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ON THE SOLUTIONS TO SOME ELLIPTIC EQUATIONS WITH NONLINEAR NEUMANN BOUNDARY CONDITIONS

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Abstract. We describe all nontrivial nonnegative solutions to the problem

$$\begin{cases} -\Delta u = a u^{\frac{n+2}{n-2}} & \text{in } H, \\ \dfrac{\partial u}{\partial \nu} = b u^{\frac{n}{n-2}} & \text{on } \partial H, \end{cases}$$

where H is the half space of $\mathbb{R}^n (n \geq 3)$.

1. Introduction. The goal of this paper is to describe all nontrivial nonnegative $C^2(H) \cap C^1(\overline{H})$-solutions to the problem

$$\begin{cases} -\Delta u = a u^{\frac{n+2}{n-2}} & \text{in } H, \\ \dfrac{\partial u}{\partial \nu} = b u^{\frac{n}{n-2}} & \text{on } \partial H, \end{cases} \tag{1.1}$$

where H is the half space of $\mathbb{R}^n (n \geq 3)$ defined by $H = \{x = (x_1,\ x_2, \ldots, x_n) : x_1 > 0\}$, a, b are two real constants, and $\nu = -e_1$ is the unit outward normal to ∂H, the boundary of H. Note that by the maximum principle if $u \not\equiv 0$, then $u > 0$ in $\overline{H}$. One is led to nonlinear Neumann problems of this kind in the study of conformal deformations of Riemannian manifolds with boundary; see [3], [6–8] and [12]. The case $a = 0, b > 0$ is related to a problem of sharp constant in a Sobolev trace inequality; see [5]. Yet a different motivation to study this kind of problems comes from parabolic equations. In the case $a = 0, b > 0$ and when the power $\frac{n}{n-2}$ is replaced by $q > 1$, nonexistence of positive solutions may serve to describe the profile near the "blow-up time" for solutions of the heat equation with a nonlinear boundary condition. For this subject we refer the reader to the recent paper [14] and the references therein. Hu has recently shown that for $a = 0, b = 1$ and $q < \frac{n}{n-2}$ there are no positive solutions ([13]). In the "limiting case" $q = \frac{n}{n-2}$, positive solutions do exist (see [14], [13] and also the theorem below).

In [6], Escobar has found all positive solutions of (1.1) which satisfy the condition $u(x) = O(|x|^{-(n-2)})$ as $|x| \to \infty$; see also the recent paper of Terracini ([19]). Here, we are able to remove this assumption and to describe all positive solutions to (1.1)

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without any a priori assumption on the behavior of u at infinity. Roughly speaking, our main result states that the set of all positive solutions consists of two main groups: the first contains the solutions found by Escobar which have the behavior $O(|x|^{-(n-2)})$ at infinity, the second contains solutions which depend on x_1 only. After completing our work we learned that Yan Yan Li and Meijun Zhu ([17]) have independently proved our Theorem 1.1 in the case $a \geq 0$ using essentially the same method. They have also established an interesting two-dimensional analogous result for an equation with exponential nonlinearity.

Our main result is the following:

Theorem 1.1. *Let u be a nontrivial nonnegative $C^2(H) \cap C^1(\overline{H})$-solution to* (1.1). *Then*

(i) *when $a > 0$, or $a \leq 0$, $b > B = \sqrt{-\frac{a(n-2)}{n}}$*

$$u(x) = \frac{\alpha}{(|x - x^0|^2 + \beta)^{\frac{n-2}{2}}}, \qquad \alpha > 0, \quad x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n, \tag{1.2}$$

where

$$x_1^0 = -\frac{b}{n-2}\alpha^{\frac{2}{n-2}}, \qquad \beta = \frac{a}{(n-2)n}\alpha^{\frac{4}{n-2}};$$

(ii) *when $a = 0$, $b = 0$*

$$u(x) = \alpha = \text{Const.} \tag{1.3}$$

(iii) *when $a = 0$, $b < 0$*

$$u(x) = \alpha x_1 + \big(-\frac{\alpha}{b}\big)^{\frac{n-2}{n}}, \qquad \alpha > 0; \tag{1.4}$$

(iv) *when $a < 0$, $b = B$*

$$u(x) = \big(\frac{2}{n-2}Bx_1 + \alpha\big)^{-\frac{n-2}{2}}, \qquad \alpha > 0. \tag{1.5}$$

In the remaining case $a < 0, b < B$, there is no nontrivial nonnegative solution to (1.1).

Remark 1.1. The constraint $b > B$ in the case $a \leq 0$ of (i) insures that the denominator in (1.2) is positive. Note also that whenever $u(x)$ is a solution to (1.1), so is $u(x + y^0)$ for every y^0 with $y_1^0 = 0$.

Remark 1.2. It is interesting to note that the solutions given in (1.5) can be recovered as a limiting case of solutions of the form (1.2). More precisely, given a solution $u(x)$ as in (1.5) corresponding to a certain $a < 0$, there exist sequences $\alpha_k \nearrow \infty$ and $b_k \searrow B$ such that the corresponding solutions $u_k(x)$ (for the same a), given by (1.2), converge towards $u(x)$ as $k \to \infty$ uniformly on compacts subsets of $\overline{H}$.

Our main device in proving Theorem 1.1 is a variant of the moving plane method. This method, which goes back to Alexandroff, has become a powerful tool to prove symmetry of positive solutions to nonlinear elliptic equations on bounded and unbounded domains. We mention in particular the works [18], [9], [10], [1], [2], [15], [16], [4]. In our variant, which we shall call "the shrinking sphere method," we use strongly the conformal invariance of the problem, and in particular its invariance with respect to reflections in spheres (rather than its invariance with respect to reflections in hyperplanes that is used in the above-mentioned works).

More precisely, we first apply to u a Kelvin transformation, with respect to some sphere, to find another solution v which has a "good" behavior at infinity. Then we compare the solution v to its Kelvin transform, denoted by v_λ, with respect to a sphere $S_\lambda(0)$, for some $\lambda > 0$. It turns out that for λ big enough we have $v_\lambda \geq v$ on $H \cap \{|x| > \lambda\}$. We then decrease λ, thus shrinking $S_\lambda(0)$, until a critical value λ_0, below which the latter inequality fails to hold. Then we are led to an alternative. Either u is a function of x_1 only, or else $v_{\lambda_0} \equiv v$. Exploiting the last identity for critical spheres of different centers on ∂H we are able to conclude that u is of the form (1.2).

The paper is divided as follows. In Section 2 we will carry out the "shrinking sphere method" establishing the alternative mentioned above. In Section 3 we find the solutions to (1.1) that depend on x_1 only. In fact we shall deal with a slightly more general problem allowing general powers of u instead of restricting ourselves to $\frac{n+2}{n-2}$ and $\frac{n}{n-2}$. Finally, in Section 4 we shall complete the proof of Theorem 1.1.

