000 | 1·000 | 1·000 | 0·999 | 0·999 | 0·998 | 0·995 | 0·990 |
| 3·6 | — | — | — | — | — | 1·000 | 1·000 | 0·999 | 0·998 | 0·995 |
| 3·8 | — | — | — | — | — | — | — | 1·000 | 0·999 | 0·998 |
| 4·0 | — | — | — | — | — | — | — | — | 1·000 | 0·999 |
| 4·2 | — | — | — | — | — | — | — | — | — | 1·000 |

Knowing $x^*$ at every point the solution is completely determined. For, in the first place, the velocity distribution is given by the corresponding one of the singly infinite system of velocity curves (some of these curves are shown in Table III and fig. 7). Furthermore, if $U$ and $dU/dx$ are respectively the velocity and the velocity gradient at the point considered

$$U = b_0(1-x^*)$$

by the definition of $b_0$ and $x^*$. Therefore

$$\frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{b_0 b_1^{\frac{1}{2}}(1-x^*)} = \frac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{U(-dU/dx)^{\frac{1}{2}}}.$$

This quantity is shown in Table II and plotted in fig. 8 as a function of $x^*$. Thus given $x^*$, $\dfrac{\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0}{U(-dU/dx)^{\frac{1}{2}}}$ is determined, and since $U$ and $dU/dx$ are given the skin friction can be evaluated.

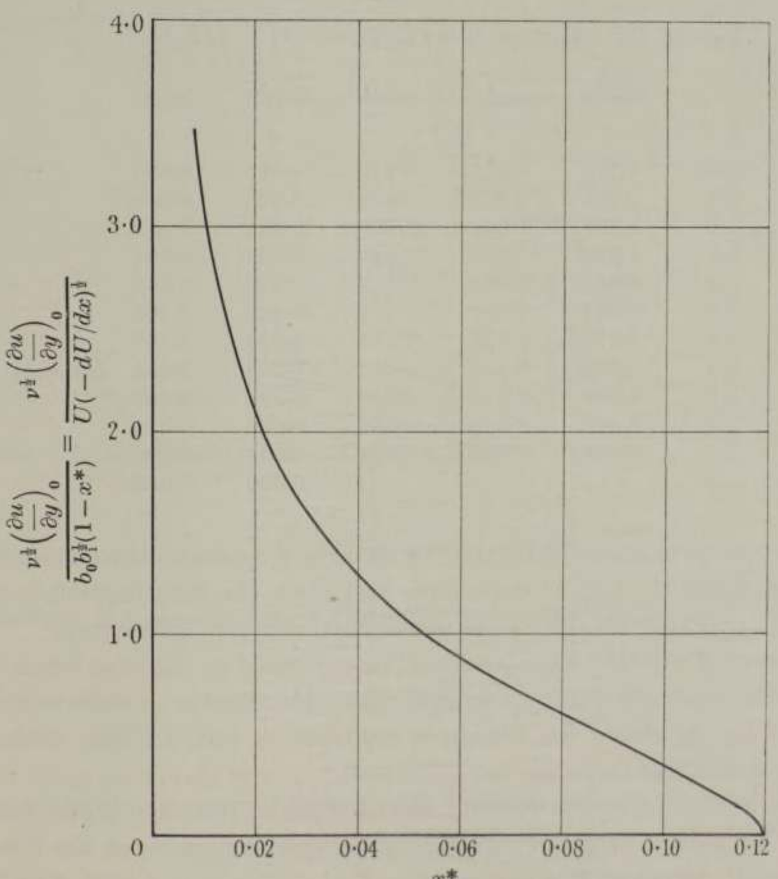


Fig. 8

We naturally obtain a continuous curve for the skin friction. This method is not all that could be desired, because making the skin friction continuous at the vertices corresponds to introducing an impulse at each vertex.

Probably a more satisfactory method would be to make $\theta$ continuous. The solution of this problem is simpler than the preceding because $b_1^{\frac{1}{2}}\theta/\nu^{\frac{1}{2}}$ is given as a function of $x^*$ by the original problem. Knowing $\theta$ initially $b_1^{\frac{1}{2}}\theta/\nu^{\frac{1}{2}}$ can be evaluated at the beginning of the first side of the polygon and the corresponding value of $x_0^*$ read off from the graph of $b_1^{\frac{1}{2}}\theta/\nu^{\frac{1}{2}}$ against $x^*$. The solution at any other point is given, as before, by putting

$$x^* = x_0^* + \frac{b_1 X}{b_0}, \quad \text{where} \quad b_0 = b_0'/(1-x_0^*).$$

The value of $\theta$ being thus determined at the second vertex the process may be repeated.

