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ON THE SOLUTION SETS OF SOME NONCONVEX
HYPERBOLIC DIFFERENTIAL INCLUSIONS

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper \mathbb{R}^q denotes a real q -dimensional Euclidean space with norm $|\cdot|$, M a separable metric space, Q the square $I \times I$ with $I = [0, 1]$. Let F be a multifunction from $Q \times \mathbb{R}^q \times M$ to the nonempty compact subsets of \mathbb{R}^q . Let $\lambda(x, y) = \alpha(x) + \beta(y) - \alpha(0)$, where α and β are continuous functions from I to \mathbb{R}^q satisfying $\alpha(0) = \beta(0)$.

Under suitable assumptions on F , we consider the Darboux problem for hyperbolic differential inclusions of the form

$$(D_{\lambda, \mu}) \quad \begin{cases} u_{xy}(x, y) \in F(x, y, u(x, y), \mu), \\ u(x, 0) = \lambda(x, 0), \quad u(0, y) = \lambda(0, y). \end{cases}$$

Denote by $\mathcal{S}(\lambda, \mu)$ the solution set of $(D_{\lambda, \mu})$. We prove that if F satisfies (among other assumptions) a Lipschitz condition with respect to u , then $\mathcal{S}(\lambda, \mu)$ is a retract of a convex subset of a Banach space. Furthermore, the retraction can be constructed as to depend continuously upon (λ, μ) . From this it follows that $\mathcal{S}(\lambda, \mu)$ is contractible in itself, and that the multifunction $(\lambda, \mu) \rightarrow \mathcal{S}(\lambda, \mu)$ admits a continuous selection. Finally it is shown that any two continuous selections of this multifunction can be joined by a homotopy with values in $\mathcal{S}(\lambda, \mu)$.

Contributions to the study of the topological structure of the solution sets to hyperbolic differential equations or inclusions of the form $(D_{\lambda, \mu})$ can be found in Górniewicz and Pruszko [6], Teodoru [12], Staicu [11]. In particular, in [6] it is shown that the solution set of $(D_{\lambda, \mu})$ with F single valued is an R_δ -set. Similar problems for other types of differential equations or inclusions have been studied by

many authors, including Himmelberg and Van Vleck [9], Cellina [3], Deimling [4], Papageorgiou [10].

2. NOTATION AND PRELIMINARIES

Let Z be a metric space with distance d_Z . For $a \in Z$ and B a nonempty subset of Z , we put $d_Z(a, B) = \inf_{b \in B} d_Z(a, b)$. We denote by $\mathcal{C}(Z)$ the space of all nonempty closed bounded subsets of Z , endowed with the Hausdorff metric

$$H_Z(A, B) = \max \left\{ \sup_{a \in A} d_Z(a, B), \sup_{b \in B} d_Z(b, A) \right\}, \quad A, B \in \mathcal{C}(Z).$$

Let Y be a measurable metric space with σ -algebra \mathcal{A} and let Z be a separable metric space. A multifunction $F: Y \rightarrow \mathcal{C}(Z)$ is called *measurable* (see Himmelberg [8]) if $\{y \in Y \mid F(y) \cap D \neq \emptyset\} \in \mathcal{A}$ for every closed subset D of Z . The Borel σ -algebra of Z is denoted by $\mathcal{B}(Z)$. In the sequel Q , as measurable space, is given the σ -algebra \mathcal{L} of the Lebesgue measurable subsets of Q .

We denote by C the Banach space of all continuous functions $u: Q \rightarrow \mathbb{R}^q$, equipped with the norm $\|u\|_C = \sup_{(x,y) \in Q} |u(x, y)|$. Given a continuous strictly positive function $a: Q \rightarrow \mathbb{R}$, we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma: Q \rightarrow \mathbb{R}^q$, endowed with the norm

$$(2.1) \quad \|\sigma\|_{L^1} = \iint_Q a(x, y) |\sigma(x, y)| \, dx \, dy.$$

Furthermore, by V we mean the linear subspace of $C(Q, \mathbb{R}^q)$ consisting of all $\lambda \in C$ such that there exist continuous functions $\alpha: I \rightarrow \mathbb{R}^q$ and $\beta: I \rightarrow \mathbb{R}^q$, with $\alpha(0) = \beta(0)$, satisfying $\lambda(x, y) = \alpha(x) + \beta(y) - \alpha(0)$ for every $(x, y) \in Q$. Observe that V , equipped with the norm of C , is a separable Banach space.

