# On the Solvability of Dynamic Elastic-visco-plastic Contact Problems

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**Abstract.** We consider two dynamic contact problems between an elastic-visco-plastic body and an obstacle, the so-called foundation. The contact is frictionless and it is modelled with normal compliance of such a type that the penetration is not restricted in the first problem, but is restricted with unilateral constraint, in the second one. We derive a variational formulation of the first problem and then prove its unique weak solvability, by using arguments on nonlinear evolution equations with monotone operators and fixed point. Then, we derive a variational formulation of the second problem and prove its weak solvability. To this end we consider a sequence of regularized problems which have a unique solution, derive *a priori* estimates and use compactness properties to obtain a solution to the original model, by passing to the limit as the regularization parameter converges to zero.

**Keywords and Phrases**: elastic-visco-plastic material, dynamic process, frictionless contact, normal compliance, Signorini condition, variational formulation, weak solution, *a priori* estimates.

## 1 Introduction

The aim of this paper is to study two frictionless contact problems for elastic-visco-plastic materials of the form

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds, \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field while $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represent the stress and the linearized strain tensor, respectively. Here $\mathscr{A}$ and $\mathscr{E}$ are linear operators describing the purely viscous and the elastic properties of the material, respectively, and $\mathscr{G}$ is a nonlinear constitutive function which describes the visco-plastic behaviour of the material. In (1.1) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable $t$.

Rheological models obtained by connecting in parallel a linear dashpot with various viscoelastic or viscoplastic models lead to one-dimensional examples of constitutive laws

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of the form (1.1). Indeed, consider first a dashpot connected in parallel with a Maxwell model; in this case an additive formula holds,

$$\sigma = \sigma^V + \sigma^R \tag{1.2}$$

where $\sigma$, $\sigma^V$ and $\sigma^R$ denote the total stress, the stress in the dashpot and the stress in the Maxwell model, respectively. We have

$$\sigma^V = A\dot{\varepsilon}, \tag{1.3}$$

and

$$\dot{\sigma}^R = E\dot{\varepsilon} - \frac{1}{\eta}\,\sigma^R \tag{1.4}$$

where $A$ and $\eta$ are positive viscosity coefficients, $E > 0$ is the Young modulus of the Maxwell material and $\varepsilon$ denotes the strain. We integrate (1.4) on $[0, t]$ with the initial conditions $\sigma^R(0) = 0$, $\varepsilon(0) = 0$ and use (1.2), (1.3) to obtain

$$\sigma(t) = A\dot{\varepsilon}(t) + E\varepsilon(t) - \frac{1}{\eta}\int_0^t (\sigma(s) - A\dot{\varepsilon}(s))\,ds, \tag{1.5}$$

which represents a constitutive equation of the form (1.1).

The previous model is a particular case of a more general rheological model, obtained by connecting in parallel a linear dashpot, (1.3), with a rate-type elastic-visco-plastic model of the form

$$\dot{\sigma}^R = E\dot{\varepsilon} + G(\sigma^R, \varepsilon) \tag{1.6}$$

in which $G$ is a nonlinear constitutive function. Indeed, we integrate (1.6) with the initial conditions $\sigma^R(0) = 0$, $\varepsilon(0) = 0$ and use (1.2), (1.3) to obtain

$$\sigma(t) = A\dot{\varepsilon}(t) + E\varepsilon(t) + \int_0^t G(\sigma(s) - A\dot{\varepsilon}(s), \varepsilon(s))\,ds, \tag{1.7}$$

which, again, represents a constitutive equation of the form (1.1).

The linear standard viscoelastic model is an example of constitutive law of the form (1.6) and in this case

$$\frac{\dot{\sigma}^R}{E} + \frac{\sigma^R}{\eta} = \Big(1 + \frac{E_1}{E}\Big)\dot{\varepsilon} + \frac{E_1}{\eta}\,\varepsilon. \tag{1.8}$$

Here $E$, $E_1$ and $\eta$ are positive constants. The one-dimensional Perzyna law is an example of nonlinear elastic-visco-plastic constitutive law of the form (1.6) and it can be written as follows,

$$\dot{\varepsilon} = \frac{1}{E}\,\dot{\sigma}^R + \frac{1}{\eta}\,(\sigma - P_K\sigma). \tag{1.9}$$

Here $\eta > 0$ is the viscosity constant, $K \subset \mathbb{R}$ is a nonempty, closed, convex set and $P_K$ is the projection mapping on $K$.