TABLE IV. TABLE OF VALUES OF $\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)\Big/ b_0 b_1^{\frac{1}{2}}$.

| $\eta$ \ $x^*$ | 0·025 | 0·050 | 0·075 | 0·100 | 0·120 |
|---|---|---|---|---|---|
| 0·0 | 1·772 | 1·011 | 0·613 | 0·315 | 0·000 |
| 0·4 | 1·866 | 1·162 | 0·801 | 0·533 | 0·240 |
| 0·8 | 1·796 | 1·200 | 0·906 | 0·691 | 0·454 |
| 1·2 | 1·467 | 1·050 | 0·860 | 0·733 | 0·604 |
| 1·6 | 0·960 | 0·741 | 0·663 | 0·632 | 0·642 |
| 2·0 | 0·481 | 0·406 | 0·401 | 0·432 | 0·504 |
| 2·4 | 0·179 | 0·168 | 0·184 | 0·229 | 0·367 |
| 2·8 | 0·049 | 0·051 | 0·064 | 0·093 | 0·193 |
| 3·2 | 0·009 | 0·011 | 0·016 | 0·028 | 0·103 |
| 3·6 | 0·002 | 0·002 | 0·004 | 0·006 | 0·023 |
| 4·0 | 0·000 | 0·000 | 0·000 | 0·001 | 0·006 |
| 4·4 | — | — | — | 0·000 | 0·000 |

The objection to this method is that by making $\theta$ continuous we make the skin friction discontinuous at each join, and since the skin friction is one of the most important results of the calculation it is not satisfactory.

This method of solution can, however, be extended to the case when the sides of the polygon are made to tend to zero, the number of sides tending to infinity. By so doing we obtain a continuous skin friction without introducing a series of impulses.

Let us approximate to the velocity distribution at the edge of the boundary layer by means of a circumscribing polygon whose sides are infinitesimally small. Suppose $B$ is one vertex of this polygon and that $A$ and $C$ are the points of contact of the sides through $B$. Suppose $A$, $B$ and $C$ have abscissae $x$, $x+\delta x_1$, $x+\delta x$ respectively. We may then write the slope of $AB$ as $(dU/dx)_x$ and that of $BC$ as $(dU/dx)_{x+\delta x}$.

The method essentially consists in determining the value of $x^*$ appropriate to any value of $x$. Let us suppose the value of $x^*$ is known at $A$. The change $\delta x_1^*$ in $x^*$ as we pass from $A$ to $B$ (considered as a point of $AB$) is, retaining the same notation as we used previously,

$$\frac{b_1\delta x_1}{b_0} = \frac{b_1\delta x_1(1-x^*)}{b_0'} = -(1-x^*)\frac{1}{U}\frac{dU}{dx}\delta x_1.$$

The change $\delta x_2^*$ in $x^*$ as we pass from $B$, considered as a point of $AB$, to $B$ considered as a point of $BC$ is given by our hypothesis that $\theta$ is continuous at $B$.

Now $$\frac{\theta}{\nu^{\frac{1}{2}}} = \frac{\chi}{b_1^{\frac{1}{2}}}.$$

Hence if $\theta$ is continuous

$$\frac{\chi(x^*+\delta x_1^*)}{\left(-\dfrac{dU}{dx}\right)_x^{\frac{1}{2}}} = \frac{\chi(x^*+\delta x_1^*+\delta x_2^*)}{\left(-\dfrac{dU}{dx}\right)_{x+\delta x}^{\frac{1}{2}}}, \tag{33}$$

i.e. $$1+\frac{1}{2}\left(\frac{d^2U/dx^2}{dU/dx}\right)_x\delta x = 1+\left(\frac{\dfrac{d\chi}{dx^*}}{\chi}\right)_{x^*+\delta x_1^*}\delta x_2^*.\dagger \tag{34}$$