In the sequel, when a product $Z = Z_1 \times \dots \times Z_n$ of metric spaces Z_i , $i = 1, \dots, n$, is considered, it is assumed that Z is given the metric $\max_{1 \leq i \leq n} d_{Z_i}(x_i, y_i)$, where $(x_1, \dots, x_n), (y_1, \dots, y_n) \in Z$.

Following Hiai and Umegaki [7], a set $K \subset L^1$ is called *decomposable* if for every $u, v \in K$ and $A \in \mathcal{L}$ we have $u\chi_A + v\chi_{Q \setminus A} \in K$, where χ_A stands for the characteristic function of A . We set $\mathcal{D}(L^1) = \{X \in \mathcal{C}(L^1) \mid X \text{ is decomposable}\}$.

Let T be a Hausdorff topological space. A subspace X of T is called a *retract* of T if there is a continuous map $\varphi: T \rightarrow X$ such that $\varphi(x) = x$ for every $x \in X$.

In order to treat problem $(D_{\lambda, \mu})$ we introduce the following

Assumption (A). The multifunction $F: Q \times \mathbb{R}^q \times M \rightarrow \mathcal{C}(\mathbb{R}^q)$ satisfies:

(a₁) F is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R}^q \times M)$ -measurable,

(a₂) for each $(x, y, u) \in Q \times \mathbb{R}^q$ the multifunction $\mu \rightarrow F(x, y, u, \mu)$ is Hausdorff continuous on M ,

(a₃) there exist positive integrable functions $h: Q \rightarrow \mathbb{R}$ and $k: Q \rightarrow \mathbb{R}$ such that

$$\begin{aligned} H_{\mathbb{R}^q}(F(x, y, u, \mu), \{0\}) &\leq h(x, y) \quad \text{for every } (x, y, u, \mu) \in Q \times \mathbb{R}^q \times M, \\ H_{\mathbb{R}^q}(F(x, y, u_1, \mu), F(x, y, u_2, \mu)) &\leq k(x, y)|u_1 - u_2| \quad \text{for every } (x, y, u_i, \mu) \\ &\in Q \times \mathbb{R}^q \times M, \quad i = 1, 2. \end{aligned}$$

For $(x, y) \in Q$ and $\varepsilon > 0$, we put:

$$\begin{aligned} Q(x, y) &= [0, x] \times [0, y], \quad R(x, y) = [x, 1] \times [y, 1], \\ P(x, y; \varepsilon) &= [x - \varepsilon, x + \varepsilon] \times [y - \varepsilon, y + \varepsilon]. \end{aligned}$$

For $(\lambda, \sigma) \in V \times L^1$, consider the following Darboux problem

$$(C_{\lambda, \sigma}) \quad \begin{cases} u_{xy}(x, y) = \sigma(x, y), \\ u(x, 0) = \lambda(x, 0), \quad u(0, y) = \lambda(0, y). \end{cases}$$

Definition 1. Let $(\lambda, \sigma) \in V \times L^1$. The function $u \in C$ given by

$$u(x, y) = \lambda(x, y) + \iint_{Q(x, y)} \sigma(\xi, \eta) \, d\xi \, d\eta \quad \text{for } (x, y) \in Q,$$

is said to be *solution* of $(C_{\lambda, \sigma})$.

Clearly $(C_{\lambda, \sigma})$ has a unique solution which, in the sequel, will be denoted by $u^{\lambda, \sigma}$.

