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On the solutions of two special types of Riccati difference equation via Fibonacci numbers

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Abstract

In this study, we investigate the solutions of two special types of the Riccati difference equation $x_{n+1} = \frac{1}{1+x_n}$ and $y_{n+1} = \frac{1}{-1+y_n}$ such that their solutions are associated with Fibonacci numbers.

MSC: 11B39; 39A10; 39A13

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1 Introduction

Nonlinear difference equations have long interested researchers in the field of mathematics as well as in other sciences. They play a key role in many applications such as the natural model of a discrete process. There have been many recent investigations and interest in the field of nonlinear difference equations by several authors [1–14]. For example, in [1], Brand defined a sequence which stems from the Riccati difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}.$$

In [6], Cinar studied the solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}.$$

In [7], Papaschinopoulos and Papadopoulos studied the fuzzy difference equation

$$x_{n+1} = A + \frac{B}{x_n},$$

which is a special case of the Riccati difference equation. In [8], Elabbasy *et al.* obtained the Fibonacci sequence in solutions of some special cases of the following difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} - cx_{n-q}}.$$

In [9], the author deals with behavior of the solution of the nonlinear difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n + dx_{n-2}}.$$

Also, he gives specific forms of the solutions of four special cases of this equation. These specific forms also contain Fibonacci numbers.

Fibonacci numbers have been interesting to the researchers for a long time to get the main theory and applications of these numbers. For instance, the ratio of two consecutive Fibonacci numbers converges to the golden section $\alpha = \frac{1+\sqrt{5}}{2}$. The applications of the golden ratio appear in many research areas, particularly in physics, engineering, architecture, nature and art. Physicists Naschie and Marek-Crnjac gave some examples of the golden ratio in theoretical physics and physics of high energy particles [15–19]. We should recall that the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ has been defined by the recursive equation

$$F_{n+2} = F_{n+1} + F_n, \tag{1}$$

with initial conditions $F_0 = 0$, $F_1 = 1$. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n. \tag{2}$$

One can clearly obtain the characteristic equation of (1) as the form $x^2 - x - 1 = 0$ such that the roots

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}. \tag{3}$$

Hence the Binet formula for Fibonacci numbers

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{4}$$

can be thought of as a solution of the recursive equation in (1). Also, the following ratio is satisfied:

$$\lim_{n\to\infty} \frac{F_{n+r}}{F_n} = \alpha^r, \tag{5}$$

where $r \in \mathbb{Z}$.

Let us consider the following lemma which will be needed for the results in this study.

Lemma 1 [15] *The following equalities hold*:

(i) *For* $n > k+1$, $n \in \mathbb{N}^+$ *and* $k \in \mathbb{N}$, $F_n = F_{k+1}F_{n-k} + F_k F_{n-(k+1)}$.
(ii) *For* $n > 0$, $\alpha^n = \alpha F_n + F_{n-1}$ *and* $\beta^n = \beta F_n + F_{n-1}$.
(iii) *For* $n > 0$, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$ (*Cassini's formula*).

In this study, we consider the Riccati difference equation

$$x_{n+1} = \frac{a + bx_n}{c + dx_n}, \quad n = 0, 1, \ldots. \tag{6}$$

Obviously, by taking $a = c = d = 1$, $b = 0$ and $a = d = 1$, $c = -1$, $b = 0$, equation (6), respectively, is transformed into the following equations:

$$x_{n+1} = \frac{1}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{7}$$

$$y_{n+1} = \frac{1}{-1 + y_n}, \quad n = 0, 1, \ldots, \tag{8}$$

where initial conditions are $x_0 \in \mathbb{R} - \{-\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty}$ and $y_0 \in \mathbb{R} - \{\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty}$, respectively, and F_m is the mth Fibonacci number.

The aim of this study is to investigate some relationships both between Fibonacci numbers and solutions of equations (7) and (8) and between the golden ratio and equilibrium points of equations (7) and (8).

2 Main results

Firstly, it is not difficult to prove that equilibrium points of equations (7) and (8) are $\overline{x}_1 = -\beta, \overline{x}_2 = -\alpha$ and $\overline{y}_1 = \alpha, \overline{y}_2 = \beta$, respectively, where $\alpha = \frac{\sqrt{5}+1}{2}$ is the golden ratio and $\beta = \frac{1-\sqrt{5}}{2}$ is the conjugate of α. Note that one of the equilibrium points of equation (8) is the golden ratio.