Moreover, $$\left(\frac{1}{\chi}\frac{d\chi}{dx^*}\right)_{x^*+\delta x_1^*}\delta x_2^* = \left(\frac{1}{\chi}\frac{d\chi}{dx^*}\right)_{x^*}\delta x_2^*,$$

retaining only first order quantities. Thus

$$\delta x_2^* = \frac{1}{2}\left\{\frac{d^2U}{dx^2}\chi\bigg/\frac{dU}{dx}\frac{d\chi}{dx^*}\right\}\delta x. \tag{35}$$

Further, the change $\delta x_3^*$ in $x^*$ as we pass from $B$ to $C$ along $BC$ is given by

$$\delta x_3^* = -(1-x^*-\delta x^*)\left(\frac{1}{U}\frac{dU}{dx}\right)_{x+\delta x}(\delta x-\delta x_1),$$

† Since $\chi$ is known to be a differentiable function of $x^*$, and since $\delta x_2^*$ will be small so long as the change in slope from $AB$ to $BC$ is small, we can write

$$\chi(x^*+\delta x_1^*+\delta x_2^*) = \chi(x^*+\delta x_1^*)+\left(\frac{\partial\chi}{\partial x^*}\right)_{x^*+\delta x_1^*}\delta x_2^*$$

to the first order. Equation (34) then follows immediately.

where $\delta x^*$ is the total change in $x^*$ from $A$ to $C$. Thus to the first order

$$\delta x_3^* = -(1-x^*)\left(\frac{1}{U}\frac{dU}{dx}\right)(\delta x - \delta x_1). \tag{36}$$

Hence the total change in $x^*$ as we pass from $A$ to $C$ is

$$\begin{aligned}\delta x^* &= \delta x_1^* + \delta x_2^* + \delta x_3^* \\ &= \left\{-(1-x^*)\frac{1}{U}\frac{dU}{dx} + \frac{1}{2}\frac{\chi}{\dfrac{d\chi}{dx^*}}\frac{d^2U/dx^2}{dU/dx}\right\}\delta x.\end{aligned}$$

Proceeding to the limit when the sides all tend to zero we see that

$$\frac{dx^*}{dx} = -\frac{1}{U}\frac{dU}{dx}(1-x^*) + \frac{1}{2}\frac{\chi(x^*)}{\dfrac{d\chi(x^*)}{dx^*}}\frac{d^2U/dx^2}{dU/dx}. \tag{37}$$

(37) is a differential equation for $x^*$ in terms of $x$, since $\chi(x^*)$ is a known function of $x^*$ shown in Table II and fig. 5, and $U$ is a known function of $x$. The solution may be obtained graphically or otherwise once the initial value of $x^*$ is known. If, as before, we suppose the solution starts from a given value $\theta_0$ of $\theta$ at $x = x_0$, then we can evaluate

$$\left(-\frac{dU}{dx}\right)_{x_0}^{\frac{1}{2}}\frac{\theta_0}{\nu^{\frac{1}{2}}}$$

and determine the initial value of $x^*$ from the graph of $\chi(x^*)$. Thus the initial value of $x^*$ corresponding to $x_0$ is determined, and hence the complete relation between $x^*$ and $x$ is known from the solution of (37). Once the value of $x^*$ appropriate to a given value of $x$ is known we can determine the skin friction and the velocity distribution. The skin friction is obtained by reading off the corresponding value of

$$\frac{\nu^{\frac{1}{2}}\left(\dfrac{\partial u}{\partial y}\right)_0}{b_0(1-x^*)\,b_1^{\frac{1}{2}}}$$

from Table II or fig. 8; since $b_0(1-x^*) = U$ and $b_1 = -dU/dx$, $\nu^{\frac{1}{2}}(\partial u/\partial y)_0$ is determined. The velocity distribution is given by the appropriate one of the singly infinite family of velocity curves given by different values of $x^*$ in the exact solution of the original problem. Some of these curves are shown in Table III and fig. 7.

So far it has been assumed that another method has been used to carry the solution just into the retarded region (i.e. just past the pressure minimum).

In all applications it would probably be most convenient to start the solution at the pressure minimum. Since $dU/dx$ vanishes at the pressure minimum it is evident that $x^*$ also vanishes there (since it contains $dU/dx$ as a factor). Further, equation (37) has a singular point when $x^* = 0$ and $dU/dx = 0$, since $\chi \sim x^{*\frac{1}{2}}$ (see below) for small values of $x^*$. The appropriate value of $dx^*/dx$ must be determined from the value of $\theta$ at the pressure minimum obtained, of course, from the method of solution used as far as the pressure minimum.