2. The shrinking sphere technique. Before starting the shrinking sphere process we present a preliminary result that will be useful in the sequel:

Lemma 2.1. *Let $y^0 \in \partial H$, N_{y^0} a neighborhood of y^0 in $\mathbb{R}^n$, v_1, v_2 in $C^2(N_{y^0} \cap H) \cap C^1(N_{y^0} \cap \overline{H} \backslash \{y^0\})$ two solutions to*

$$-\Delta v = a v^{\frac{n+2}{n-2}}, \quad v \geq 0 \quad \text{in} \ \ N_{y^0} \cap H, \tag{2.1}$$

$$\frac{\partial v}{\partial \nu} = b v^{\frac{n}{n-2}} \quad \text{on} \ \ N_{y^0} \cap \partial H \backslash \{y^0\}. \tag{2.2}$$

Denote by $B_{\delta_0}(y^0)$ the ball of center y^0 and radius δ_0. If

$$v_1 - v_2 \geq 0, \quad v_1 - v_2 \not\equiv 0 \quad \text{on} \ \ \overline{B_{\delta_0}(y^0)} \cap \overline{H} \backslash \{y^0\}$$

and if there exists M such that

$$v_2 \leq M \quad \text{on} \ \ \overline{B_{\delta_0}(y^0)} \cap \overline{H} \backslash \{y^0\} \tag{2.3}$$

then there exists $\alpha > 0$ such that

$$v_1 - v_2 \geq \alpha \quad \text{on} \ \ \overline{B_{\delta_0}(y^0)} \cap \overline{H} \backslash \{y^0\}. \tag{2.4}$$

Proof. Note that by the mean value theorem $w = v_1 - v_2$ satisfies

$$-\Delta w = a\{v_1^{\frac{n+2}{n-2}} - v_2^{\frac{n+2}{n-2}}\} = a\frac{n+2}{n-2}\Psi^{\frac{4}{n-2}} w \quad \text{in} \ \ N_{y^0} \cap H, \tag{2.5}$$

$$\frac{\partial w}{\partial \nu} = b\{v_1^{\frac{n}{n-2}} - v_2^{\frac{n}{n-2}}\} = b\frac{n}{n-2}\eta^{\frac{2}{n-2}} w \quad \text{on} \quad N_{y^0} \cap \partial H \backslash \{y^0\}, \tag{2.6}$$

where $\Psi = \Psi(x)$, $\eta = \eta(x)$ belong to the interval $(v_2(x), v_1(x))$. By the maximum principle and Hopf boundary lemma one has

$$w > 0 \quad \text{on} \quad \overline{B_{\delta_0}(y^0)} \cap \overline{H} \backslash \{y^0\}. \tag{2.7}$$

The main point then is to find a positive lower bound for w. In what follows, δ denotes a positive number such that $\delta \leq \delta_0$. We set

$$m(\delta) = \min_{S_\delta(y^0)\cap\overline{H}} w(x) \wedge 1, \tag{2.8}$$

where $S_\delta(y^0)$ denotes the sphere of center y^0 and radius δ, and $\wedge$ is the minimum function between two numbers. For any $0 < \epsilon < \delta$ set

$$h_\epsilon(x) = h(x) = \big\{|x - y^0|^2 + x_1 + \delta - \frac{\epsilon^{n-2}}{|x - y^0|^{n-2}}\big\} m(\delta). \tag{2.9}$$

Notice that

$$h(x) \leq (\delta^2 + 2\delta) m(\delta) \quad \forall x \in \overline{B_\delta(y^0)} \backslash \{y^0\}, \tag{2.10}$$

$$h(x) \leq (\delta^2 + 2\delta - 1) m(\delta) \quad \forall x \in S_\epsilon(y^0) \cap \overline{H}. \tag{2.11}$$

So, provided δ is small enough, say $\delta \leq \delta_1 = \delta_0 \wedge \frac{1}{3}$, one has clearly

$$h(x) \leq (\delta^2 + 2\delta) m(\delta) \leq m(\delta) \leq 1 \quad \forall x \in \overline{B_\delta(y^0)} \backslash \{y^0\} \tag{2.12}$$

$$h(x) \leq 0 \quad \forall x \in S_\epsilon(y^0) \cap \overline{H}. \tag{2.13}$$

Remark also that

$$\Delta h = 2nm(\delta) \quad \text{in} \quad H \tag{2.14}$$

$$\frac{\partial h}{\partial \nu} = -m(\delta) \quad \text{on} \quad \partial H \backslash \{y^0\}. \tag{2.15}$$

Denote by $A = A_{\delta,\epsilon}$ the half-annulus

$$A = \{x \in H : \epsilon < |x - y^0| < \delta\}. \tag{2.16}$$

We claim that for δ small enough

$$z = w - h \geq 0 \quad \text{on} \quad \overline{A_{\delta,\epsilon}} \quad \forall \epsilon \in (0, \delta)\,. \tag{2.17}$$

For $\delta \leq \delta_1$ it is clear from (2.8), (2.12), (2.13) that

$$z \geq 0 \quad \text{on} \quad \partial A \cap H. \tag{2.18}$$

Let us denote by $\overline{x}$ a point of $\overline{A}$ where z achieves its minimum. If $z(\overline{x}) \geq 0$, we are done—so without loss of generality we can assume

$$w(\overline{x}) < h(\overline{x}). \tag{2.19}$$

First we show that $\overline{x}$ cannot lie in A. Indeed, by (2.5), (2.14) one has

$$-\Delta z = a\frac{n+2}{n-2}\Psi(x)^{\frac{4}{n-2}}w + 2nm(\delta) \quad \text{in } \ A. \tag{2.20}$$

If $a \geq 0$ then (2.20) implies that $-\Delta z(\overline{x}) > 0$ which is impossible for an interior minimum point. If $a < 0$, then at $\overline{x}$ we have

$$0 \leq \Delta z(\overline{x}) = -a\frac{n+2}{n-2}\Psi(\overline{x})^{\frac{4}{n-2}}w(\overline{x}) - 2nm(\delta),$$

$$w(\overline{x}) < h(\overline{x}) \leq (\delta^2 + 2\delta)m(\delta), \tag{2.21}$$

and also, since $w \geq 0$,

$$v_2(\overline{x}) \leq v_1(\overline{x}) \leq v_2(\overline{x}) + h(\overline{x}) \leq M + 1. \tag{2.22}$$

Hence, combining the three above inequalities we obtain, since $\Psi(\overline{x}) \in (v_2(\overline{x}), v_1(\overline{x}))$,

$$2nm(\delta) \leq -a\frac{n+2}{n-2}(M+1)^{\frac{4}{n-2}}(\delta^2 + 2\delta)m(\delta)\ ,$$

which is impossible for δ small enough, say $\delta \leq \delta_2 \leq \delta_1$. So either $z(\overline{x}) \geq 0$ or $\overline{x} \notin A$.

We proceed in the same way to show that either $z(\overline{x}) \geq 0$ or $\overline{x} \notin \partial A \cap \partial H$. Indeed, at a point $\overline{x} \in \partial A \cap \partial H$ where a negative minimum is achieved we must have:

$$\frac{\partial z}{\partial \nu}(\overline{x}) \leq 0. \tag{2.23}$$

Due to (2.6), (2.15) one has

$$\frac{\partial z}{\partial \nu} = b\frac{n}{n-2}\eta(x)^{\frac{2}{n-2}}w + m(\delta) \quad \text{on } \ \partial A \cap \partial H. \tag{2.24}$$

If $b \geq 0$, then $\frac{\partial z}{\partial \nu}(\overline{x}) > 0$ which contradicts (2.23). If $b < 0$, (2.23), (2.24) imply

$$m(\delta) \leq -b\frac{n}{n-2}\eta(\overline{x})^{\frac{2}{n-2}}w(\overline{x})\ , \tag{2.25}$$

and using (2.21), (2.22) we derive

$$m(\delta) \leq -b\frac{n}{n-2}(M+1)^{\frac{2}{n-2}}(\delta^2 + 2\delta)m(\delta)$$

which is impossible for $\delta \leq \delta_3 \leq \delta_2$. So, fixing $\delta \leq \delta_3$ we have (2.17); i.e.,

$$w(x) \geq h_\epsilon(x) \quad \text{on} \quad \overline{A_{\delta,\epsilon}} \;\; \forall \epsilon \in (0, \delta)\ .$$