Theorem 1 *For $n = 0, 1, 2, \ldots$, the solutions of equations* (7) *and* (8) *are as follows*:

(i) *For $x_0 \in \mathbb{R} - (\{\frac{1}{\alpha}, \frac{1}{\beta}\} \cup \{-\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty})$, $x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}$.*

(ii) *For $y_0 \in \mathbb{R} - (\{\alpha, \beta\} \cup \{\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty})$, $y_n = \frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0}$.*

Proof Firstly, in here we will just prove (ii) since (i) can be thought in the same manner.

(ii) We will prove this theorem by induction. For $k = 0$,

$$\frac{F_0 + F_1 y_0}{F_{-1} + F_0 y_0} = \frac{0 + 1y_0}{1 + 0y_0} = y_0.$$

Now assume that

$$y_k = \frac{F_{-k} + F_{-(k-1)}y_0}{F_{-(k+1)} + F_{-k}y_0}, \tag{9}$$

is true for all positive integers k. Therefore, we have to show that it is true for $k + 1$. Taking into account (2) and (9), we write

$$\begin{aligned} y_{k+1} &= \frac{1}{-1 + y_k} \\ &= \frac{F_{-(k+1)} + F_{-k}y_0}{F_{-k} - F_{-(k+1)} + (F_{-(k-1)} - F_{-k})y_0} \\ &= \frac{F_{-(k+1)} + F_{-k}y_0}{F_{-(k+2)} + F_{-(k+1)}y_0}, \end{aligned}$$

which ends the induction and the proof. □

Theorem 2 *Let the solutions of equations* (7) *and* (8) *be $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$, respectively and $x_0 \in \mathbb{R} - \{-\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty}$. Therefore, $\{x_n\}_{n=0}^{\infty} = \{-y_n\}_{n=0}^{\infty}$ is satisfied if and only if the initial conditions are $x_0 = -y_0$.*

Proof First, assume that $\{x_n\}_{n=0}^{\infty} = \{-y_n\}_{n=0}^{\infty}$. Taking into account (2), we can write

$$\begin{aligned}\frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0} &= -\frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0} \\ &= \frac{F_n - F_{n-1}y_0}{F_{n+1} - F_n y_0}.\end{aligned}$$

By using simple mathematical operations and the well-known Cassini's formula for Fibonacci numbers, we have

$$\begin{aligned}&\left(F_{n-1}F_{n+1} - F_n^2\right)x_0 = \left(F_n^2 - F_{n-1}F_{n+1}\right)y_0, \\ &(-1)^n x_0 = (-1)^{n+1} y_0, \\ &x_0 = -y_0.\end{aligned}$$

Second, assume that $x_0 = -y_0$. By considering the solutions of equation (7), we get

$$\begin{aligned}x_n &= \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0} \\ &= \frac{(-1)^{n+1}F_n - (-1)^{n+1}F_{n-1}y_0}{(-1)^{n+1}F_{n+1} - (-1)^{n+1}F_n y_0} \\ &= \frac{F_{-n} + F_{-(n-1)}y_0}{-F_{-(n+1)} - F_{-n}y_0} \\ &= \frac{F_{-n} + F_{-(n-1)}y_0}{-(F_{-(n+1)} + F_{-n}y_0)} \\ &= -y_n,\end{aligned}$$

which is desired. □

Theorem 3 *The following statements hold*:

(i) *For the initial condition* $x_0 = \frac{1}{\alpha}$ (*or* $x_0 = \frac{1}{\beta}$), *equation* (7) *has the fixed solution* $x_n = \frac{1}{\alpha}$ (*or* $x_n = \frac{1}{\beta}$).

(ii) *For the initial condition* $y_0 = \alpha$ (*or* $y_0 = \beta$), *equation* (8) *has the fixed solution* $y_n = \alpha$ (*or* $y_n = \beta$).

Proof Here we will just prove (i) since the proof of (ii) can be done quite similarly.

