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ON THE SOLUTIONS OF QUARTIC DIOPHANTINE EQUATIONS

MAHNAZ AHMADI, ALI SARBAZ JANFADA, and KAMRAN NABARDI

Abstract. In this article, first, using the elliptic curve method, it is proved that the quartic Diophantine equations $x^4 - y^4 = k(t^\lambda - u^4 \pm v^4)$ for positive even λ and integers k and t have infinitely many non-trivial rational solutions. Then, by direct ways, parametric solutions for equations $x^4 - y^4 = k(t^3 \pm u^4 - v^4)$ are found.

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Key words. Quartic Diophantine equations, number of solutions of Diophantine equations, elliptic curves, rank of elliptic curves.

1. INTRODUCTION

Diophantine equations, i.e., equations with integer coefficients for which integer solutions are sought, are among the oldest subjects in mathematics. In the following we would like to review some efforts done for the quartic Diophantine equation.

A quartic Diophantine equation is called $m-n$, if a sum of m quartics equals to a sum of n fourth powers. For simplicity, we omit the word 'quartic'. The $2-1$ equation $x^4 + y^4 = z^4$, is a case of Fermat's Last Theorem and therefore has no solutions. Generally, the equations $x^4 \pm y^4 = z^2$ have no integer solutions.

In 1772, Euler proposed that the $3-1$ equation $A^4 + B^4 + C^4 = D^4$ has no integer solution. This assertion is known as the Euler Quartic Conjecture and was disproved by Elkies [3]. Using a geometric construction, he showed that many solutions existed, however, it is not known if a parametric solution exists.

There are many known solutions of the equation $A^4 + B^4 + C^4 = 2D^4$. Parametric solutions to the $2-3$ equation $A^4+B^4 = C^4+D^4+E^4$ are known [5]. The smallest solution is $7^4 + 7^4 = 3^4 + 5^4 + 8^4$.

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Corresponding author: Kamran Nabardi.

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Ramanujan [11] gave the general expression to the $3-3$ equation $A^4+B^4+C^4=D^4+E^4+F^4$ as:

$$3^4+(2x^4-1)^4+(4x^5+x)^4=(4x^4+1)^4+(6x^4-3)^4+(4x^5-5x)^4.$$

Some numerical integer solutions were found for $A^4+B^4+C^4+D^4=E^4$ [8, 9, 10], however, the existence of any parametric solution is not known.

The equation $A^4+hB^4=C^4+hD^4$ was considered by several number theorists. Probably, Gerardin [2, pp. 647] was the first to study it but, in the cases $h=2$ and $h=5$, Grigorief and Werebrusow recorded solutions earlier than Gerardin. Choudhry [1], considered this equation and solved it for 75 positive integer values $h\leq 101$.

The equation $A^4+B^4=2(C^4+D^4)$ and the general case $A^4+B^4=n(C^4+D^4)$ were studied in [6, 7]. The former one was solved by the elliptic curve method, while the solutions of the latter one are found by some rational transformations imposing extra conditions.

In this article, all rank computations are implemented by the 'mwrank' software. Moreover, we assume that all solutions are nontrivial. By a trivial solution of the equation $ax^m+by^n+cz^k=au^m+bv^n+cw^k$ we mean either of the following cases as well as similar circumstances.

(i) $x=u, y=v, z=w$;
(ii) $x=r^{nk}, y=s^{mk}, z=t^{mn}, u=t^{nk}, v=r^{mk}, w=s^{mn}$, for some rationals r,s,t, when $a=b=c$.

2. PRELIMINARIES

Let K be a field and C be the algebraic curve defined over K by

$$v^2=au^4+bu^3+cu^2+du+e,\ a\neq 0.$$

Consider the K-rational affine point $(u,v)=(p,q)$ on C. We may assume $p=0$ by changing u to $u+p$, if necessary. Then $e=q^2$ and the above equation turns to

$$v^2=au^4+bu^3+cu^2+du+q^2,\ a\neq 0. \tag{1}$$

Let $q=0$. If $d=0$, the curve (1) will have a singularity at $(u,v)=(0,0)$. Therefore, assume $d\neq 0$. Dividing both side of (1) by u^4 we get

$$\Big(\frac{v}{u^2}\Big)^2=d\Big(\frac{1}{u}\Big)^3+c\Big(\frac{1}{u}\Big)^2+b\Big(\frac{1}{u}\Big)+a.$$

Put $X=1/u$ and $Y=1/u^2$. Then, we obtain the elliptic curve $Y^2=dX^3+cX^2+bX+a$ in the Weierstrass form. The harder case is when $q\neq 0$ in which case we have the following result [12, Theorem 2.17].