Letting $\epsilon \to 0$ we obtain

$$w(x) \geq \lim_{\epsilon \to 0} h_\epsilon(x) \geq \delta m(\delta) \quad \text{on} \quad \overline{B_\delta(y^0)} \cap \overline{H} \backslash \{y^0\}.$$

The result follows by setting

$$\alpha = \delta m(\delta) \wedge \min\{w(x) : x \in \overline{B_{\delta_0}(y^0)} \backslash B_\delta(y^0) \cap \overline{H}\}. \qquad \square$$

Let us now apply our shrinking sphere technique which is a variant of the moving plane method. We shall use some of the main ideas of [4] which simplify the original arguments of [10]. We will denote all along by I_y^R the inversion with respect to the sphere $S_R(y)$, i.e., the transformation defined by

$$I_y^R(x) = y + R^2 \frac{x - y}{|x - y|^2} \tag{2.26}$$

for every x in $\mathbb{R}^n$. Let u be a positive solution to (1.1). Since we do not have any information on the behavior of u at infinity we introduce a new function v via a Kelvin transformation. More precisely, for some fixed $y \in \partial H$ we set

$$v(x) = \frac{1}{|x - y|^{n-2}} u\left(I_y^1(x)\right). \tag{2.27}$$

Note that v may be singular at y. In what follows, we will say that a function v is singular at a point y if $\lim_{x \to y} v(x)$ does not exist or is not finite.

It is easy to check that v satisfies the same equation as u; that is,

$$\begin{cases} -\Delta v = a v^{\frac{n+2}{n-2}} & \text{in} \;\; H, \\ -\dfrac{\partial v}{\partial x_1} = b v^{\frac{n}{n-2}} & \text{on} \;\; \partial H \backslash \{y\}. \end{cases} \tag{2.28}$$

Moreover, when $|x| \to +\infty$, $I_y^1(x) \to y$ and thus $v(x) \sim \frac{u(y)}{|x|^{n-2}}$ which for $|x|$ large enough implies:

$$0 < v(x) \leq \frac{C_1}{|x|^{n-2}}, \tag{2.29}$$

for some positive constant C_1.

Next we choose as the origin of our coordinate system a point on ∂H different from y. So we assume

$$y \neq 0. \tag{2.30}$$

For $\lambda > 0$ we set

$$\Sigma_\lambda = \{x \in H : |x| > \lambda\}. \tag{2.31}$$

For $x \in \Sigma_\lambda$ we consider

$$x^\lambda = I_0^\lambda(x) = \frac{\lambda^2}{|x|^2}x, \qquad v_\lambda(x) = \frac{\lambda^{n-2}}{|x|^{n-2}}v(x^\lambda) \tag{2.32}$$

$$w_\lambda = v_\lambda - v, \qquad \overline{w}_\lambda = w_\lambda / g, \tag{2.33}$$

where g is defined by

$$g(x) = |x + \beta e_1|^{-\alpha}, \qquad e_1 = (1, 0, \dots, 0) \tag{2.34}$$

with

$$0 < \alpha < n-2, \qquad |b|\frac{n}{n-2}C_1^{\frac{n}{n-2}} < \alpha\beta. \tag{2.35}$$

The reason for this choice of α and β will become clear later. Let us notice that w_λ and $\overline{w}_\lambda$ may be singular at y and $y^\lambda = I_0^\lambda(y)$. If we take $\lambda > |y|$, y^λ is the only possible singular point for these functions on $\overline{\Sigma}_\lambda$, the closure of Σ_λ.

In the following we will consider mainly the case where

$$\lambda > |y| \ , \quad v \text{ singular at } \ y.$$

We will indicate how to modify the proof in the case when v is nonsingular at y. In this latter case we will apply the shrinking sphere technique to u instead of v.

Step 1: For every $\mu > 0$ there exists $R_\mu > 0$ such that whenever $\inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda < 0$ with $\lambda \geq \mu$, one has

$$\inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda = \inf_{\overline{\Sigma}_\lambda \cap \overline{B}_{R_\mu}} \overline{w}_\lambda$$

(B_{R_μ} denotes the ball of center 0 and radius R_μ; in the case where v is singular at y we have denoted for simplicity by $\overline{\Sigma}_\lambda$ what should be $\overline{\Sigma}_\lambda \backslash \{y^\lambda\}$).

a) The case when v is singular at y. Using (2.29) we have for $|x|$ large

$$-\frac{C_1}{|x|^{n-2}} \leq w_\lambda(x). \tag{2.36}$$

Since for $|x|$ large enough $g(x) = |x + \beta e_1|^{-\alpha} \geq C_2|x|^{-\alpha}$, we obtain by (2.33), (2.35), (2.36)

$$\overline{w}_\lambda(x) = w_\lambda(x)/g(x) \geq -C_3|x|^{\alpha-(n-2)} > \inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda \tag{2.37}$$

for $|x|$ large enough, say $|x| \geq R_\mu$.

b) The case when v is nonsingular at y. In this case one replaces v by u in the definition of w_λ and $\overline{w_\lambda}$. By (2.26), (2.27) one has, since $I_y^1(x) \to y$ when $|x| \to +\infty$,

$$0 < u(x) = \frac{v(I_y^1(x))}{|x-y|^{n-2}} \leq \frac{C_1'}{|x|^{n-2}}, \tag{2.38}$$

for $|x|$ large, for some constant C_1'. This replaces (2.29) and one can then argue as above to obtain the desired conclusion. This completes the proof of Step 1.

Step 2: There exists $\lambda_1 > 0$ such that $\inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda < 0$ for some $\lambda \geq \lambda_1$ implies that $\inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda$ is achieved.

a) The case when v is singular at y. First note that by Step 1 for $\lambda > 1$

$$\inf_{\overline{\Sigma}_\lambda} \overline{w}_\lambda = \inf_{\overline{\Sigma}_\lambda \cap \overline{B}_R} \overline{w}_\lambda$$

for some R. Next, remark that if we choose $\delta_0 < |y|$, then applying Lemma 2.1 with $v_1 = v$, $v_2 = 0$, $y^0 = y$ we obtain for some constant α:

$$v \geq \alpha \quad \text{on} \quad \overline{B_{\delta_0}(y)} \cap \overline{H} \backslash \{y\}. \tag{2.39}$$

Then, for λ large enough $x \in B_{\delta_0}(y^\lambda)$ implies $x^\lambda \in B_{\delta_0}(y)$ so that by (2.39)

$$\begin{aligned} w_\lambda(x) = v_\lambda(x) - v(x) &= \frac{\lambda^{n-2}}{|x|^{n-2}} v_\lambda(x^\lambda) - v(x) \geq \frac{\lambda^{n-2}}{|x|^{n-2}} \alpha - \frac{u\left(I_y^1(x)\right)}{|x-y|^{n-2}} \\ &= \frac{1}{|x|^{n-2}} \left(\lambda^{n-2}\alpha - \frac{|x|^{n-2}}{|x-y|^{n-2}} u\left(I_y^1(x)\right)\right). \end{aligned}$$

When $|x| \to +\infty$ one has

$$\frac{|x|^{n-2}}{|x-y|^{n-2}} u\left(I_y^1(x)\right) \to u(y).$$

Also, as $\lambda \to +\infty$ one has $|x| \to +\infty$ for $x \in B_{\delta_0}(y^\lambda)$. For λ large enough, say $\lambda \geq \lambda_1$ it follows that

$$w_\lambda(x) \geq 0 \quad \text{on} \quad B_{\delta_0}(y^\lambda) \cap \overline{H} \backslash \{y^\lambda\}\ . \tag{2.40}$$