We may proceed here as we did before. Let us suppose that $A$ is the velocity maximum (the pressure minimum) and suppose that $AB$ and $BC$ are two adjacent sides of the circumscribing polygon ($AB$ will therefore be parallel to the $x$-axis). As before we take the abscissae of $A$, $B$ and $C$ to be $x$, $x+\delta x_1$ and $x+\delta x$. Thus the slope of $BC$ is, to the first order, $(d^2U/dx^2)_x\,\delta x$. The value $\theta_0$ of $\theta$ is supposed given at $A$ by the method of solution used in the accelerated region and will be used as our starting point. Apart from a first order quantity the value of $\theta$ at $B$ considered as a point of $AB$ will be $\theta_0$. Therefore the value of $\chi$ at $B$ considered as a point of $BC$ is determined by making $\theta$ continuous at $B$ and is $\dfrac{\theta_0}{\nu^{\frac{1}{2}}}\left(-\dfrac{d^2U}{dx^2}\,\delta x\right)^{\frac{1}{2}}$ neglecting quantities of the order of $(\delta x)^{\frac{3}{2}}$. Now $\chi$ vanishes at $A$, so we may write, at $C$,

$$\delta\chi = \frac{\theta_0}{\nu^{\frac{1}{2}}}\left(-\frac{d^2U}{dx^2}\,\delta x\right)^{\frac{1}{2}}.\dagger \tag{38}$$

Moreover, for small values of $x^*$, $\chi$ is of the form $Kx^{*\frac{1}{2}}$,‡ where $K = 0{\cdot}664$. Therefore to the same order

$$\delta\chi = K(\delta x^*)^{\frac{1}{2}}, \tag{39}$$

since $\chi = \delta\chi$ when $x^* = \delta x^*$. Equating the values of $\delta\chi$ from (38) and (39) we see that

$$\frac{dx^*}{dx} = -2{\cdot}269\,\frac{d^2U}{dx^2}\,\frac{\theta_0^2}{\nu}. \tag{40}$$

Thus, starting from the pressure minimum with a known value of $\theta_0$ it is necessary to solve the equation

$$\frac{dx^*}{dx} = -\frac{1}{U}\frac{dU}{dx}(1-x^*) + \tfrac{1}{2}\chi\,\frac{d^2U}{dx^2}\bigg/\frac{dU}{dx}\,\frac{d\chi}{dx^*}$$

† Since the change in $\chi$ from $B$ to $C$ is $0(\delta x)^2$.

‡ From the solution in series for $u$ we see that

$$\chi = \frac{b_1^{\frac{1}{2}}\theta}{\nu^{\frac{1}{2}}} = 2x^{*\frac{1}{2}}\int_0^\infty \tfrac{1}{2}f_0(1-\tfrac{1}{2}f_0)\,d\eta + 0(x^{*\frac{3}{2}}).$$

with the boundary conditions $x^* = 0, \frac{dx^*}{dx} = -2{\cdot}269 \frac{d^2U}{dx^2}\left(\frac{\theta_0^2}{\nu}\right)$ (the additional boundary condition being required since $x^* = 0$ is a singular point with an infinite number of integrals passing through it). The function $\chi \Big/ \frac{d\chi}{dx^*}$ is shown in Table II and fig. 9, tabulated or plotted against $x^*$.