Definition 2. Let (A) be satisfied. Let $(\lambda, \mu) \in V \times M$. A function $u \in C$ is said to be *solution* of $(D_{\lambda, \mu})$ if there exists a function $\sigma \in L^1$ such that:

$$\begin{aligned} \sigma(x, y) &\in F(x, y, u(x, y), \mu) \quad \text{for } (x, y) \in Q \text{ a.e.}, \\ u(x, y) &= \lambda(x, y) + \iint_{Q(x, y)} \sigma(\xi, \eta) \, d\xi \, d\eta \quad \text{for every } (x, y) \in Q. \end{aligned}$$

We denote by $\mathcal{S}(\lambda, \mu)$ the *solution set* of $(D_{\lambda, \mu})$, i.e. the set of all solutions of $(D_{\lambda, \mu})$.

Proposition 1. Let $k: Q \rightarrow \mathbb{R}$ be a positive integrable function. Then there exists a continuous strictly positive function $a: Q \rightarrow \mathbb{R}$ which, for each $(x, y) \in Q$, satisfies

$$(2.2) \quad \iint_{R(x, y)} k(\xi, \eta) a(\xi, \eta) \, d\xi \, d\eta = \frac{1}{2} (a(x, y) - 1).$$

Proof. For $n \in \mathbb{N}$ set $x_i = i/n$, $i = 0, 1, \dots, n$. Fix $n \in \mathbb{N}$ so that

$$\iint_{[x_{i-1}, x_i] \times I} 2k(\xi, \eta) \, d\xi \, d\eta < 1, \quad i = 1, 2, \dots, n.$$

By using the Banach-Caccioppoli fixed point theorem, it is easy to show that there is a continuous strictly positive function $a_n: [x_{n-1}, x_n] \times I \rightarrow \mathbb{R}$ satisfying (2.2) (with a_n in the place of a) for every $(x, y) \in [x_{n-1}, x_n] \times I$. Then, recursively, one can construct continuous strictly positive functions $a_i: [x_{i-1}, x_i] \times I \rightarrow \mathbb{R}$, $i = 1, 2, \dots, n-1$, satisfying

$$\iint_{[x, x_i] \times [y, 1]} k(\xi, \eta) a_i(\xi, \eta) \, d\xi \, d\eta = \frac{1}{2} (a_i(x, y) - a_{i+1}(x_i, y)),$$

for every $(x, y) \in [x_{i-1}, x_i] \times I$. Define $a: Q \rightarrow \mathbb{R}$ by $a(x, y) = \sum_{i=1}^n a_i(x, y) \chi_{U_i}(x, y)$, where $U_1 = [x_0, x_1] \times I$ and $U_i = (x_{i-1}, x_i] \times I$, $i = 2, \dots, n$. It is routine to verify that the function a is continuous, strictly positive, and that a satisfies (2.2) for every $(x, y) \in Q$. This completes the proof. \square

Proposition 2. *The map $T: V \times L^1 \rightarrow C$ given by $T(\lambda, \sigma) = u^{\lambda, \sigma}$, where $u^{\lambda, \sigma}$ is the solution of $(C_{\lambda, \sigma})$, is linear and one-to-one.*

Proof. Clearly T is linear. To show that T is one-to-one, suppose that $T(\lambda_1, \sigma_1) = T(\lambda_2, \sigma_2)$ for some $(\lambda_i, \sigma_i) \in V \times L^1$, $i = 1, 2$. This implies $\lambda_1 = \lambda_2$ and thus, setting $\sigma = \sigma_1 - \sigma_2$, we have

$$(2.3) \quad \iint_{Q(x, y)} \sigma(\xi, \eta) \, d\xi \, d\eta = 0 \quad \text{for every } (x, y) \in Q.$$

Let L be the set of all Lebesgue points of σ belonging to the interior of Q , and observe that $Q \setminus L$ has Lebesgue measure zero. Let $(\xi, \eta) \in L$ be arbitrary. For $\varepsilon > 0$ sufficiently small, we have

$$(2.4) \quad \sigma(\xi, \eta) = \frac{1}{4\varepsilon^2} \iint_{P(\xi, \eta; \varepsilon)} (\sigma(\xi, \eta) - \sigma(x, y)) \, dx \, dy + \frac{1}{4\varepsilon^2} \iint_{P(\xi, \eta; \varepsilon)} \sigma(x, y) \, dx \, dy.$$