More details on the one-dimensional models (1.4), (1.6), (1.8) and (1.9) as well as on the construction of rheological models obtained by connecting springs and dashpots can be found in [9] and [13, Ch. 6].

Following the previous one-dimensional examples we see that at each time moment $t$, the stress tensor $\boldsymbol{\sigma}(t)$ in (1.1) is split into two parts,

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^V(t) + \boldsymbol{\sigma}^R(t), \tag{1.10}$$

where

$$\boldsymbol{\sigma}^V(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) \tag{1.11}$$

represents the purely viscous part of the stress and the remainder part, $\boldsymbol{\sigma}^R(t)$, satisfies a rate-type elastic-visco-plastic equation,

$$\boldsymbol{\sigma}^R(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}^R(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds. \tag{1.12}$$

Various results, examples and mechanical interpretations in the study of elastic-visco-plastic materials of the form (1.12) can be found in [8, 14] and references therein. Note also that when $\mathscr{G} = \mathbf{0}$ the constitutive law (1.1) becomes the Kelvin-Voigt viscoelastic constitutive relation,

$$\boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}). \tag{1.13}$$

Quasistatic contact problems for materials of the form (1.12) and (1.13) were investigated in a large number of papers, see e.g. [1, 2, 3, 12, 24, 25] and the references therein. A survey of these results can be found in [13]. There, both the variational analysis and the numerical approach of the problems, including the study of semi-discrete and fully discrete schemes, were provided. Existence results in the study of dynamic problems with Kelvin-Voigt materials of the form (1.13) can be found in in [15, 17, 19]. The case of viscoelastic materials with singular memory was considered in [16, 18] and, for more details, we send the reader to the monograph [11].

In the present paper we consider two dynamic contact problem for rate-type materials of the form (1.1); we assume that the contact is frictionless and it is modelled with normal compliance of such a type that the penetration could be infinite in the first problem, but is limited and associated to an unilateral constraint, in the second one. The normal compliance contact condition was first considered in [22] in the study of dynamic problems with linearly elastic and viscoelastic materials. This condition allows the interpenetration of the body's surface into the obstacle and it was justified by considering the interpenetration and deformation of surface asperities. On occasions, the normal compliance condition has been employed as a mathematical regularization of Signorini's nonpenetration condition and used as such in numerical solution algorithms. Contact problems with normal compliance have been discussed in numerous papers, e.g. [4, 5, 6, 20, 21, 24] and the references therein. In particular, the first existence result in the study of quasistatic contact problems with normal compliance and friction was obtained in [4] in the case of linearly elastic materials and in [24] in the case of nonlinear Kelvin-Voigt viscoelastic materials. Unlike the up-to-now research, however, the method we present in this paper allows to treat also such normal compliance models in which the compliance term do not necessarily need to represent a compact perturbation of the original problem, without contact. This will allow to study such models, where a strictly limited penetration is allowed and/or to perform the limit procedure to the Signorini contact condition.

The paper is organized as follows. In Section 2 we introduce some notation and preliminaries. In Section 3 we describe the two contact problems and list the assumption on the data. In Section 4 we state and prove the unique weak solvability of the problem with infinite penetration, Theorem 4.1. To this end we use arguments on nonlinear evolution equations with monotone operators and fixed point methods. Then, in Section 5 we state and prove the weak solvability of the problem with finite penetration and unilateral constraint, Theorem 5.1. To this end we consider a sequence of regularized problems which have a unique solution, derive *a priori* estimates and use compactness properties to obtain a solution to the model, by passing to the limit as the regularization parameter converges to zero.

# 2 Notation and preliminaries

In this short section we present the notation we shall use and some preliminary material. For further details, we refer the reader to [10, 11, 13, 23].

We denote by $r_+$ and $r_-$ the positive and negative part of $r$, i.e. $r_+ = max\,\{0, r\}$, $r_- = max\,\{0, -r\}$. We also denote by $S^N$ the space of second order symmetric tensors on $\mathbb{R}^N$ ($N = 2, 3$), while "$\cdot$" and $\|\cdot\|$ will represent the inner product and the Euclidean norm on $S^N$ and $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\Gamma$ and let $\boldsymbol{\nu}$ denote the unit outer normal on $\Gamma$. We assume that $\Gamma$ is partitioned into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$. Everywhere in what follows the index $i$ and $j$ run from 1 to $N$, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable.