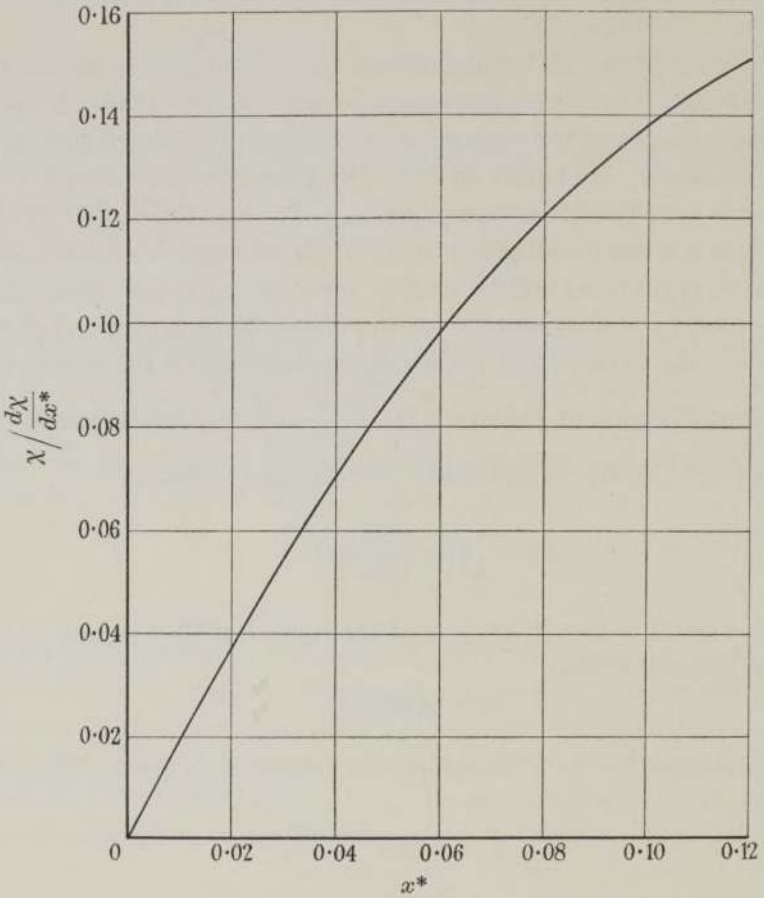


FIG. 9

## Comparison with a known solution

Falkner and Skan have given the exact solution of the boundary layer equation for a velocity distribution $U = x^{-m}$ at the edge of the boundary layer for values of $m$ between 0·09 and 0 (they also give the solution for negative values of $m$, but that need not concern us here). The present method of solution does not lend itself to starting from the infinite value of $U$ at

$x = 0$; we can, however, start from the exact value of $\theta$ at any point further downstream and continue by means of the present solution in order to test the adequacy of this method of solution. Equation (40) reduces to

$$\frac{dx^*}{dx} = \frac{1}{x}\left[m(1-x^*) - \tfrac{1}{2}(m+1)\frac{\chi}{\frac{d\chi}{dx^*}}\right]. \tag{41}$$

Falkner and Skan do not, however, give sufficient information to enable us to calculate $\theta$ at any point for a given value of $m$, but we may choose the initial value of $x^*$ for starting our solution so that it leads to the right skin friction.

For the value $m = 0{\cdot}09$ Falkner and Skan find a solution in which the skin friction vanishes for all values of $x$. Let us consider the same problem and start our solution from the value of $x^*$ which gives the right skin friction initially (zero), i.e. we start from $x^* = 0{\cdot}120$.

With $x^* = 0{\cdot}120$ we find from (41) that

$$\frac{dx^*}{dx} = \frac{1}{x}\left[m(0{\cdot}880) - \frac{m+1}{2}0{\cdot}151\right].$$

In order that the skin friction should vanish everywhere $dx^*/dx$ must vanish everywhere so that $x^*$ retains the value $0{\cdot}120$ for all values of $x$; we require therefore

$$\frac{m}{m+1} = \frac{0{\cdot}0755}{0{\cdot}880},$$

which leads to a value $m = 0{\cdot}0938$.

Thus the present method gives a value $0{\cdot}0938$ for $m$ compared with the value $0{\cdot}09$ obtained by Falkner and Skan. Hartree, using the differential analyser, has recently given a value $0{\cdot}0905$ for this quantity (1937).

Pohlhausen's equation for $\lambda\left(=\dfrac{dU}{dx}\dfrac{\delta^2}{\nu}\right)$ may be put in the form

$$\frac{d\lambda}{dx} = \frac{1}{U}\frac{dU}{dx}f(\lambda) + \frac{1}{\frac{dU}{dx}}\frac{d^2U}{dx^2}h(\lambda) \quad \text{(Howarth 1934, p. 14).}$$

This method gives separation when $\lambda = -12$, so that applying a similar analysis here to determine the appropriate value of $m$ we find

$$\frac{m}{m+1} = -\frac{h(-12)}{f(-12)},$$

which leads to a result $m = 0{\cdot}100$.

The agreement given by the present method with the exact solution is much better than that given by Pohlhausen's method and may, I think, be considered quite good for the following reason. The problem to which it has been applied is probably the one most likely to show up any defects because the pressure gradient is decreasing in the direction of the flow. It is evident from the values already given that with this type of flow the present method gives separation too late; an examination of equation (37) suggests that when the pressure gradient is increasing in the direction of the flow (as it usually does in practical cases between pressure minimum and separation) the present method may give separation, if anything, too early since the sign of the last term in the equation depends on the curvature of the pressure curve.