The first integral vanishes as $\varepsilon \rightarrow 0$ by virtue of a result from [5, p. 217]. The second one is zero, as consequence of (2.3) and of the equality

$$\begin{aligned} \iint_{P(\xi, \eta; \varepsilon)} \sigma(x, y) \, dx \, dy &= \iint_{Q(\xi+\varepsilon, \eta+\varepsilon)} \sigma(x, y) \, dx \, dy + \iint_{Q(\xi-\varepsilon, \eta-\varepsilon)} \sigma(x, y) \, dx \, dy \\ &\quad - \iint_{Q(\xi-\varepsilon, \eta+\varepsilon)} \sigma(x, y) \, dx \, dy - \iint_{Q(\xi+\varepsilon, \eta-\varepsilon)} \sigma(x, y) \, dx \, dy. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$, (2.4) gives $\sigma(\xi, \eta) = 0$, thus $\sigma_1 = \sigma_2$. Hence $(\lambda_1, \sigma_1) = (\lambda_2, \sigma_2)$, which implies that T is one-to-one. This completes the proof. \square

3. MAIN RESULTS

Let assumption (A) be satisfied. Let $(\lambda, \mu, \sigma) \in V \times M \times L^1$. Let $u^{\lambda, \sigma}: Q \rightarrow \mathbb{R}^q$ be the solution of $(C_{\lambda, \sigma})$. We put

$$(3.1) \quad \mathcal{V}(\lambda, \mu, \sigma) = \{\varrho \in L^1 \mid \varrho(x, y) \in F(x, y, u^{\lambda, \sigma}(x, y), \mu), (x, y) \in Q \text{ a.e.}\},$$

$$(3.2) \quad \mathcal{F}(\lambda, \mu) = \{\varrho \in L^1 \mid \varrho \in \mathcal{V}(\lambda, \mu, \varrho)\}.$$

Observe that $\mathcal{V}(\lambda, \mu, \sigma)$ is a decomposable closed bounded subset of L^1 , thus (3.1) defines a multifunction $\mathcal{V}: V \times M \times L^1 \rightarrow \mathcal{D}(L^1)$.

Furthermore, set

$$W = \{u \in C \mid u = u^{\lambda, \sigma} \text{ for some } (\lambda, \sigma) \in V \times L^1\}.$$

By Proposition 2, for each $u \in W$ there is one and only one $(\lambda, \sigma) \in V \times L^1$ such that $u = u^{\lambda, \sigma}$. In view of that, we write $u^{\lambda, \sigma}$ to denote an arbitrary member of W . Let k be the positive integrable function occurring in assumption (A). By Proposition 1, there is a continuous strictly positive function $a: Q \rightarrow \mathbb{R}$ satisfying (2.2) for every $(x, y) \in Q$. With this choice of a , for arbitrary $u^{\lambda, \sigma} \in W$ we set

$$(3.3) \quad \|u^{\lambda, \sigma}\|_W = \|u^{\lambda, \sigma}\|_C + \|\sigma\|_{L^1},$$

where $\|\sigma\|_{L^1}$ is given by (2.1). By using Proposition 2, it is easy to check that (3.3) defines a norm on W and that, under this norm, W is a Banach space.

For $\lambda \in V$, set

$$W(\lambda) = \{u \in W \mid u(x, 0) = \lambda(x, 0) \text{ for } x \in I, u(0, y) = \lambda(0, y) \text{ for } y \in I\}.$$

We observe that $W(\lambda)$ is a nonempty convex closed subset of W satisfying

$$\mathcal{T}(\lambda, \mu) \subset W(\lambda) \quad \text{for every } \mu \in M.$$

Theorem 1. *Let assumption (A) be satisfied. Let $G = \{(\lambda, \mu, u) \in V \times M \times W \mid (\lambda, \mu) \in V \times M, u \in W(\lambda)\}$. Then there exists a continuous function $\Phi: G \rightarrow W$ satisfying, for each $(\lambda, \mu) \in V \times M$, the following properties:*

$$(3.4) \quad \Phi(\lambda, \mu, u) \in \mathcal{T}(\lambda, \mu) \quad \text{for every } u \in W(\lambda),$$

$$(3.5) \quad \Phi(\lambda, \mu, u) = u \quad \text{for every } u \in \mathcal{T}(\lambda, \mu).$$

Proof. Let $\mathcal{V}: V \times M \times L^1 \rightarrow L^1$ be defined by (3.1).