We use the standard notation for Lebesgue ($L_p$, $\boldsymbol{L}_p \equiv (L_p)^N$, $p \in [1, \infty]$) and Sobolev spaces $W_p^k$, $H^k \equiv W_2^k$, $\boldsymbol{H}^k \equiv (H^k)^N$, $k \geq 0$, $p \in [1, \infty]$) associated to $\Omega$ and $\Gamma$ and their duals. For the spaces with zero traces $\mathring{H}^k$, $\mathring{\boldsymbol{H}}^k = (\mathring{H}^k)^N$ is used if $k \notin \frac{1}{2} + \mathbb{N}$. Moreover, we use also the spaces

$$\mathcal{H} = \{\ \boldsymbol{\sigma} = (\sigma_{ij})\ :\ \sigma_{ij} = \sigma_{ji} \in L_2(\Omega)\ \},$$
$$H_1 = \{\ \boldsymbol{u} = (u_i)\ :\ \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\ \},$$
$$\mathcal{H}_1 = \{\ \boldsymbol{\sigma} \in \mathcal{H}\ :\ \mathrm{Div}\,\boldsymbol{\sigma} \in \boldsymbol{L}_2(\Omega)\ \}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and the divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The spaces $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_{\Omega} \sigma_{ij}\tau_{ij}dx,$$
$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}},$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_{\boldsymbol{L}_2(\Omega)}.$$

In general, we denote by $\|\cdot\|_X$ the norm on a Banach space $X$, this holds, in particular, for the associated norms on the spaces $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$.

For every element $\boldsymbol{v} \in H_1$ we also use the notation $\boldsymbol{v}$ to denote the trace of $\boldsymbol{v}$ on $\Gamma$ and we denote by $v_\nu$ and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on $\Gamma$ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

We also denote by $\sigma_\nu$ and $\boldsymbol{\sigma}_\tau$ the normal and the tangential traces of a function $\boldsymbol{\sigma} \in \mathcal{H}_1$, and we note that when $\boldsymbol{\sigma}$ is a regular function then

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu},$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} = \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v}\, da \qquad \forall \boldsymbol{v} \in H_1. \tag{2.1}$$

Now, let $V$ be the closed subspace of $H_1$ given by

$$V = \{\ \boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1\ \}.$$

We denote by $(\cdot,\cdot)_V$ the restriction of the inner product $(\cdot,\cdot)_{H_1}$ to $V$, i.e

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \tag{2.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows that $(V, \|\cdot\|_V)$ is a real Hilbert space; moreover, by the Sobolev trace theorem, there exists a positive constant $c_B$ depending only on the domain $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}_2(\Gamma_3)} \le c_B \|\boldsymbol{v}\|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{2.3}$$

Let $T>0$. For each $t \in [0,T]$ we use the notation $Q_t = (0,t) \times \varOmega$, $S_{ti} = (0,t) \times \varGamma_i$ and, if $t = T$ we write $Q \equiv Q_T = (0,T) \times \varOmega$, $S_i \equiv S_{Ti} = (0,T) \times \varGamma_i$. Also, for every real Banach space $X$ we use the notation $C([0,T];X)$ and $C^1([0,T];X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to $X$, respectively; $C([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

while $C^1([0,T];X)$ is a real Banach space with the norm

$$\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.$$

Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L_p(0,T;X)$ and for the Sobolev spaces $W_p^k(0,T;X)$.

We end this section with a standard existence and uniqueness result which may be found in [7, p. 64].

**Theorem 2.1** *Let $V \subset H \subset V'$ be a Gelfand triple and denote by $\|\cdot\|_V$, $\|\cdot\|_H$, $\|\cdot\|_{V'}$ and $\langle\cdot,\cdot\rangle_{V'\times V}$ the norm on the spaces $V$, $H$, $V'$ and the duality pairing between $V'$ and $V$, respectively. Assume that $A : V \to V'$ is a linear continuous operator which satisfies*

$$\langle Av, v\rangle_{V'\times V} + \alpha \|v\|_H^2 \geq \omega \|v\|_V^2 \quad \forall\, v \in V, \tag{2.4}$$

*for some constants $\omega > 0$ and $\alpha \in \mathbb{R}$. Then, given $u_0 \in H$ and $f \in L_2(0,T;V')$, there exists a unique function $u$ which satisfies*

$$u \in L_2(0,T;V) \cap C([0,T];H), \;\; \dot{u} \in L_2(0,T;V'),$$
$$\dot{u}(t) + Au(t) = f(t) \quad \text{a.e. } t \in (0,T),$$
$$u(0) = u_0.$$

Theorem 2.1 will be used in Section 4 in the proof of the unique solvability of the frictionless contact problems with normal compliance and infinite penetration.