This completes the proof of this step since for $\lambda \geq \lambda_1$ the (negative) infimum of $\overline{w}_\lambda$ must lie in the compact set

$$K_\lambda = \overline{\Sigma}_\lambda \cap \{x : |x| \leq R,\ |x - y^\lambda| \geq \delta_0\}. \tag{2.41}$$

b) The case when v is nonsingular at y. It is clear in this case that the infimum of

$$\overline{w}_\lambda = (u_\lambda - u)/g \tag{2.42}$$

is achieved on the compact set

$$K_\lambda = \overline{\Sigma}_\lambda \cap \{x : |x| \leq R\ \}. \tag{2.43}$$

Step 3: There exists an R_0, independent of λ ($\lambda > |y|$ when v is singular at y), such that if x^0 is a point of $\overline{\Sigma}_\lambda \backslash \{y^\lambda\}$ ($\overline{\Sigma}_\lambda$ in the case where v is nonsingular) where $\overline{w}_\lambda$ achieves a negative minimum, then $|x^0| \leq R_0$.

a) The case when v is singular at y. Since v_λ satisfies the same equation as v, from (2.28) one deduces by the mean value theorem that

$$-\Delta w_\lambda = a\{v_\lambda^{\frac{n+2}{n-2}} - v^{\frac{n+2}{n-2}}\} = C(x) w_\lambda \quad \text{in} \ \ \Sigma_\lambda \tag{2.44}$$

with

$$C(x) = a\frac{n+2}{n-2}\Psi(x)^{\frac{4}{n-2}},$$

where $\Psi(x)$ denotes a number between $v(x)$ and $v_\lambda(x)$.

We may also write the above equation as

$$-\Delta(g\overline{w}_\lambda) = C(x)g\overline{w}_\lambda,$$

or

$$\Delta\overline{w}_\lambda + 2\frac{\nabla g\nabla\overline{w}_\lambda}{g} + \big(\frac{\Delta g}{g} + C(x)\big)\overline{w}_\lambda = 0. \tag{2.45}$$

A simple calculation gives

$$\frac{\Delta g}{g} = \alpha(\alpha - (n-2))|x + \beta e_1|^{-2}.$$

Moreover, if $\overline{w}_\lambda(x) < 0$ and $|x|$ is large enough then we have in light of (2.29)

$$v_\lambda(x) \leq \Psi(x) \leq v(x) \leq \frac{C_1}{|x|^{n-2}},$$

so

$$|C(x)| \leq |a|\frac{n+2}{n-2}C_1^{\frac{4}{n-2}}|x|^{-4}$$

and thus for $|x| \geq R_1$, using the first inequality of (2.35),

$$\frac{\Delta g}{g} + C(x) < 0. \tag{2.46}$$

It follows from (2.45), (2.46) that if x^0 is an interior point of $\overline{\Sigma}_\lambda$ where $\overline{w}_\lambda$ achieves a negative minimum then $|x| \leq R_1$.

Next we deal with the possibility that $x^0 \in \partial\Sigma_\lambda \cap \partial H\backslash\{y^\lambda\}$ (note that x^0 cannot belong to $\partial\Sigma_\lambda \cap \{x_1 > 0\}$ since on this set we have $\overline{w}_\lambda = 0$). We have

$$-\frac{\partial v_\lambda}{\partial x_1} = bv_\lambda^{\frac{n}{n-2}} \quad \text{on} \quad \partial H\backslash\{y^\lambda\}$$

and thus by the mean value theorem

$$-\frac{\partial w_\lambda}{\partial x_1} = D(x)w_\lambda \quad \text{on} \quad \partial H\backslash\{y^\lambda\}, \tag{2.47}$$

where

$$D(x) = b\frac{n}{n-2}\eta(x)^{\frac{2}{n-2}} \tag{2.48}$$

with $\eta(x)$ lying between $v(x)$ and $v_\lambda(x)$.

We then deduce that

$$-\frac{\partial \overline{w}_\lambda}{\partial x_1} = \big(D(x) + \frac{g_{x_1}}{g}\big)\overline{w}_\lambda \quad \text{on} \quad \partial H \backslash \{y^\lambda\}. \tag{2.49}$$

If $w_\lambda(x) < 0$ and $|x|$ is large enough, then by (2.29)

$$v_\lambda(x) \le \eta(x) \le v(x) \le \frac{C_1}{|x|^{n-2}}$$

and thus by (2.48)

$$|D(x)| \le |b| \frac{n}{n-2} C_1^{\frac{n}{n-2}} |x|^{-2}.$$

But $\frac{g_{x_1}}{g} = -\alpha\beta|x+\beta e_1|^{-2}$, so by the second inequality of (2.35) we conclude that for $|x|$ large enough, say $|x| > R_2$,

$$\frac{g_{x_1}}{g} + D(x) < 0\,. \tag{2.50}$$

Thus, by (2.49) and (2.50), if x^0 is a point of $\partial H \cap \overline{\Sigma}_\lambda \backslash \{y^\lambda\}$ where $\overline{w}_\lambda$ achieves a negative minimum then we must have $|x_0| \le R_2$. The conclusion of Step 2 follows by taking $R_0 = \max(R_1, R_2)$.

b) The case when v is nonsingular at y. In this case the proof is identical—except that x^0 can be located at y^λ—but then (2.47) holds on ∂H and one can conclude as above, α, β chosen as in (2.35) with C_1' instead of C_1.

Step 4: Decreasing λ until a critical value. Let us set

$$\lambda_0 = \inf\{\lambda > 0 \ (\lambda > |y| \text{ if } v \text{ is singular at } y) : w_\lambda(x) \ge 0 \text{ on } \Sigma_\lambda\}. \tag{2.51}$$

For $\lambda > \max(R_0, \lambda_1)$ it is clear by combining Step 2 and Step 3 that

$$w_\lambda \ge 0 \quad \text{on} \quad \Sigma_\lambda\,. \tag{2.52}$$

Thus, λ_0 is well defined.

a) In the case where v is singular at y one has $\lambda_0 = |y|$ and $v_{|y|} = v$. Looking for a contradiction we assume that $\lambda_0 > |y|$. Since for $\lambda > \lambda_0$ one has

$$v_\lambda \ge v \quad \text{on} \quad \Sigma_\lambda\,, \tag{2.53}$$

letting $\lambda \searrow \lambda_0$ we obtain

$$v_{\lambda_0} \ge v \quad \text{on} \quad \Sigma_{\lambda_0}. \tag{2.54}$$

Let us show that

$$v_{\lambda_0} = v. \tag{2.55}$$

Note that it is enough to establish this equality on Σ_{λ_0} in order to have it on H. Let us assume that (2.55) fails. Then, by the maximum principle and (2.54) one has (see (2.44), (2.47))

$$w_{\lambda_0} > 0 \quad \text{on} \quad \Sigma_{\lambda_0}. \tag{2.56}$$

Then, a contradiction with the definition of λ_0 will follow if we can show that for some $\lambda < \lambda_0$

$$w_\lambda > 0 \quad \text{on} \ \ \Sigma_\lambda. \tag{2.57}$$

By the definition of λ_0 there exists a sequence $\lambda_k \nearrow \lambda_0$, $|y| < \lambda_k$ with $\inf_{\overline{\Sigma}_{\lambda_k}} \overline{w}_{\lambda_k} < 0$. By Step 1 we know that for some R

$$\inf_{\overline{\Sigma}_{\lambda_k}} \overline{w}_{\lambda_k} = \inf_{\overline{\Sigma}_{\lambda_k} \cap \overline{B}_R} \overline{w}_{\lambda_k}, \quad \forall k\ . \tag{2.58}$$