THEOREM 2.1. *Let K be a field of characteristic not 2 and C be the algebraic curve defined over K by*

$$C: v^2=au^4+bu^3+cu^2+du+q^2,\ q\neq 0.$$

Suppose C has a K-rational point (p,q). Let

$$X=\frac{2q(v+q)+du}{u^2},\ Y=\frac{4q^2(v+q)+2q(du+cu^2)-(d^2u^2/2q)}{u^3}.$$

Define

$$a_1=d/q,\ a_2=c-(d^2/4q^2),\ a_3=2qb,\ a_4=-4q^2a,\ a_6=a_2a_4.$$

Then the curve C is in a one to one correspondence with the elliptic curve

$$E:Y^2+a_1XY+a_3Y=X^3+a_2X^2+a_4X+a_6.$$

The inverse transformation is

$$u=\frac{2q(X+c)-(d^2/2q)}{Y},\ v=-q+\frac{u(uX-d)}{2q}.$$

The point $(u,v)=(0,q)$ on C corresponds to the point $(X,Y)=\infty$ on E and $(u,v)=(0,-q)$ on C corresponds to $(X,Y)=(-a_2,a_1a_2-a_3)$ on E.

3. THE EQUATIONS $x^4-y^4=k(t^\lambda-u^4\mp v^4)$ FOR EVEN λ

Consider the Diophantine equations

$$x^4-y^4=k(t^{2r}-u^4\mp v^4),\ k\in\mathbb{Z}.$$

Put $x=u+v$, and $y=u-v$. Then, after some straightforward computations, we get

$$\left(\frac{t^r}{v^2}\right)^2=\pm\left(\frac{u}{v}\right)^4+\frac{8}{k}\left(\frac{u}{v}\right)^3+\frac{8}{k}\left(\frac{u}{v}\right)+1.$$

Now, taking $h=t^r/v^2$ and $g=u/v$ we arrive at

$$h^2=\pm g^4+\frac{8}{k}g^3+\frac{8}{k}g+1$$

which, by Theorem 2.1, corresponds to the elliptic curve in generalized Weierstrass forms

$$E_k^{\mp}:\mathcal{Y}^2+\frac{8}{k}\mathcal{X}Y+\frac{16}{k}\mathcal{Y}=\mathcal{X}^3-\frac{16}{k^2}\mathcal{X}^2\mp4\mathcal{X}\pm\frac{64}{k^2}.$$

These curves may be changed to the (short) Weierstrass forms as follows.

$$E_k^-:\quad y^2=x^3+\left(\frac{64}{k^2}-4\right)x+\frac{128}{k^2}$$

$$E_k^+:\quad y^2=x^3+\left(\frac{64}{k^2}+4\right)x.$$

Clearly $P=(2,\frac{16}{k})$ is on E_k^- and Considering $k=\frac{8(65t^2-18t+1)}{1-65t^2}$, where $t\in\mathbb{Q}-\{\frac{1}{13},\frac{1}{5}\}$, one can show that $Q=\left(\frac{1}{4},\frac{585t^2-130t+9}{8(13t-1)(5t-1)}\right)$ is a point on E_k^+. We need k to be an integer. To reconcile with that, we can take $t=\pm\frac{m}{n}$, where $n^2-65m^2=\pm1$. Thus, $(n,m)=(X,Y)$ is a positive integer solution of the Pell equation $X^2-65Y^2=\pm1$. The smallest solution of which is

$(n, m) = (8, 1)$. This has infinitely many solutions (n, m) so there are infinitely many possibilities for such t not in $\{\frac{1}{13}, \frac{1}{5}\}$ which lead to integer values of k for which E_k^+ has a rational point presumably of infinite order. For a large number of values of k, except for $k = 8$ in which case the curve is singular, rank records in Table 1 show that all ranks are positive.

k	rank(E_k^-)	rank(E_k^+)	k	rank(E_k^-)	rank(E_k^+)	k	rank(E_k^-)	rank(E_k^+)
1	1	2	11	1	1	21	3	2
2	1	1	12	2	2	22	2	2
3	2	1	13	3	2	23	2	1
4	1	1	14	1	2	24	2	2
5	2	2	15	2	3	25	1	2
6	1	2	16	1	2	26	2	1
7	1	1	17	2	2	27	2	1
8	singular	1	18	1	1	28	2	2
9	2	2	19	1	1	29	1	2
10	2	1	20	1	1	30	2	2