# 3 Problems statement

In this section we present the two problems which describe the frictionless contact process and present the assumption on the data.

The physical setting is as follows. An elastic-visco-plastic body occupies a bounded domain $\varOmega \subset \mathbb{R}^N$ $(N = 2,3)$ with a regular boundary $\varGamma$ that is partitioned into three disjoint measurable parts $\varGamma_1$, $\varGamma_2$ and $\varGamma_3$. Let $T > 0$ and let $[0,T]$ denote the time interval of interest. The body is clamped on $S_1 = (0,T) \times \varGamma_1$ and thus the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $Q = (0,T) \times \varOmega$ and a surface traction of density $\boldsymbol{f}_2$ acts on $S_2 = (0,T) \times \varGamma_2$. In the reference configuration the body is in frictionless contact on $S_3 = (0,T) \times \varGamma_3$ with an obstacle, the so-called foundation.

In the first problem the contact is modelled with normal compliance in such a way that the penetration is not limited. Under these conditions, the classical formulation of the problem is the following.

**Problem** $\mathscr{P}_1$ *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^N$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S^N$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{in } Q, \tag{3.1}$$

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } Q, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_1, \tag{3.3}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } S_2, \tag{3.4}$$

$$-\sigma_\nu = p(u_\nu) \quad \text{on } S_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } S_3, \tag{3.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \quad \text{in } \Omega. \tag{3.7}$$

Here (3.1) is the elastic-visco-plastic constitutive law already presented in Section 1, (3.2) represents the equation of motion in which $\rho$ denotes the density of mass, (3.3) and (3.4) are the displacement and traction boundary conditions, respectively. Condition (3.6) shows that the tangential shear, denoted $\boldsymbol{\sigma}_\tau$, vanishes on the contact surface, i.e. the process is frictionless. Finally, the functions $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$ in (3.7) denote the initial displacement and the initial velocity, respectively.

We now describe the contact conditions (3.5) in which our main interest is. Here $\sigma_\nu$ denotes the normal stress, $u_\nu$ is the normal displacement and $p$ is a Lipschitz continuous increasing function which vanish for a negative argument, i.e.

$$\left\{\begin{array}{l} \text{(a) } p : \mathbb{R} \to \mathbb{R}. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p|r_1 - r_2| \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1,\, r_2 \in \mathbb{R}. \\ \text{(d) } p(r) = 0 \ \text{ for all } r < 0. \end{array}\right. \tag{3.8}$$

Condition (3.5) combined with assumption (3.8) shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); also, when there is penetration (i.e. when $u_\nu \ge 0$), then the reaction of the foundation is towards the body (since $\sigma_\nu \le 0$) and it is increasing with the penetration (since $p$ is an increasing function). Finally, note that in this condition the penetration is not restricted and the normal stress is uniquely determined by the normal displacement.

A first example of normal compliance function $p$ which satisfies condition (3.8) is

$$p(r) = c_\nu r_+ \tag{3.9}$$

where $c_\nu$ is a positive constant. In this case condition (3.5) shows that the reaction of the foundation is proportional to the penetration and therefore (3.5), (3.8) model the contact with a linearly elastic foundation. A second example of normal compliance function $p$ which satisfies condition (3.8) is given by

$$p_\nu(r) = \begin{cases} c_\nu r_+ & \text{if} \quad r \le \alpha, \\ c_\nu \alpha & \text{if} \quad r > \alpha, \end{cases} \tag{3.10}$$

where $\alpha$ is a positive coefficient related to the wear and hardness of the surface and, again, $c_\nu > 0$. In this case the contact condition (3.5) means that when the penetration is too large, i.e. when it exceeds $\alpha$, the obstacle backs off and offers no additional resistance to the penetration. We conclude that in this case the foundation has an elastic-plastic behavior.