From Lemma 2.1 with $v_1 = v_{\lambda_0}$, $v_2 = v$, $y^0 = y^{\lambda_0}$, $\delta_0 < \lambda_0 - |y^{\lambda_0}|$ we see that for some $\alpha > 0$ one has

$$\overline{w}_{\lambda_0} \geq \alpha \quad \text{on} \ \ \overline{B_{\delta_0}(y^{\lambda_0})} \cap \overline{H} \backslash \{y^{\lambda_0}\}.$$

Thus, by continuity, for λ_k close to λ_0 one has

$$\overline{w}_{\lambda_k} \geq \frac{\alpha}{2} \quad \text{on} \ \ B_\delta(y^{\lambda_0}) \cap \overline{H} \backslash \{y^{\lambda_k}\} \tag{2.59}$$

for some small δ, $\delta < \delta_0$. Then, define

$$\Omega_k = (\Sigma_{\lambda_k} \cap B_R) \backslash B_\delta(y^{\lambda_0}).$$

Clearly, $\overline{w}_{\lambda_k}$ achieves its minimum at some point $x_k \in \overline{\Omega_k}$. Consider

$$D = D_\epsilon = \Omega_k \backslash B_{\lambda_0 + \epsilon}$$

with $0 < \epsilon < |y^{\lambda_0}| - \lambda_0 - \delta$ to be chosen later. By the maximum principle

$$\min_{\overline{D}} \overline{w}_{\lambda_0} = \gamma = \gamma(\epsilon) > 0$$

hence for k large enough (note that $\overline{w}_{\lambda_k}$ converges toward $\overline{w}_{\lambda_0}$ uniformly on $\overline{D}$)

$$\min_{\overline{D}} \overline{w}_{\lambda_k} = \frac{\gamma}{2} > 0\ . \tag{2.60}$$

It follows that $x_k \in \overline{E_k}$ where $E_k = \{B_{\lambda_0+\epsilon} \backslash B_{\lambda_k}\} \cap H$. Using the maximum principle we will show that

$$w_{\lambda_k} \geq 0 \quad \text{on } E_k \text{ for } k \text{ large enough,} \tag{2.61}$$

hence the desired contradiction. We will need the following lemma:

Lemma 2.2. *Assume $\delta > 0$. Given $M > 0$ there exists $\varepsilon_0 = \varepsilon_0(M) > 0$ such that for any $\varepsilon < \varepsilon_0$ on the domain $A = A_\varepsilon = \{B_{\delta+\epsilon} \backslash B_{\delta-\epsilon}\} \cap H$ the following maximum principle holds. Whenever $w \in C^2(A) \cap C^1(\overline{A})$ satisfies*

$$-\Delta w \geq c(x) w \quad \text{on} \ \ A, \tag{2.62}$$

$$-\frac{\partial w}{\partial x_1} \geq d(x)w \quad \text{on} \;\; \partial A \cap \partial H, \tag{2.63}$$

with

$$\sup_A |c(x)|, \qquad \sup_{\partial A \cap \partial H} |d(x)| \leq M,$$

and

$$w \geq 0 \quad \text{on} \;\; \partial A \cap H \; ,$$

then $w \geq 0$ *on* $\overline{A}$.

Proof. We shall first construct a positive function z on $\overline{A}$ satisfying

$$-\Delta z > Mz \quad \text{on} \;\; A \tag{2.64}$$

$$-\frac{\partial z}{\partial x_1} > Mz \quad \text{on} \;\; \partial A \cap \partial H. \tag{2.65}$$

Indeed, denoting $r = |x|$ we define

$$z = e^{-\beta x_1} \cos(\mu(r-\delta))$$

with $\mu \equiv \frac{1}{\varepsilon}$ and β to be determined. First note that on ∂H we have

$$-\frac{\partial z}{\partial x_1} = \beta e^{-\beta x_1} \cos(\mu(r-\delta)) = \beta z \, .$$

Choosing $\beta = M+1$, (2.65) is satisfied. Next, a simple calculation gives

$$\begin{aligned}
-\Delta z &= (\mu^2 - \beta^2) z + \mu e^{-\beta x_1} \sin(\mu(r-\delta)) \big(\frac{n-1}{r} - 2\beta \frac{x_1}{r}\big) \\
&= \big[(\mu^2 - (M+1)^2) + \mu \tan(\mu(r-\delta)) \big(\frac{n-1}{r} - 2(M+1)\frac{x_1}{r}\big)\big] z.
\end{aligned}$$

Hence for ε small enough (i.e., μ large) we have on A

$$-\Delta z \geq \big[(\mu^2 - (M+1)^2) - \mu \tan(1) \big|\frac{n-1}{r} - 2(M+1)\frac{x_1}{r}\big|\big] z > Mz \, .$$

Next we set $f = w/z$. It satisfies on A:

$$\begin{aligned}
\Delta f &= \frac{1}{z}\Delta w - \frac{2}{z^2} \nabla z \cdot \nabla w + \frac{2}{z^3} w |\nabla z|^2 - \frac{w}{z^2} \Delta z \\
&= \frac{1}{z}(\Delta w + c(x)w) - \frac{2}{z} \nabla z \cdot \nabla f - \frac{\Delta z + c(x) z}{z} \frac{w}{z} \leq -\frac{2}{z} \nabla z \cdot \nabla f + \overline{c}(x) f,
\end{aligned} \tag{2.66}$$

where $\overline{c}(x) = -\frac{\Delta z + c(x) z}{z} > 0$ by (2.64). Moreover, on $\partial A \cap \partial H$ we have

$$\begin{aligned}
-\frac{\partial f}{\partial x_1} &= -\frac{1}{z}\frac{\partial w}{\partial x_1} + \frac{w}{z^2}\frac{\partial z}{\partial x_1} \\
&= \frac{1}{z}\big(-\frac{\partial w}{\partial x_1} - d(x)w\big) + d(x)\frac{w}{z} + \frac{w}{z^2}\frac{\partial z}{\partial x_1} \geq \overline{d}(x) f,
\end{aligned} \tag{2.67}$$

where $\overline{d}(x) = \frac{1}{z}(\frac{\partial z}{\partial x_1} + d(x)z) < 0$ by (2.65). By the maximum principle we see that $f \geq 0$ on $\overline{A}$, hence the result. □

We return to the proof of (2.61). By (2.44), (2.47) we have for each k

$$-\Delta w_{\lambda_k} = C_k(x) w_{\lambda_k} \quad \text{on} \quad E_k,$$

$$-\frac{\partial w_{\lambda_k}}{\partial x_1} = D_k(x) w_{\lambda_k} \quad \text{on} \quad \partial E_k \cap \partial H.$$

For some constant M we clearly have

$$\sup_{E_k} |C_k(x)|, \qquad \sup_{\partial E_k \cap \partial H} |D_k(x)| \leq M \quad \forall\ k.$$

Applying Lemma 2.2 for w_{λ_k} with $\delta = \lambda_0$ we find that for k large enough so that $\lambda_k > \lambda_0 - \varepsilon_0$, (2.61) holds. Hence the desired contradiction and (2.55) follows.