Table 1 – The rank of $E_k^{\mp}$

4. THE EQUATION $x^4 - y^4 = k(t^3 + u^4 - v^4)$

We want to find a family of integer solutions for

$$x^4 - y^4 = k(t^3 + u^4 - v^4). \tag{2}$$

Put

$$x = at + k, \ y = bt - k, \ u = ct + 1, \ v = t + 1. \tag{3}$$

Plugging (3) in (2), one can get

$$At^4 + Bt^3 + Ct^2 + Dt = 0,$$

where,

$$\begin{aligned} A &= a^4 - b^4 - (c^4 - 1)k, \\ B &= (4a^3 + 4b^3 - 4c^3 + 3)k, \\ C &= 6(a^2 - b^2)k^2 - 6(c^2 - 1)k, \\ D &= 4(a + b)k^3 - 4(c - 1)k. \end{aligned}$$

Now, let $D = 0$. Then, $c = (a + b)k^2 + 1$. Inserting this in (3) and then in (2) we have

$$A't^4 + B't^3 + C't^2 = 0,$$

where A', B', and C' are in terms of a, b, c, k and

$$C' = -6b^2k^2 - 12ak^5b - 12bk^3 - 6b^2k^5 - 6a^2k^5 - 12ak^3 + 6a^2k^2.$$

Again, let $C' = 0$. So, $b = -\frac{ak^3 - a + 2k}{k^3+1}$. Finally, from (3), we obtain

$$t = -\frac{1}{H}\Big((k+1)(k^2-k+1)(-7k^6 + 24ak^5 - 24a^2k^4 2 + (10+8a^3)k^3 - 24ak^+ 24a^2k + 1 - 8a^3)\Big),$$

and

$$x = -\frac{1}{H}\Big(ak^9 + 8k^7 - 29ak^6 + 48a^2k^5 - (32a^3+8)k^4 + (35a+8a^4)k^3 - 48a^2k^2 + 32a^3k - 8a^4 + a\Big),$$

$$y = -\frac{1}{H}\Big(-ak^9a + 6k^7 - 33ak^6 + 48a^2k^5 - (32a^3+12)k^4 + (8a^4 - 8a^4 + 33a)k^3 - 48a^2k^2 + (32a^3 - 2)k\Big),$$

$$u = -\frac{1}{H}\Big(7k^9 - 30ak^8 + 48a^2k^7 - (32a^3+9)k^6 + (36a+8a^4)k^5 - 48a^2k^4 + (32a^3+1)k^3 - (8a^4+2a)k^2 + 1\Big),$$

$$v = \frac{1}{H}\Big(7k^9 - 32k^8a + 48a^2k^7 - (32a^3+11)k^6 + (8a^4+32a)k^5 - 48a^2k^4 + (32a^3-3)k^3 - 8a^4k^2 - 1\Big),$$

where, $H = 8(k-1)(k^2+k+1)(ak^2+1)(a-k)^3$, in which the factors $k - 1, a-k, ak^2+1$ are assumed nonzero.

Example 4.1. Take $k = 3$. Then we get $x^4 - y^4 = 3(t^3 + u^4 - v^4)$. Then, by letting $a = -1$ we have

$$x = \frac{489}{1664},\ y = \frac{1077}{1664},\ t = \frac{5481}{1664},\ u = \frac{6949}{1664},\ v = \frac{7145}{1664}.$$

5. THE EQUATION $x^4 - y^4 = k(t^3 - u^4 - v^4)$

We are dealing with Diophantine equation

$$x^4 - y^4 = k(t^3 - u^4 - v^4). \tag{4}$$

Let

$$x = as + k,\ y = bs - k,\ t = cs + 1,\ u = s + 1,\ v = s.$$

Putting these in (4), we have,

$$As^4 + Bs^3 + Cs^3 + Ds = 0,$$

where,

$$(5)\qquad \begin{aligned} A &= a^4 - b^4 + 2k, \\ B &= 4a^3k + 4b^3k - kc^3 + 4k, \\ C &= 6a^2k^2 - 6b^2k^2 - 3kc^2 + 6k, \\ D &= 4ak^3 + 4bk^3 - 3kc + 4k, \end{aligned}$$

where, $c = \frac{4(k^2+1)}{3}$. Now if, $D = 0$, one will get $b = -\frac{4-3c+4ak^2}{4k^2}$. Considering this and plugging in (5), we have

$$\begin{aligned} A &= 4a^3 - 6a^2 + 4a + 2k - 1, \\ B &= 12ka^2 - 12ka - \frac{64}{27}k^7 - \frac{64}{9}k^5 - \frac{64}{9}k^3 + \frac{152}{27}k, \\ C &= 12ak^2 - \frac{16}{3}k^5 - \frac{32}{3}k^3 - 6k^2 + \frac{2}{3}k. \end{aligned}$$