In the second problem the contact is again modelled with normal compliance but in such way that the penetration is limited and associated to a unilateral constraint. The classical formulation of the problem is the following.

**Problem** $\mathscr{P}_2$ *Find a displacement field* $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^N$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S^N$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{in } Q, \tag{3.11}$$

$$\rho\ddot{\boldsymbol{u}} = \operatorname{Div}\boldsymbol{\sigma} + \boldsymbol{f}_0 \quad \text{in } Q, \tag{3.12}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } S_1, \tag{3.13}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \quad \text{on } S_2, \tag{3.14}$$

$$u_\nu \le g, \quad \sigma_\nu + p(u_\nu) \le 0, \quad (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \quad \text{on } S_3, \tag{3.15}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on } S_3, \tag{3.16}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \ \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 \quad \text{in } \Omega. \tag{3.17}$$

Here $g \ge 0$ is given and $p$ is a function which satisfies

$$\begin{cases} \text{(a) } p : ]-\infty, g] \to \mathbb{R}. \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \quad \forall r_1, r_2 \le g. \\ \text{(c) } (p(r_1) - p(r_2))(r_1 - r_2) \ge 0 \quad \forall r_1, r_2 \le g. \\ \text{(d) } p(r) = 0 \ \text{ for all } r < 0. \end{cases} \tag{3.18}$$

Condition (3.15) combined with assumption (3.18) shows that when there is separation between the body and the obstacle (i.e. when $u_\nu < 0$), then the reaction of the foundation vanishes (since $\sigma_\nu = 0$); moreover, the penetration is limited (since $u_\nu \le g$) and $g$ represents its maximum value. When $0 \le u_\nu < g$ then the reaction of the foundation is uniquely determined by the normal displacement (since $-\sigma_\nu = p(u_\nu)$) and, when $u_\nu = g$, the normal stress is not uniquely determined but is submitted to the restriction $-\sigma_\nu \ge p(g)$. Such a condition shows that the contact follows a normal compliance condition of the form (3.5) but up to the limit $g$ and then, when this limit is reached, the contact follows a Signorini-type unilateral condition with the gap $g$. For this reason we refer to the contact condition (3.5) as to a normal compliance contact condition with finite penetration and unilateral constraint, and we conclude that the foundation has an elastic-rigid behavior. Also, note that when $g = 0$ condition (3.15) becomes the classical Signorini contact condition in a form with a zero gap function,

$$u_\nu \le 0, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p = 0$, condition (3.5) becomes the Signorini contact condition in a form with a gap function,

$$u_\nu \le g, \qquad \sigma_\nu \le 0, \qquad \sigma_\nu (u_\nu - g) = 0.$$

The last two conditions model the contact with a perfectly rigid foundation.

A carefully examination of contact conditions (3.5) and (3.15) shows that both of them can be cast in the abstract formulation

$$-\sigma_\nu \in \partial P(u_\nu) \qquad \text{on } (0,T) \times \varGamma_3, \tag{3.19}$$

in which $P$ is a prescribed function which satisfies

$$\left\{ \begin{array}{l} \text{(a) } P : \mathbb{R} \to (-\infty, +\infty]. \\ \text{(b) } P \text{ is convex and lower semicontinuous.} \\ \text{(c) } P(r) = 0 \ \text{ for all } r < 0. \end{array} \right. \tag{3.20}$$

Here $\partial P$ denotes the subdifferential of $P$, i.e. $\partial P : \mathbb{R} \to 2^{\mathbb{R}}$ is the multivalued operator given by

$$\partial P(r) = \{\ f \in \mathbb{R} \mid P(s) - P(r) \ge f(s - r) \quad \forall s \in \mathbb{R}\ \}.$$

Indeed, the contact condition with normal compliance and infinite penetration (3.5) can be recovered from (3.19) by taking

$$P(r) = \int_0^r p(s)\, ds,$$

whereas the contact condition with normal compliance finite penetration (3.15) can be recovered from (3.19) by taking

$$P(r) = \begin{cases} \displaystyle\int_0^r p(s)\, ds & \text{if} \quad r \le g, \\ +\infty & \text{if} \quad r > g. \end{cases}$$