(i) \mathcal{V} is Hausdorff continuous. To this end we prove first that \mathcal{V} is Hausdorff lower semicontinuous. Suppose the contrary. Then there exist an $\varepsilon > 0$, a sequence $\{(\lambda_n, \mu_n, \sigma_n)\}$ converging to $(\lambda_0, \mu_0, \sigma_0)$ in $V \times M \times L^1$, and a sequence $\{\varrho_n\} \subset L^1$, with $\varrho_n \in \mathcal{V}(\lambda_0, \mu_0, \sigma_0)$ for each $n \in \mathbb{N}$, such that

$$(3.6) \quad d_{L^1}(\varrho_n, \mathcal{V}(\lambda_n, \mu_n, \sigma_n)) \geq \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ define $M_n: Q \rightarrow \mathcal{C}(\mathbb{R}^q)$ by

$$M_n(x, y) = F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n) \\ \cap B_{\mathbb{R}^q}(\varrho_n(x, y), d_{\mathbb{R}^q}(\varrho_n(x, y), F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n))),$$

where, for $a \in \mathbb{R}^q$ and $r \geq 0$, $B_{\mathbb{R}^q}(a, r) = \{x \in \mathbb{R}^q \mid |x - a| \leq r\}$. As M_n is measurable, there exists a measurable selection $\tilde{\varrho}_n \in \mathcal{V}(\lambda_n, \mu_n, \sigma_n)$ such that

$$|\varrho_n(x, y) - \tilde{\varrho}_n(x, y)| = d_{\mathbb{R}^q}(\varrho_n(x, y), F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n)) \quad \text{for } (x, y) \in Q \text{ a.e.}$$

From this, observing that $\varrho_n(x, y) \in F(x, y, u^{\lambda_0, \sigma_0}(x, y), \mu_0)$, one has:

$$\begin{aligned} & \iint_Q a(x, y) |\varrho_n(x, y) - \tilde{\varrho}_n(x, y)| \, dx \, dy \\ &= \iint_Q a(x, y) d_{\mathbb{R}^q}(\varrho_n(x, y), F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n)) \, dx \, dy \\ &\leq \iint_Q a(x, y) H_{\mathbb{R}^q}(F(x, y, u^{\lambda_0, \sigma_0}(x, y), \mu_0), F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n)) \, dx \, dy \\ &\leq \iint_Q a(x, y) H_{\mathbb{R}^q}(F(x, y, u^{\lambda_n, \sigma_n}(x, y), \mu_n), F(x, y, u^{\lambda_0, \sigma_0}(x, y), \mu_n)) \, dx \, dy \\ &\quad + \iint_Q a(x, y) H_{\mathbb{R}^q}(F(x, y, u^{\lambda_0, \sigma_0}(x, y), \mu_n), F(x, y, u^{\lambda_0, \sigma_0}(x, y), \mu_0)) \, dx \, dy. \end{aligned}$$

Denoting by $w_n(x, y)$ the function under the sign of the last integral, and using assumption (A) (a₃), it follows that

$$\|\varrho_n - \tilde{\varrho}_n\|_{L^1} \leq \iint_Q a(x, y) k(x, y) |u^{\lambda_n, \sigma_n}(x, y) - u^{\lambda_0, \sigma_0}(x, y)| \, dx \, dy \\ + \iint_Q w_n(x, y) \, dx \, dy.$$

Let $n \rightarrow +\infty$. The first integral vanishes, for $\{u^{\lambda_n, \sigma_n}\}$ converges to u^{λ_0, σ_0} in C . Likewise does the second integral, because of the Lebesgue dominated convergence

theorem. Therefore, there is $n_0 \in \mathbb{N}$ such that $\|\varrho_n - \tilde{\varrho}_n\|_{L^1} < \frac{1}{2}\varepsilon$ for $n \geq n_0$. A fortiori

$$d_{L^1}(\varrho_n, \mathcal{V}(\lambda_n, \mu_n, \sigma_n)) < \frac{\varepsilon}{2} \quad \text{for } n \geq n_0,$$

which contradicts (3.6). Consequently \mathcal{V} is Hausdorff lower semicontinuous. The proof that \mathcal{V} is Hausdorff upper semicontinuous is similar, and thus it is omitted. Hence \mathcal{V} is Hausdorff continuous.