(i) Firstly, let $x_0 = \frac{1}{\alpha} = \frac{\sqrt{5}-1}{2}$ be the initial condition of equation (7). Then, by using Lemma 1(ii), we have

$$x_n = \frac{F_n + \frac{F_{n-1}}{\alpha}}{F_{n+1} + \frac{F_n}{\alpha}} = \frac{\alpha F_n + F_{n-1}}{\alpha F_{n+1} + F_n} = \frac{\alpha^n}{\alpha^{n+1}} = \frac{1}{\alpha}.$$

Secondly, let $x_0 = \frac{1}{\beta} = -\frac{\sqrt{5}+1}{2}$ be the initial condition of equation (7). Then, by considering Lemma 1(ii), we obtain

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0} = \frac{F_n + \frac{F_{n-1}}{\beta}}{F_{n+1} + \frac{F_n}{\beta}} = \frac{\beta F_n + F_{n-1}}{\beta F_{n+1} + F_n} = \frac{\beta^n}{\beta^{n+1}} = \frac{1}{\beta},$$

which is desired. □

Theorem 4 *The following statements hold*:

(i) *For* $x_0 \in \mathbb{R} - (\{\frac{1}{\beta}\} \cup \{-\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty})$, *all the solutions of equation* (7) *converge to* $-\beta$, *where* $\beta = \frac{1-\sqrt{5}}{2}$. *That is,* $\lim_{n\to\infty} x_n = -\beta$.

(ii) *For* $y_0 \in \mathbb{R} - (\{\alpha\} \cup \{\frac{F_{m+1}}{F_m}\}_{m=1}^{\infty})$, *all the solutions of equation* (8) *converge to* β, *where* $\beta = \frac{1-\sqrt{5}}{2}$. *That is,* $\lim_{n\to\infty} y_n = \beta$.

Proof To prove, we use the solutions of (7) and (8).

(i) By using Theorem 1(i), we can write

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0} = \lim_{n\to\infty} \frac{1 + \frac{F_{n-1}}{F_n}x_0}{\frac{F_{n+1}}{F_n} + x_0}.$$

Thus, from (5), we have

$$\lim_{n\to\infty} x_n = \frac{1 + \frac{1}{\alpha}x_0}{\alpha + x_0} = \frac{1}{\alpha} = -\beta.$$

(ii) The proof can be seen easily in a similar manner to Theorem 4(i). □

Theorem 5 *Let* $\{x_n\}_{n=0}^{\infty}$ *be the solution of* (7). *Then, we have*

$$\lim_{n\to\infty} \prod_{i=0}^{n} x_i = F_0.$$

Proof For $x_0 = F_0$, the result is trivial. If $x_0 \neq F_0$, by Theorem 1, then we can write

$$x_0 = \frac{F_0 + F_{-1}x_0}{F_1 + F_0 x_0},$$
$$x_1 = \frac{F_1 + F_0 x_0}{F_2 + F_1 x_0},$$
$$\vdots$$
$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}.$$

By multiplying both sides of the above equalities, we obtain

$$\prod_{i=0}^{n} x_i = \frac{F_0 + F_{-1}x_0}{F_{n+1} + F_n x_0} = \frac{x_0}{F_{n+1} + F_n x_0}. \tag{10}$$

Letting $n \to \infty$, the last equality gives the following result

$$\lim_{n\to\infty} \prod_{i=0}^{n} x_i = F_0.$$

Consequently, the proof is completed. □

The following theorem establishes that the Fibonacci numbers can be obtained by using the solutions of (7).

Theorem 6 *Let the initial condition of equation* (7) *be* $x_0 = \frac{F_k}{F_{k+1}}$, *where* F_k *is the kth Fibonacci number. For* $n > k+1$ *and* $k, n \in \mathbb{Z}^+$, *we have*

$$F_n = \frac{F_{k+1}}{x_1 x_2 \cdots x_{n-(k+1)}}.$$

Proof Firstly, taking $n-(k+1)$ instead of n in (10), we obtain

$$\prod_{i=0}^{n-(k+1)} x_i = \frac{x_0}{F_{n-k} + F_{n-(k+1)} x_0}. \tag{11}$$

Secondly, dividing both sides in (11) with x_0, we get

$$\prod_{i=1}^{n-(k+1)} x_i = \frac{1}{F_{n-k} + F_{n-(k+1)} x_0}. \tag{12}$$

Finally, by considering Lemma 1(i), we obtain

$$\begin{aligned} x_1 x_2 \cdots x_{n-(k+1)} &= \frac{1}{F_{n-k} + F_{n-(k+1)} x_0} \\ &= \frac{1}{F_{n-k} + F_{n-(k+1)} \frac{F_k}{F_{k+1}}} \\ &= \frac{F_{k+1}}{F_n}, \end{aligned}$$

from which the result follows. □

3 Conclusion

In this study, we mainly obtained the relationship between the solutions of Riccati difference equations (given in (7), (8)) and Fibonacci numbers. We also presented that the nontrivial solutions of equations in (7) and (8) actually converge to $-\beta$ and β, respectively, so that β is conjugate to the golden ratio. We finally note that the results in this paper are given in terms of Fibonacci numbers.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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