We see that in both cases above, if $p$ satisfies conditions (3.8) or (3.18), then the corresponding function $P$ satisfies condition (3.20). However, the contact condition described by the multivalued relation (3.19) is more general. Indeed, taking in (3.19)

$$P(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{1}{\lambda(\alpha+1)}\, r^{\alpha+1} & \text{if } r \geq 0, \end{cases}$$

where $\alpha$ and $\lambda$ are positive parameters, leads to the contact condition

$$-\sigma_\nu = \frac{1}{\lambda}\,(u_\nu)_+^\alpha. \tag{3.21}$$

We note that (3.21) is of the form (3.5) however, if $\alpha \neq 1$, the corresponding function $p$ does not satisfies assumption (3.8)(b). Also, taking in (3.19)

$$P(r) = \begin{cases} 0 & \text{if } r < 0, \\ -\lambda \ln\left(\cos \frac{r}{\lambda}\right) & \text{if } r \in \left[0, \dfrac{\lambda\pi}{2}\right), \\ \infty & \text{if } r \geq \dfrac{\lambda\pi}{2}, \end{cases}$$

where $\lambda$ is a positive constant, leads to the contact condition

$$-\sigma_\nu = \begin{cases} 0 & \text{if } u_\nu < 0, \\ \tan \frac{u_\nu}{\lambda} & \text{if } u_\nu \in \left[0, \dfrac{\lambda\pi}{2}\right). \end{cases} \tag{3.22}$$

In this last condition the penetration is allowed, but limited, since it does not exceed $\dfrac{\lambda\pi}{2}$; however, (3.22) can not be cast on the form (3.19) with $p$ satisfying (3.18).

In this paper we restrict ourselves to the study of the dynamic frictionless contact problems $\mathscr{P}_1$ and $\mathscr{P}_2$. Considering more general problems involving the contact condition (3.19), which contains as special cases (3.21) and (3.22), leads to important mathematical difficulties, would represent an important extension of this work, and will be treated in a furthcoming paper.

We now describe the assumptions on the data we consider in the study of the mechanical problems (3.1)–(3.7) and (3.11)–(3.17). We assume that the operators $\mathscr{A}$ and $\mathscr{E}$ are linear whereas the operator $\mathscr{G}$ may be nonlinear and they satisfy the following conditions.

$$\begin{cases} \text{(a) } \mathscr{A} = (\mathscr{A}_{ijk\ell}) : \Omega \times S^N \to S^N. \\ \text{(b) } \mathscr{A}_{ijk\ell} \in L_\infty(\Omega),\ 1 \leq i,j,k,\ell \leq N. \\ \text{(c) } \mathscr{A}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathscr{A}\boldsymbol{\tau},\ \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S^N, \text{ a.e. in } \Omega. \\ \text{(d) There exists } a_0 > 0 \text{ such that} \\ \qquad \mathscr{A}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \geq a_0 \|\boldsymbol{\tau}\|^2\ \forall \boldsymbol{\tau} \in S^N, \text{ a.e. in } \Omega. \end{cases} \tag{3.23}$$

$$
\text{(3.24)}\qquad \begin{cases} \text{(a) } \mathscr{E} = (\mathscr{E}_{ijk\ell}) : \Omega \times S^N \to S^N. \\ \text{(b) } \mathscr{E}_{ijk\ell} \in L_\infty(\Omega),\ 1 \le i,j,k,\ell \le N. \\ \text{(c) } \mathscr{E}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathscr{E}\boldsymbol{\tau},\ \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in S^N,\ \text{a.e. in } \Omega. \\ \text{(d) There exists } e_0 > 0 \ \text{ such that} \\ \qquad \mathscr{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge e_0 \|\boldsymbol{\tau}\|^2 \ \ \forall\, \boldsymbol{\tau} \in S^N,\ \text{a.e. in } \Omega. \end{cases}
$$

$$
\text{(3.25)}\qquad \begin{cases} \text{(a) } \mathscr{G} : \Omega \times S^N \times S^N \to S^N. \\ \text{(b) There exists } L_{\mathscr{G}} > 0 \text{ such that} \\ \qquad \|\mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1) - \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2)\| \\ \qquad\quad \le L_{\mathscr{G}} \left(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|\right) \\ \qquad\qquad \forall\, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S^N,\ \text{a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S^N,\ \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) \\ \qquad \text{is measurable on } \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathscr{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}) \text{ belongs to } \mathcal{H}. \end{cases}
$$