(ii) For every $(\lambda, \mu, \sigma_1), (\lambda, \mu, \sigma_2) \in V \times M \times L^1$,

$$(3.7) \quad H_{L^1}(\mathcal{V}(\lambda, \mu, \sigma_1), \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}.$$

Indeed, let $(\lambda, \mu, \sigma_i) \in V \times M \times L^1$, $i = 1, 2$. Denoting by u^{λ, σ_1} and u^{λ, σ_2} the solutions of (C_{λ, σ_1}) and (C_{λ, σ_2}) , respectively, one has

$$(3.8) \quad |u^{\lambda, \sigma_1}(x, y) - u^{\lambda, \sigma_2}(x, y)| \leq \iint_{Q(x, y)} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \, d\xi \, d\eta, \quad \text{for } (x, y) \in Q.$$

Let $\varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1)$ be arbitrary. Take $\varrho_2 \in \mathcal{V}(\lambda, \mu, \sigma_2)$ so that

$$|\varrho_1(x, y) - \varrho_2(x, y)| = d_{R^i}(\varrho_1(x, y), F(x, y, u^{\lambda, \sigma_2}(x, y), \mu)), \quad \text{for } (x, y) \in Q \text{ a.e.}$$

From this, observing that $\varrho_1(x, y) \in F(x, y, u^{\lambda, \sigma_1}(x, y), \mu)$, and using (A) (a_3) , (3.8) and Proposition 1, one has:

$$\begin{aligned} \|\varrho_1 - \varrho_2\|_{L^1} &\leq \iint_Q a(x, y) H_{R^i}(F(x, y, u^{\lambda, \sigma_1}(x, y), \mu), F(x, y, u^{\lambda, \sigma_2}(x, y), \mu)) \, dx \, dy \\ &\leq \iint_Q a(x, y) k(x, y) |u^{\lambda, \sigma_1}(x, y) - u^{\lambda, \sigma_2}(x, y)| \, dx \, dy \\ &\leq \iint_Q a(x, y) k(x, y) \left(\iint_{Q(x, y)} |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \, d\xi \, d\eta \right) \, dx \, dy \\ &= \iint_Q |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| \left(\iint_{R(\xi, \eta)} k(x, y) a(x, y) \, dx \, dy \right) \, d\xi \, d\eta \\ &\leq \frac{1}{2} \iint_Q |\sigma_1(\xi, \eta) - \sigma_2(\xi, \eta)| a(\xi, \eta) \, d\xi \, d\eta \\ &= \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}. \end{aligned}$$

A fortiori, $d_{L^1}(\varrho_1, \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}$ and thus, since $\varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1)$ is arbitrary,

$$\sup_{\varrho_1 \in \mathcal{V}(\lambda, \mu, \sigma_1)} d_{L^1}(\varrho_1, \mathcal{V}(\lambda, \mu, \sigma_2)) \leq \frac{1}{2} \|\sigma_1 - \sigma_2\|_{L^1}.$$

From this, and the analogous inequality obtained by interchanging the roles of ϱ_1 and ϱ_2 , (3.7) follows.