Returning now to the method suggested in this paper we can test it against the exact solution by choosing some small value of $m$, say, 0·04. Then

$$\frac{dx^*}{dx} = \frac{1}{x}\left[0\cdot04(1-x^*) - 0\cdot52\frac{\chi}{\dfrac{d\chi}{dx^*}}\right]. \tag{42}$$

With this value for $m$ the graph given by Falkner and Skan shows that

$$\nu^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_0 = 0\cdot26x^{-\frac{(3m+1)}{2}}.\dagger \tag{43}$$

Starting from the value $x = 1$ we see that

$$\nu^{\frac{1}{2}}\frac{\partial u}{\partial y}\bigg/ U\left(-\frac{dU}{dx}\right)^{\frac{1}{2}} = \frac{0\cdot26}{0\cdot2} = 1\cdot3.$$

The appropriate value of $x^*$ determined from fig. 8 is 0·040. When $x = 1$ and $x^* = 0\cdot040$, $dx^*/dx = 0\cdot015$. Now $dx^*/dx = 0$ for all values of $x$ when $x^* = 0\cdot042$. It is made clear by trial that the solution is given by $x^* = 0\cdot042$ for all values of $x$ apart from a small initial region of the order $x = 0\cdot2$ when $x^*$ passes from 0·040 to 0·042. Now when $x^* = 0\cdot042$,

$$\nu^{\frac{1}{2}}\frac{\partial u}{\partial y}\bigg/ U\left(-\frac{dU}{dx}\right)^{\frac{1}{2}} = 1\cdot25,$$

so that

$$\nu^{\frac{1}{2}}\frac{\partial u}{\partial y} = 0\cdot25x^{-\frac{(3m+1)}{2}}, \tag{44}$$

for all values of $x$.

† This value was obtained (more precisely it was half this value) from a small scale graph, so that its accuracy is somewhat doubtful.

Comparing (43) and (44) we see that the agreement may be considered excellent especially in view of the determination of the 0·26 in equation (43) from a small scale graph. It is, in fact, almost impossible to distinguish between the abscissae corresponding to the values 0·25 and 0·26.

## Comparison with experiment

Schubauer (1935) has measured the pressure distribution around an elliptic cylinder of fineness ratio 2·96 : 1. It is doubtful, however, whether the Reynolds number he used was sufficiently high.

If $V$ is the velocity at infinity, if $u$ is the ratio of the velocity at a point in the boundary layer to $V$ and if $p$ is the ratio of the pressure to $\frac{1}{2}\rho V^2$ the large terms in the second boundary layer equation (the equation of flow perpendicular to the wall) are

$$\frac{\partial p}{\partial y} = -\frac{2u^2}{r},$$

where $y$ is the ratio of the normal distance from the boundary to the minor axis $B$ and $r$ is the ratio of the radius of curvature of the boundary at the point considered to $B$. Therefore the difference between the value of $p$ at a point $y$ and the value $p_s$ of $p$ at the surface is given by

$$p - p_s = -\frac{2U^2\delta}{r}\int_0^{y/\delta}\frac{u^2}{U^2}d\left(\frac{y}{\delta}\right),$$

where $U$ is the value of $u$ and $\delta$ is the value of $y$ at the edge of the boundary layer.

Now in Schubauer's measurements, when $x = 1{\cdot}946$ (he found separation at 1·99), the value of $y$ given by $u = 0{\cdot}95U$ is approximately 0·04 so that the change in $p$ across the boundary layer is approximately $\frac{2 \times 1{\cdot}5 \times 0{\cdot}04}{4} \times \frac{1}{3}$, i.e. 0·010. The value 1/3 which has been used here for the integral is slightly low compared with the one obtained by using Schubauer's velocity distribution (a value 0·4 was obtained by the present writer by numerical integration); Schubauer also gives the result that $r$ varies between 0·17 at the end of the major axis to 4·4 at the end of the minor axis ($x = 1{\cdot}6$), so that the value 4 for $r$ at a point well past the end of the minor axis is probably on the large side. Thus the value 0·010 is a very conservative estimate of the change in $p$ across the layer. Now the third figure in $p$ is significant in determining the pressure gradient to the accuracy with which it is required in the retarded region. The average pressure gradient over the region from the pressure minimum ($x = 1{\cdot}3$) to the observed point of separation ($x = 1{\cdot}99$) is 0·190 approximately. Moreover the change in $p$ across the layer is not a constant but depends primarily on $\delta$ so that the pressure distribution at the edge of the boundary layer probably gives an entirely different pressure gradient distribution. (Schubauer estimates that $\delta$ does not vary greatly in the retarded region, he gives a variation of from 0·040 to 0·049, but Millikan who has also examined Schubauer's experimental data gives a considerably greater variation from 0·032 and 0·051.) There is therefore a reasonable doubt whether the pressure gradient obtained by Schubauer is adequate for the purpose of testing the accuracy of the method of solution suggested above.