We suppose that the mass density satisfies

$$
\text{(3.26)}\qquad \rho \in L_\infty(\Omega), \quad \text{there exists } \rho^* > 0 \ \text{ such that } \ \rho(\boldsymbol{x}) \ge \rho^* \text{ a.e. } \boldsymbol{x} \in \Omega,
$$

the body forces and surface tractions have the regularity

$$
\text{(3.27)}\qquad \boldsymbol{f}_0 \in L_2(0,T; \boldsymbol{L}_2(\Omega)), \quad \boldsymbol{f}_2 \in L_2(0,T; \boldsymbol{L}_2(\Gamma_2)),
$$

and the initial data satisfy

$$
\text{(3.28)}\qquad \boldsymbol{u}_0 \in V, \qquad \boldsymbol{u}_1 \in \boldsymbol{L}_2(\Omega).
$$

We finish this section with further notation which are needed in the study of Problem $\mathscr{P}_1$ and $\mathscr{P}_2$. Thus, in the rest of the paper we use a modified inner product on the Hilbert space $H = \boldsymbol{L}_2(\Omega)$, given by

$$
\text{(3.29)}\qquad (\boldsymbol{u}, \boldsymbol{v})_H = (\rho\, \boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}_2(\Omega)} \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in H,
$$

that is, it is weighted with $\rho$, and we let $\|\cdot\|_H$ be the associated norm, i.e.,

$$
\text{(3.30)}\qquad \|\boldsymbol{v}\|_H = (\rho\, \boldsymbol{v}, \boldsymbol{v})^{1/2}_{\boldsymbol{L}_2(\Omega)} \quad \forall\, \boldsymbol{v} \in H.
$$

It follows from assumption (3.26) that $\|\cdot\|_H$ and $\|\cdot\|_{\boldsymbol{L}_2(\Omega)}$ are equivalent norms on $H$, and also the inclusion mapping of $(V, \|\cdot\|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by $V'$ the dual space of $V$. Identifying $H$ with its own dual, we can write the Gelfand triple

$$
V \subset H \subset V'.
$$

We use the notation $\langle \cdot, \cdot \rangle_{V' \times V}$ to represent the duality pairing between $V'$ and $V$ and we recall that

$$
\text{(3.31)}\qquad \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{V' \times V} = (\boldsymbol{u}, \boldsymbol{v})_H \qquad \forall\, \boldsymbol{u} \in H,\ \boldsymbol{v} \in V.
$$

Finally, we denote by $\|\cdot\|_{V'}$ the norm on $V'$.

Assumptions (3.27) allow us, for a.e. $t \in (0,T)$, to define $\boldsymbol{f}(t) \in V'$ by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V'\times V} = \int_\Omega \boldsymbol{f}_0(t)\cdot \boldsymbol{v}\,dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot \boldsymbol{v}\,da \quad \forall\, \boldsymbol{v} \in V, \tag{3.32}$$

and note that

$$\boldsymbol{f} \in L_2(0,T;V'). \tag{3.33}$$

Also, in the study of Problem $\mathscr{P}_1$ we need the set of admissible displacements field

$$K = \{\ \boldsymbol{v} \in V \mid v_\nu \le g \quad \text{a.e. on } \Gamma_3\ \} \tag{3.34}$$

and we reinforce assumption (3.28) with

$$\boldsymbol{u}_0 \in K. \tag{3.35}$$

Finally, assumption (3.8) or (3.18) allow to consider the functional defined by

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu) v_\nu\, da, \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V. \tag{3.36}$$

Note that $j$ is defined on $V \times V$ in the case of Problem $\mathscr{P}_1$ and on $K \times V$ in the case of Problem $\mathscr{P}_2$.