Since the multifunction $\mathcal{V}: V \times M \times L^1 \rightarrow \mathcal{D}(L^1)$ satisfies (i) and (ii), by a result of Bressan, Cellina and Fryszkowski [2] there is continuous map $\varphi: V \times M \times L^1 \rightarrow L^1$ satisfying, for each $(\lambda, \mu) \in V \times M$, the following properties:

$$(3.9) \quad \varphi(\lambda, \mu, \sigma) \in \mathcal{F}(\lambda, \mu) \quad \text{for every } \sigma \in L^1,$$

$$(3.10) \quad \varphi(\lambda, \mu, \sigma) = \sigma \quad \text{for every } \sigma \in \mathcal{F}(\lambda, \mu).$$

Let $(\lambda, \mu, u) \in G$ be arbitrary. Since $u \in W(\lambda)$, for some $\sigma \in L^1$ we have $u = u^{\lambda, \sigma}$, where $u^{\lambda, \sigma}$ is the solution of $(C_{\lambda, \sigma})$. Hence $(\lambda, \mu, u) = (\lambda, \mu, u^{\lambda, \sigma})$. Let $\Phi(\lambda, \mu, u^{\lambda, \sigma}): Q \rightarrow \mathbb{R}^q$ be given by

$$(3.11) \quad \Phi(\lambda, \mu, u^{\lambda, \sigma})(x, y) = \lambda(x, y) + \iint_{Q(x, y)} \varphi(\lambda, \mu, \sigma)(\xi, \eta) \, d\xi \, d\eta.$$

As $\Phi(\lambda, \mu, u^{\lambda, \sigma}) = u^{\lambda, \varphi(\lambda, \mu, \sigma)}$, this equality defines a map

$$\Phi: G \rightarrow W.$$

It will be shown that Φ is continuous and that satisfies (3.4) and (3.5).

The multifunction Φ is continuous. To see this, let $\varepsilon > 0$. For arbitrary $(\lambda_0, \mu_0, u^{\lambda_0, \sigma_0})$, $(\lambda, \mu, u^{\lambda, \sigma})$ in G , we have

$$(3.12) \quad \begin{aligned} & \|\Phi(\lambda, \mu, u^{\lambda, \sigma}) - \Phi(\lambda_0, \mu_0, u^{\lambda_0, \sigma_0})\|_W \\ &= \|u^{\lambda, \varphi(\lambda, \mu, \sigma)} - u^{\lambda_0, \varphi(\lambda_0, \mu_0, \sigma_0)}\|_C + \|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} \\ &\leq \|\lambda - \lambda_0\|_C + \left(1 + \frac{1}{m}\right) \|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1}, \end{aligned}$$

where m denotes the absolute minimum of the continuous strictly positive function $a: Q \rightarrow \mathbb{R}$. As φ is continuous, there is $0 < \delta < \varepsilon$ so that $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$, provided that $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|\sigma - \sigma_0\|_{L^1} < \delta$. Now, let $(\lambda, \mu, u^{\lambda, \sigma}) \in G$ satisfy $\|\lambda - \lambda_0\|_C < \delta$, $d_M(\mu, \mu_0) < \delta$ and $\|u^{\lambda, \sigma} - u^{\lambda_0, \sigma_0}\|_W < \delta$. Since $\|\sigma - \sigma_0\|_{L^1} \leq \|u^{\lambda, \sigma} - u^{\lambda_0, \sigma_0}\|_W < \delta$, we have $\|\varphi(\lambda, \mu, \sigma) - \varphi(\lambda_0, \mu_0, \sigma_0)\|_{L^1} < \varepsilon$. Hence, from (3.12),

$$\|\Phi(\lambda, \mu, u^{\lambda, \sigma}) - \Phi(\lambda_0, \mu_0, u^{\lambda_0, \sigma_0})\|_W < \delta + \left(1 + \frac{1}{m}\right) \varepsilon < \left(2 + \frac{1}{m}\right) \varepsilon,$$

and thus Φ is continuous.

Let $(\lambda, \mu) \in V \times M$. Let $u \in V(\lambda)$ be arbitrary, thus $u = u^{\lambda, \sigma}$ for some $\sigma \in L^1$. By (3.9), $\varphi(\lambda, \mu, \sigma) \in \mathcal{F}(\lambda, \mu)$ and hence $u^{\lambda, \varphi(\lambda, \mu, \sigma)} \in \mathcal{T}(\lambda, \mu)$. As $\Phi(\lambda, \mu, u^{\lambda, \sigma}) = u^{\lambda, \varphi(\lambda, \mu, \sigma)}$ and $u^{\lambda, \sigma} = u$, it follows that $\Phi(\lambda, \mu, u) \in \mathcal{T}(\lambda, \mu)$, proving (3.4).