We next show that (2.55) leads to a contradiction. Indeed if $\lambda_0 > |y|$ as we assumed, then v is singular at two points: y and y^{λ_0}, impossible. It follows then that $\lambda_0 = |y|$. To complete this case of Step 4 we need only to show that (2.55) holds for $\lambda_0 = |y|$. For that matter we consider $\tilde{v} = v_{\lambda_0}$, $\lambda_0 = |y|$. The function $\tilde{v}$ satisfies (2.28) and is singular at y only. Indeed, $\tilde{v}$ is not singular at 0 since when $|x| \to 0$

$$\tilde{v}(x) = \frac{\lambda_0^{n-2}}{|x|^{n-2}} v(x^{\lambda_0}) \sim \frac{\lambda_0^{n-2}}{|x|^{n-2}} \frac{C}{|x^{\lambda_0}|^{n-2}} = \frac{C}{\lambda_0^{n-2}}.$$

So we can apply the shrinking sphere procedure as described above to $\tilde{v}$ and define

$$\tilde{\lambda}_0 = \inf\{\lambda > |y| : \tilde{v}_\lambda(x) \geq \tilde{v}(x) \quad \text{on} \quad \Sigma_\lambda\}.$$

Since the unique singularity of $\tilde{v}$ is at y, we must have $\tilde{\lambda}_0 = |y|$ as above. So,

$$v_{\lambda_0} \geq v, \quad \tilde{v}_{\lambda_0} \geq \tilde{v}.$$

But $\tilde{v}_{\lambda_0} = v$ and $\tilde{v} = v_{\lambda_0}$ so,

$$v_{\lambda_0} = v \tag{2.68}$$

and the proof of Step 4 is complete in this case.

b) In the case when v is nonsingular at y, (2.55) holds for some $\lambda_0 > 0$.

Recall that in this case λ_0 is defined by:

$$\lambda_0 = \inf\{\lambda > 0 : u_\lambda(x) \geq u \quad \text{on} \quad \Sigma_\lambda\}.$$

Next, we claim that $\lambda_0 > 0$. Indeed, if not, fixing any $x \in H$, we would have for some sequence $\lambda_k \to 0$

$$w_{\lambda_k}(x) = \frac{\lambda_k^{n-2}}{|x|^{n-2}} u\Big(\frac{\lambda_k^2}{|x|^2} x\Big) - u(x) \geq 0\ ,$$

which is impossible for λ_k small enough (since $u(x) > 0$).

Next we show exactly as in the first part of Step 4 that (2.55) holds. Some simplifications occur due to the absence of singularity.

Step 5: If v is singular at y then u is a function of x_1 only. From the previous steps we have $v_{\lambda_0} = v$ with $\lambda_0 = |y|$. Let us rewrite (2.68) in terms of u. We have :

$$v(x) = \frac{1}{|x-y|^{n-2}} u\left(I_y^1(x)\right), \quad x^{\lambda_0} = I_0^{\lambda_0}(x) = \frac{\lambda_0^2}{|x|^2} x, \quad v_{\lambda_0}(x) = \frac{\lambda_0^{n-2}}{|x|^{n-2}} v(x^{\lambda_0}).$$

Hence

$$v_{\lambda_0}(x) = \frac{\lambda_0^{n-2}}{|x|^{n-2}} \frac{1}{|x^{\lambda_0} - y|^{n-2}} u\left(I_y^1(x^{\lambda_0})\right).$$

But,

$$\begin{aligned} |x|^2 |x^{\lambda_0} - y|^2 &= |x|^2 \Big(\frac{\lambda_0^4}{|x|^2} - \frac{2\lambda_0^2 (x,y)}{|x|^2} + \lambda_0^2 \Big) \\ &= \lambda_0^2 (|y|^2 - 2(x,y) + |x|^2) = \lambda_0^2 |x-y|^2 \end{aligned} \tag{2.69}$$

since $\lambda_0 = |y|$. Thus, combining (2.68) and (2.69) we find

$$v_{\lambda_0}(x) = \frac{1}{|x-y|^{n-2}} u\left(I_y^1(x^{\lambda_0})\right)$$

and (2.55) is equivalent to

$$u\left(I_y^1(x)\right) = u\left(I_y^1(x^{\lambda_0})\right) \quad \forall\, x \in \overline{H} \backslash \{y\}. \tag{2.70}$$

Now the points x and x^{λ_0} are symmetric with respect to reflection in the sphere $S_{\lambda_0}(0)$. So the points $I_y^1(x)$ and $I_y^1(x^{\lambda_0})$ are symmetric with respect to reflection in the "generalized sphere" $I_y^1(S_{\lambda_0}(0))$. Since this latter "generalized sphere" passes through $I_y^1(y) = \infty$, it is in fact a hyperplane which is in addition orthogonal to ∂H, as this was the case for $S_{\lambda_0}(0)$. So (2.70) just says that u is symmetric with respect to reflection in this hyperspace. But the choice of origin is arbitrary. We may apply the "shrinking sphere" method with spheres centered at $x^0 \in \partial H$ as long as $x^0 \neq y$ (see (2.30)). The critical sphere will be $S_{|y-x^0|}(x^0)$ and the corresponding analogue of the identity (2.70) will mean that u is symmetric with respect to the hyperplane $I_y^1(S_{|y-x^0|}(x^0))$. Varying x^0 in this way we find that u is symmetric with respect to each hyperplane orthogonal to ∂H which does not pass through y. By continuity we get this symmetry also for orthogonal hyperplanes that pass through y. It follows that u depends on x_1 only. This completes the proof of Step 5.

In this case where v is nonsingular we know that $u_{\lambda_0} = u$ for some λ_0. So we are finally ending up with the following symmetry property:

Lemma 2.3. *Assume u is a solution to* (1.1) *that is not a function of x_1 only; then*

$$\begin{aligned} &\forall x^0 \in \partial H,\ \exists R_0 = R_0(x^0) > 0 \quad \text{such that} \\ &u(x) = \frac{{R_0}^{n-2}}{|x-x^0|^{n-2}} u\big(I_{x^0}^{R_0}(x)\big) \quad \forall x \in H. \end{aligned} \tag{2.71}$$

3. Solutions depending on x_1 only. As we saw in Section 2, either u satisfies the symmetry property (2.71), or it is a function of x_1 only. In this section we shall treat the latter possibility by finding all the solutions of (1.1) which are depending on x_1 only. In fact, we shall consider a more general problem allowing different powers then $\frac{n+2}{n-2}$, $\frac{n}{n-2}$. So, for $a, b \in \mathbb{R}$, $p > 1, q > 0$ consider the system

$$\begin{cases} u''(x) = -au^p(x) \quad x > 0, \\ u'(0) = -bu^q(0), \end{cases} \tag{3.1}$$

where u is a nonnegative C^2-function. Then we have:

Lemma 3.1. *Let u be a nontrivial nonnegative solution of* (3.1).

(i) *If $a = 0$ and $b = 0$, then $u = \alpha =$ Const.*

(ii) *If $a = 0$ and $b < 0$, then $u = \alpha x + (-\frac{\alpha}{b})^{\frac{1}{q}}$, $\alpha > 0$.*

(iii) *If $a < 0$, $b > 0$ and $q \neq \frac{p+1}{2}$, then*

$$u = \Big(\frac{p-1}{2}Ax + \big(\frac{A}{b}\big)^{-\frac{p-1}{2q-(p+1)}}\Big)^{-\frac{2}{p-1}}, \qquad A = \sqrt{-2a/(p+1)}.$$

(iv) *If $a < 0$, $b < 0$, $q = \frac{p+1}{2}$, and $b = A$, then $u = (\frac{p-1}{2}Ax + \alpha)^{-\frac{2}{p-1}}$, $\alpha > 0$.*

In all the other cases there are no nontrivial nonnegative solutions.