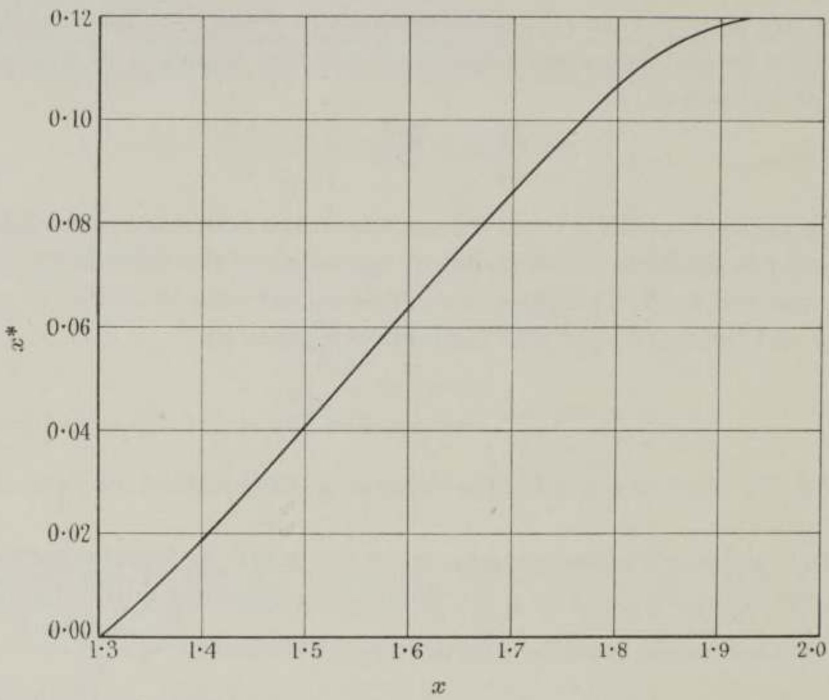


FIG. 10

However, using the velocity distribution at the edge of the boundary layer deduced by Schubauer from his observed pressure distribution, I estimated $dU/dx$ and $d^2U/dx^2$ graphically in the retarded region and applied the method suggested above starting from the value of $\theta$ at the pressure minimum given by Pohlhausen's method. The integral curve for $x^*$ is shown in fig. 10; it will be noticed that the value of $x$ at the point of separation is 1·925 and that $x^*$ does not rise much above the separation value (since $dx^*/dx$ is small at the point of separation). The values of the skin friction in the retarded region are shown graphically in fig. 11.† Schubauer using smoke to locate the separation point found $x = 1{\cdot}99$ for separation, whilst Millikan using Kármán and Millikan's method gives $x = 1{\cdot}88$.