Let $(\lambda, \mu) \in V \times M$. Let $u \in \mathcal{T}(\lambda, \mu)$ be arbitrary, that is $u = u^{\lambda, \sigma}$ for some $\sigma \in \mathcal{V}(\lambda, \mu, \sigma)$. Hence $\sigma \in \widehat{\mathcal{F}}(\lambda, \mu)$ and so, by (3.10), $\varphi(\lambda, \mu, \sigma) = \sigma$. From this and (3.11) it follows that

$$\Phi(\lambda, \mu, u) = \Phi(\lambda, \mu, u^{\lambda, \sigma}) = u^{\lambda, \varphi(\lambda, \mu, \sigma)} = u^{\lambda, \sigma} = u,$$

proving (3.5). This completes the proof of the theorem. \square

Corollary 1. *Let assumption (A) be satisfied. Then, for each $(\lambda, \mu) \in V \times M$, $\mathcal{T}(\lambda, \mu)$ is an absolute retract. Furthermore, $\mathcal{T}(\lambda, \mu)$ is a contractible closed subspace of W .*

Proof. By Theorem 1, $\mathcal{T}(\lambda, \mu)$ is a retract of $W(\lambda)$. As $W(\lambda)$ is a convex subset of W , by a result of Borsuk [1, p. 85] $\mathcal{T}(\lambda, \mu)$ is an absolute retract. Consequently $\mathcal{T}(\lambda, \mu)$ is a contractible closed subspace of W , completing the proof. \square

The following result is of a type proved by Cellina [3].

Corollary 2. *Let assumption (A) be satisfied. Then there exists a continuous map $\tau: V \times M \rightarrow W$ satisfying*

$$(3.13) \quad \tau(\lambda, \mu) \in \mathcal{T}(\lambda, \mu) \quad \text{for every } (\lambda, \mu) \in V \times M.$$

Proof. For $\lambda \in V$ set $u(\lambda) = u^{\lambda, 0}$, where $u^{\lambda, 0}$ denotes the solution of $(C_{\lambda, 0})$. Define $\tau: V \times M \rightarrow W$ by $\tau(\lambda, \mu) = \Phi(\lambda, \mu, u(\lambda))$, where Φ is the map constructed in Theorem 1. The function τ is well defined, since $u(\lambda) \in W(\lambda)$. Furthermore, τ is continuous, as $\|u(\lambda) - u(\lambda_0)\|_W = \|\lambda - \lambda_0\|_C$ for $\lambda, \lambda_0 \in V$, and satisfies (3.13), by virtue of (3.4). Hence the result. \square

Corollary 3. *Let assumption (A) be satisfied. For $i = 1, 2$, let $\tau_i: V \times M \rightarrow W$ be a continuous map such that $\tau_i(\lambda, \mu) \in \mathcal{T}(\lambda, \mu)$, for every $(\lambda, \mu) \in V \times M$. Then there exists a continuous map $h: V \times M \times I \rightarrow W$ satisfying:*

- (i) $h(\lambda, \mu, 0) = \tau_1(\lambda, \mu)$ and $h(\lambda, \mu, 1) = \tau_2(\lambda, \mu)$, for every $(\lambda, \mu) \in V \times M$,
- (ii) $h(\lambda, \mu, s) \in \mathcal{T}(\lambda, \mu)$ for every $(\lambda, \mu, s) \in V \times M \times I$.

Proof. Define $h: V \times M \times I \rightarrow W$ by

$$(3.14) \quad h(\lambda, \mu, s) = \Phi(\lambda, \mu, (1-s)\tau_1(\lambda, \mu) + s\tau_2(\lambda, \mu)),$$

where Φ is the map constructed in Theorem 1. By using (3.14), (3.5) and (3.4), it is routine to see that h has the required properties. Hence the result. \square

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