Let $C = 0$. Therefore, $a = \frac{8k^4+16k^2+9k-1}{18k}$ and then from (4) we have

$$s = \frac{54k^2(64k^6 + 48k^4 + 39k^2 + 1)}{512k^{12} + 3072k^{10} + 5952k^8 + 3976k^6 + 3468k^4 - 33k^2 - 1}.$$

And following that,

$$\begin{aligned} x &= k\big(-1053k^3 - 27k - 1296k^5 - 1728k^7 + 1716k^4 + 36k^2 + 928k^6 \\ &\qquad + 1728k^8 + 1536k^{10} + 512k^{12} + 2\big)G^2, \\ y &= -k(1053k^3 + 27k + 1296k^5 + 1728k^7 + 1716k^4 + 36k^2 \\ &\qquad + 928k^6 + 1728k^8 + 1536k^{10} + 512k^{12} + 2)G^2, \\ t &= (-2288k^6 + 588k^4 - 2112k^8 - 1536k^{10} - 105k^2 + 512k^{12} - 1)G^3, \\ u &= (1362k^4 - 87k^2 + 1384k^6 + 2496k^8 + 3072k^{10} + 512k^{12} - 1)G^2, \\ v &= -54k^2(39k^2 + 1 + 48k^4 + 64k^6)G^2, \end{aligned}$$

where $G = 512k^{12} + 3072k^{10} + 5952k^8 + 3976k^6 + 3468k^4 - 33k^2 - 1$.

EXAMPLE 5.1. Taking $k = 3$, we obtain

$$\begin{aligned} x &= 273386918368014786254081976, \\ y &= 279465527282684531065966392, \\ t &= 20212790587004292934996436166624016, \\ u &= 115743586666025623792077800, \\ v &= 6078608914669744811884416. \end{aligned}$$

REFERENCES

[1] A. Choudhry, *On the Diophantine equation* $A^4+hB^4=C^4+hD^4$, Indian J. Pure Appl. Math., **26** (1995), 1057–1061.
[2] L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea Publishing Company, New York, 1952.
[3] N. D. Elkies, *On* $A^4+B^4+C^4=D^4$, Math. Comp., **51** (1988), 825–835.
[4] L. Euler, *Resolutio formulae Diophanteae* $ab(maa+nbb)=cd(mcc+ndd)$ *per numeros rationales*, in *Nova acta Academiae scientiarum imperialis petropolitanae*, Vol. 13 (1795–1796), Typis Academiae Scientiarum, Petropolis, 1802, 45–63.
[5] F. Ferari, L'intermédiaire des mathématiciens, **20** (1913), 105–106.
[6] F. A. Izadi and K. Nabardi, *Diophantine equation* $x^4+y^4=2(u^4+v^4)$, Math. Slovaca, **66** (2016), 1–4.
[7] A. S. Janfada and K. Nabardi, *On Diophantine equation* $x^4+y^4=n(u^4+v^4)$, Math. Slovaca, **69** (2019), 1245–1248.
[8] J. Leech, *On* $A^4+B^4+C^4+D^4=E^4$, Math. Proc. Cambridge Philos. Soc., **54** (1958), 554–555.
[9] R. Norrie, *On the algebraical solution of indeterminate cubic and quartic equations*, in *University of St. Andrews* 500^{th} *Anniversary Memorial Volume of Scientific Papers*, University of St. Andrews, Edinburgh, 1911, 47–92.
[10] J. O. Patterson, *A note on the Diophantine problem of finding four biquadrates whose sum is a biquadrate*, Bull. Amer. Math. Soc., **48** (1942), 736–737.
[11] W. Ramanujan, *Notebooks*, Springer-Verlag, New York, 1987, 385–386.
[12] L. C. Washington, *Elliptic curves. Number theory and cryptography*, 2^{nd} edition, Chapman and Hall/CRC, Boca Raton, FL, 2008.

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Urmia University
Department of Mathematics
Urmia, Iran
E-mail: `ahmadi5296@gmail.com`
E-mail: `a.sjanfada@urmia.ac.ir`
E-mail: `k.nabardi@urmia.ac.ir`
`https://orcid.org/0000-0002-4073-9358`