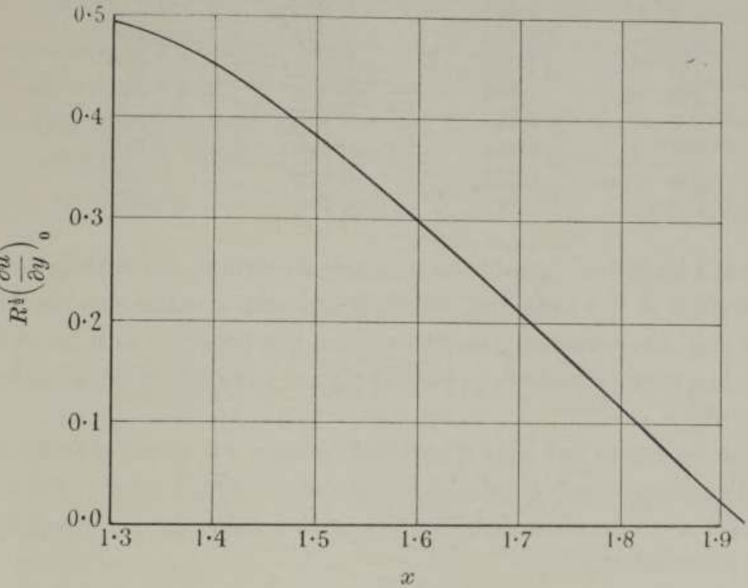


Fig. 11

It should be remarked, however, that using the values Schubauer estimated for $dU/dx$ and $d^2U/dx^2$ the present method failed to give separation; $x^*$ did not rise above 0·10. The values estimated by Schubauer and those I used are shown in Tables V*a* and V*b* respectively. The reason for failure was largely the very early inflexion in the velocity curve given by Schubauer

Table V*a*

| $x$ | $U$ | $\frac{dU}{dx}$ | $\frac{d^2U}{dx^2}$ |
|---|---|---|---|
| 1·250 | 1·295 | 0·002 | −0·170 |
| 1·350 | 1·295 | −0·012 | −0·183 |
| 1·457 | 1·292 | −0·033 | −0·192 |
| 1·600 | 1·284 | −0·070 | −0·208 |
| 1·700 | 1·275 | −0·094 | −0·203 |
| 1·832 | 1·261 | −0·121 | −0·012 |
| 1·900 | 1·252 | −0·120 | +0·108 |

† It is impossible to estimate at all accurately the skin friction from Schubauer's measurements. In addition to the large correction he finds it necessary to apply near to the surface, the points are scattered in the retarded region. I have not attempted, therefore, to include his results.

Table V$b$

| $x$ | $U$ | $\frac{dU}{dx}$ | $\frac{d^2U}{dx^2}$ |
|---|---|---|---|
| 1·3 | 1·295 | 0·000 | −0·215 |
| 1·4 | 1·294 | −0·024 | −0·255 |
| 1·5 | 1·290 | −0·050 | −0·260 |
| 1·6 | 1·284 | −0·075 | −0·255 |
| 1·7 | 1·275 | −0·100 | −0·220 |
| 1·8 | 1·264 | −0·119 | −0·140 |
| 1·9 | 1·252 | −0·126 | −0·005 |
| 1·925 | 1·249 | −0·126 | +0·055 |

and for which I have been unable to find any evidence. My estimation gave a value 1·90 for $x$ at the inflexion whilst Schubauer's value was about 1·84. In any case the comparatively large value of the pressure difference across the layer makes it very doubtful whether we are justified in determining the second derivative at all.

It may also be remarked that when Pohlhausen's method was applied to the values I estimated for $dU/dx$ and $d^2U/dx^2$ the maximum value for $-\lambda$ was about 7·5 a value considerably below the value 12 required by this method for separation.

I wish to express my gratitude to the Air Ministry for providing me with a computer to perform much of the mechanical labour necessary in the solution of the equations (7)–(11).

## Summary

*Part I*. The problem of the flow along a flat plate placed edgewise to a steady stream, when a retarding pressure gradient varying linearly as the distance $x$ from the leading edge of the plate is superposed is discussed. If $y$ denotes distance measured perpendicular to the plate, a solution is obtained in the form of a power series in $x$ whose coefficients are functions of $y/x^{\frac{1}{2}}$. Differential equations are obtained for these coefficients. Seven of the coefficients have been obtained with reasonable accuracy and the eighth and ninth roughly. Unfortunately it appears that about eight more terms are required to carry the solution to the point of separation; the work involved in their determination is prohibitive. Two approximate methods have been developed for determining the error when the first seven terms of the series are used as an approximation. These methods lead to the determination of the point of separation and are in agreement as to

its position. If $b_0$ is the velocity at the edge of the boundary layer at the leading edge of the plate and $b_1$ is the velocity gradient, separation is found when $b_1x/b_0 = 0{\cdot}120$.

*Part II.* A method is developed for the solution of the boundary layer equations in any retarded region. It is obtained by replacing the velocity distribution at the edge of the boundary layer by a circumscribing polygon of infinitesimal sides and applying the preceding solution to each of these sides, making the momentum integral continuous at each vertex. The problem is thereby reduced to the solution of a first order differential equation.

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