Proof. We distinguish two cases according to the sign of a.

I. $a \geq 0$. In this case $u'' \leq 0$ and u' is nonincreasing.

If $u'(0) = 0$, $b \neq 0$, then due to the second equation of (3.1), $u(0) = 0$ and $u \equiv 0$ by uniqueness of the solution of the Cauchy problem.

If $u'(0) = 0$, $b = 0$, then we have two cases:

(i) $u' \equiv 0$. This implies $u'' \equiv 0$ and if $a \neq 0$, then $u \equiv 0$. If $a = 0$, then $u = Cst.$.

(ii) $u' \not\equiv 0$. Then there exists an x_0 such that $u'(x_0) < 0$.

Since u' is nonincreasing

$$u(x) = u(x_0) + \int_{x_0}^{x} u'(s)\,ds \leq u(x_0) + u'(x_0)(x - x_0) < 0$$

for x large enough, which is impossible.

If $u'(0) < 0$, then we can argue as in (ii) to show that (3.1) has no solution. So, there remains only the case when $u'(0) > 0$; this imposes $b < 0$. Of course, for the reason we just pointed out, we must have $u' \geq 0$. Since u' is nonincreasing $l' = \lim_{x\to+\infty} u'(x) \geq 0$, exists.

From the first equation of (3.1), multiplying by u' and integrating, we obtain

$$\frac{{u'}^2}{2} + \frac{a}{p+1}u^{p+1} \equiv C = \text{Const.} \tag{3.2}$$

Assume first that $a \neq 0$. Then, u has a limit l at $+\infty$ and necessarily $l' = 0$. If $l \neq 0$ then from the first equation of (3.1) it follows that $u'' \leq -\epsilon < 0$ for x large. Then, for x_0, x large

$$u'(x) = u'(x_0) + \int_{x_0}^{x} u''(s)\,ds \leq u'(x_0) - \epsilon(x - x_0)$$

which contradicts $l' = 0$. Thus, $l = l' = C = 0$. But then $\{\frac{u'^2}{2} + \frac{a}{p+1}u^{p+1}\}(0) = 0$, which is impossible.

Assume now that $a = 0$. Then $u' \equiv l' > 0$, $l' = -bu(0)^q$ and $u(0) = (-\frac{l'}{b})^{\frac{1}{q}}$. Thus, in this case we obtain $u = l'x + (-\frac{l'}{b})^{\frac{1}{q}}$.

II. $a < 0$. In this case $u'' \geq 0$ hence u' is nondecreasing. We distinguish two cases.

(i) There is a point x_0 such that $u'(x_0) \geq 0$. Then for $x > x_0$ we have $u'(x) > 0$. Indeed, if $u'(x_0) > 0$ this is clear. If $u'(x_0) = 0$, then $u(x_0) > 0$ or else $u \equiv 0$ by uniqueness for the Cauchy problem. But then $u''(x_0) > 0$ and $u'(x) > 0$ for $x > x_0$. From the equation (3.2) we then derive for $x > x_0$

$$u' = \sqrt{A^2u^{p+1} + 2C}$$

hence,

$$\int_{x_0}^{x} \frac{u'(s)}{\sqrt{A^2u(s)^{p+1} + 2C}}\, ds = x - x_0.$$

Since $u' > 0$ for $x > x_0$ we may change variables to get

$$\int_{u(x_0)}^{u(x)} \frac{du}{\sqrt{A^2u^{p+1} + 2C}} = x - x_0.$$

Since $p > 1$, the above integral converges at $+\infty$ and so u must blow up. Thus, the only possible case is when:

(ii) $u' < 0$. This imposes $b > 0$. Since u' is nondecreasing, u' has a limit at $+\infty$, and u too since $u \geq 0$, and u is nonincreasing.

Set

$$l' = \lim_{x\to+\infty} u'(x) \leq 0, \quad l = \lim_{x\to+\infty} u(x) \geq 0.$$

The only possibility is of course $l' = 0$ (otherwise l would not exist) but then $l = 0$. Indeed, if not, for x large $u'' \geq \epsilon > 0$ which contradicts the existence of l'. Then, going back to (3.2) we get $l = l' = C = 0$. Hence

$$u' = -Au^{\frac{p+1}{2}}. \tag{3.3}$$

From (3.3) at 0 we get

$$u^{q-\frac{p+1}{2}}(0) = \frac{A}{b}. \tag{3.4}$$

When $q \neq \frac{p+1}{2}$ we deduce from (3.3), (3.4) (using $u(0) \neq 0$ otherwise $u \equiv 0$) that u solves (3.1) if and only if it solves

$$u' = -Au^{\frac{p+1}{2}}, \quad u(0) = (\frac{A}{b})^{\frac{2}{2q-(p+1)}}. \tag{3.5}$$

When $q = \frac{p+1}{2}$, (3.4) implies that the problem has a solution only when $b = A$ and any solution to (3.1) is a solution to

$$u' = -Au^{\frac{p+1}{2}}, \quad u(0) = \alpha, \quad \text{for some } \alpha > 0. \tag{3.6}$$

The first equation of (3.5) can be integrated to give

$$u = \big(\frac{p-1}{2}Ax + u(0)^{-\frac{p-1}{2}}\big)^{-\frac{2}{p-1}}.$$

Combining this with (3.5), (3.6) we obtain the last two cases of Lemma 3.1.

4. The proof of Theorem 1.1 completed. By the previous steps we need only to show that the solutions to (1.1) not depending on x_1 only are all of the form (1.2). Note that for those solutions we have the conclusion of Lemma 2.3. We will need the following lemma.

Lemma 4.1. *For $p \geq 1$ let $g \in C^1(\mathbb{R}^p)$ be a function with the following property:*

$$\begin{aligned} &\forall\, x^0 \in \mathbb{R}^p,\ \exists R_0 = R_0(x^0) > 0 \quad \textit{such that} \\ &g(x) = \frac{|x-x^0|^2}{{R_0}^2} g\big(I_{x^0}^{R_0}(x)\big) \quad \forall x \in \mathbb{R}^p. \end{aligned} \tag{4.1}$$

Then there exist constants $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, $v \in \mathbb{R}^p$ such that

$$g(x) = \alpha|x|^2 + \beta(x, v) + \gamma. \tag{4.2}$$

Proof. First note that the value of $R_0 = R_0(x^0)$ can be easily determined. Indeed, letting $|x| \to \infty$ in (4.1) we find that $\alpha = \lim_{|x|\to+\infty} \frac{g(x)}{|x|^2} = \frac{g(x^0)}{R_0^2}$. The case $\alpha = 0$ yields $g \equiv 0$ so we shall assume in the sequel that $\alpha \neq 0$. The identity (4.1) now reads

$$\frac{g(x^0)g(x)}{\alpha} = |x-x^0|^2 g\left(I_{x^0}^{R_0}(x)\right) \quad \forall x \in \mathbb{R}^p. \tag{4.3}$$

Denoting by

$$\{I_{x^0}^{R_0}(x)\}_j = x_j^0 + R_0^2 \frac{x_j - x_j^0}{|x-x^0|^2} = x_j^0 + \frac{g(x^0)}{\alpha}\frac{x_j - x_j^0}{|x-x^0|^2}$$

the j^{th} entry of $I_{x^0}^{R_0}(x)$, we find by a simple calculation

$$\frac{\partial}{\partial x_i}\{I_{x^0}^{R_0}(x)\}_j = \frac{g(x^0)}{\alpha}\frac{|x-x^0|^2\delta_{ij} - 2(x_i - x_i^0)(x_j - x_j^0)}{|x-x^0|^4}. \tag{4.4}$$

We denote $y^0 = I_{x^0}^{R_0}(x)$ and then differentiate (4.3) with respect to x_i. Using (4.3) and (4.4) we obtain

$$\begin{aligned} &\frac{g(x^0)}{\alpha}\frac{\partial g}{\partial x_i}(x) = 2(x_i - x_i^0)g(y^0) + |x-x^0|^2 \sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^0)\frac{\partial}{\partial x_i}\{I_{x^0}^{R_0}(x)\}_j \\ &= 2(x_i - x_i^0)g(y^0) + |x-x^0|^2 \sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^0)\frac{g(x^0)}{\alpha}\frac{|x-x^0|^2\delta_{ij} - 2(x_i - x_i^0)(x_j - x_j^0)}{|x-x^0|^4} \\ &= 2(x_i - x_i^0)\frac{g(x^0)g(x)}{\alpha|x-x^0|^2} + \frac{g(x^0)}{\alpha}\frac{\partial g}{\partial x_i}(y^0) - \frac{2g(x^0)}{\alpha}\sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^0)\frac{(x_i - x_i^0)(x_j - x_j^0)}{|x-x^0|^2}\ , \end{aligned}$$

hence

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial g}{\partial x_i}(y^0) + \frac{2(x_i - x_i^0)}{|x-x^0|^2}\Big(g(x) - \sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^0)(x_j - x_j^0)\Big). \tag{4.5}$$

We can do the same calculation for any $x^0 \in \mathbb{R}^p$. For $x^1 \in \mathbb{R}^p$, (4.5) reads

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial g}{\partial x_i}(y^1) + \frac{2(x_i - x_i^1)}{|x-x^1|^2}\Big(g(x) - \sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^1)(x_j - x_j^1)\Big)\,, \tag{4.6}$$

with $y^1 = I_{x^1}^{R_1}(x), R_1 = R_1(x^1) = \sqrt{g(x^1)/\alpha}$. Next we subtract (4.5) from (4.6) to get

$$\begin{aligned} 0 &= \frac{\partial g}{\partial x_i}(y^1) - \frac{\partial g}{\partial x_i}(y^0) + 2g(x)\left(\frac{x_i - x_i^1}{|x-x^1|^2} - \frac{x_i - x_i^0}{|x-x^0|^2}\right) \\ &+ 2\sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^0)\frac{(x_i - x_i^0)(x_j - x_j^0)}{|x-x^0|^2} - 2\sum_{j=1}^p \frac{\partial g}{\partial x_j}(y^1)\frac{(x_i - x_i^1)(x_j - x_j^1)}{|x-x^1|^2}. \end{aligned} \tag{4.7}$$

Next we pass to the limit in (4.7) with $x = x^{(m)}$ such that $x_i^{(m)} \to \infty$ while $x_j^{(m)} = 0$ for $i \neq j$. Since $y^0 \to x^0$, $y^1 \to x^1$ and $\frac{g(x)}{|x|^2} \to \alpha$, we find

$$\frac{\partial g}{\partial x_i}(x^1) - \frac{\partial g}{\partial x_i}(x^0) = 2\alpha(x_i^1 - x_i^0)\,, \qquad i = 1, \dots, p\,.$$

Taking $x^0 = 0$ and x^1 equal to any $x \in \mathbb{R}^p$, we find

$$\nabla g(x) = v + 2\alpha x \quad \forall x \in \mathbb{R}^p\,, \qquad \text{with } v = \nabla g(0) \in \mathbb{R}^p\,,$$

hence (4.2) .

Lemma 4.2. *A function $u \in C^2(H) \cap C^1(\overline{H})$ satisfies the property* (2.71) *if and only if the following holds*: *there exists a point $z \in \mathbb{R}^n \setminus \overline{H}$ such that on $B_r(y^0) = I_z^1(H)$, the Kelvin transform of u, $w(x) = \frac{1}{|x-z|^{n-2}} u\left(I_z^1(x)\right)$, is a radially symmetric C^2-function.*

Proof. Let us define $g = u^{-\frac{2}{n-2}}$ on $\overline{H}$. By (2.71) we deduce in particular that g satisfies (4.1) on $\partial H = \mathbb{R}^{n-1}$. Applying Lemma 4.1 we find that the restriction of g to ∂H is of the form (4.2) with $\alpha > 0$ and $\beta^2|v|^2 < 4\alpha\gamma$. Next we claim that all the spheres $\{S_{R_0(x^0)}(x^0)\,;\, x^0 \in \partial H\}$ pass through a point $z \in \mathbb{R}^n \setminus \overline{H}$ (and also through its symmetric point in H). In fact, by the proof of Lemma 4.1 we may write for $x^0 \in \partial H$:

$$R_0^2 = \frac{g(x^0)}{\alpha} = |x^0|^2 + \frac{\beta}{\alpha}(x^0, v) + \frac{\gamma}{\alpha} = \left|x^0 + \frac{\beta}{2\alpha}v \pm \sqrt{\frac{\gamma}{\alpha} - (\frac{\beta}{2\alpha})^2|v|^2}\; e_1\right|^2 .$$

Hence all the spheres pass through the two points $-\frac{\beta}{2\alpha}v \pm \sqrt{\frac{\gamma}{\alpha} - (\frac{\beta}{2\alpha})^2|v|^2}\; e_1$. We define z as the point out of these two which does not belong to $\overline{H}$.

Now consider the function $w(x) = \frac{1}{|x-z|^{n-2}} u\left(I_z^1(x)\right)$ which is a C^2-function defined on the closure of the ball $B_r(y^0) = I_z^1(H)$. Note that w is regular at z since (2.71) implies that u has the required regularity at infinity. Next we interpret the symmetry property (2.71) in terms of w. Arguing as in Step 5 of Section 2 we find that w satisfies

$$w\big(I_z^1(x)\big) = w\Big(I_z^1\big(I_{x^0}^{|z-x^0|}(x)\big)\Big) \quad \forall x^0 \in \partial H,\ \forall x \in H. \tag{4.8}$$

From (4.8) we deduce that w is symmetric with respect to reflections in the"generalized spheres" which are the images by I_z^1 of the spheres $\{S_{R_0(x^0)}(x^0)\ ;\ x^0 \in \partial H\}$. Since all those spheres pass through z and z goes to ∞, these images are hyperplanes. Moreover these hyperplanes pass through y^0 since they should be orthogonal to $S_r(y^0)$ as the original spheres are orthogonal to ∂H. Hence (4.8) just means that w is radially symmetric. We can apply the same argument in the other direction to get the complete result. □

Now we are ready to finish the proof of Theorem 1.1. Let $u > 0$ be a solution of (1.1) which is not a function of x_1 only. By Lemma 2.3 and Lemma 4.2 we conclude that $w(x) = \frac{1}{|x-z|^{n-2}} u\left(I_z^1(x)\right)$ is a positive radially symmetric function on $B_r(y^0) = I_z^1(H)$. But w satisfies the same equation as u, namely,

$$-\Delta w = a w^{\frac{n+2}{n-2}} \quad \text{in } B_r(y^0)\ . \tag{4.9}$$

It is well known that all the radially symmetric solutions of (4.9) are of the form

$$w(x) = \frac{A}{(B|x-y^0|^2 + C)^{\frac{n-2}{2}}} \quad \text{for some} \quad A, B, C \in \mathbb{R}. \tag{4.10}$$

Going back to u we find that it is of the form (1.2).

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