# 4 The problem with infinite penetration

In this section we study the weak solvability of problem $\mathscr{P}_1$. We start with a brief description of the steps in the derivation of a variational formulation for this mechanical problem. To this end, assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are smooth functions satisfying (3.1)–(3.7) and let $t \in [0,T]$. We take the dot product of equation (3.2) with $\boldsymbol{w}$ where $\boldsymbol{w}$ is an arbitrary element of $V$, integrate the result over $\Omega$, and use Green's formula (2.1) to obtain

$$(\rho\, \ddot{\boldsymbol{u}}(t), \boldsymbol{w})_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} = \int_\Omega \boldsymbol{f}_0(t)\cdot \boldsymbol{w}\,dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot \boldsymbol{w}\,da. \tag{4.1}$$

Applying the boundary conditions (3.4) and (3.6) and noting that $\boldsymbol{w} = \boldsymbol{0}$ on $\Gamma_1$, we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot \boldsymbol{w}\,da = \int_{\Gamma_2} \boldsymbol{f}(t)\cdot \boldsymbol{w}\,da + \int_{\Gamma_3} \sigma_\nu(t)\, w_\nu\, da. \tag{4.2}$$

Moreover, (3.5) combined with (3.36) lead to

$$\int_{\Gamma_3} \sigma_\nu(t)\, w_\nu\, da = -j(\boldsymbol{u}(t), \boldsymbol{w}). \tag{4.3}$$

We now use (4.1)–(4.3) and the equalities (3.29), (3.31) and (3.32) to find

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V}. \tag{4.4}$$

Finally, we combine (3.1), (4.4), and (3.7) to derive the following variational formulation of Problem $\mathscr{P}_1$.

**Problem** $\mathscr{P}_1^V$ *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds \quad \text{a.e. } t \in (0,T), \tag{4.5}$$

$$\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j(\boldsymbol{u}(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V}, \quad \forall \boldsymbol{w} \in V,\ \text{a.e. } t \in (0,T), \tag{4.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1. \tag{4.7}$$

The main result of this section is the following.

**Theorem 4.1** *Assume that conditions* (3.18), (3.23)–(3.28) *hold. Then, Problem* $\mathscr{P}_1^V$ *has a unique solution. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W_2^1(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}} \in L_2(0,T;V'), \tag{4.8}$$

$$\boldsymbol{\sigma} \in L_2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L_2(0,T;V'). \tag{4.9}$$

We conclude by Theorem 4.1 that the frictionless contact problem with normal compliance and infinite penetration (3.1)–(3.7) has a unique *weak solution* and it satisfies (4.8)–(4.9).

The proof of Theorem 4.1 will be carried out in several steps. We assume in the rest of this section that (3.18), (3.23)–(3.28) hold; below in this section $c$ will denote a generic positive constant which may depend on $\Omega$, $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\mathscr{A}$, $\mathscr{E}$, $\mathscr{G}$, $p$ and $T$, but does not depend on $t$ nor on the rest of the input data, and whose value may change from place to place.

Let $\boldsymbol{\eta} \in L_2(0,T;V')$ be given. In the first step we consider the following variational problem.

**Problem** $\mathscr{P}_1^{\eta-disp}$ *Find a displacement field* $\boldsymbol{u}_\eta : [0,T] \to V$ *such that*

$$\langle \ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{w}\rangle_{V'\times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \langle \boldsymbol{\eta}(t), \boldsymbol{w}\rangle_{V'\times V} = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall \boldsymbol{w} \in V,\ \text{a.e. } t \in (0,T), \tag{4.10}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0. \tag{4.11}$$

To solve Problem $\mathscr{P}_1^{\eta-disp}$ we apply the abstract existence and uniqueness result contained in Theorem 2.1.

**Lemma 4.1** *There exists a unique solution to Problem $\mathscr{P}_1^{\eta-disp}$ and it has the regularity expressed in* (4.8)*. Moreover, if $\boldsymbol{u}_i$ represents the solution of Problem $\mathscr{P}_1^{\eta_i-disp}$ for $\boldsymbol{\eta}_i \in L_2(0,T;V')$, $i=1,2$, there exists $c>0$ such that*

$$\int_0^t \|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V^2\,ds \le c\int_0^t \|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_{V'}^2\,ds \quad \forall t\in[0,T]. \tag{4.12}$$

**Proof.** We define the operator $A: V\to V'$ by

$$\langle A\boldsymbol{v},\boldsymbol{w}\rangle_{V'\times V} = (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{v}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \qquad \forall\,\boldsymbol{v},\,\boldsymbol{w}\in V. \tag{4.13}$$

It follows from (3.23) and (2.2) that $A$ is a linear continuous operator which satisfies condition (2.4) with $\alpha=\omega=a_0$ and recall that by (3.33), (3.28) we have $\boldsymbol{f}-\boldsymbol{\eta}\in L_2(0,T;V')$ and $\boldsymbol{u}_1\in H$. It follows now from Theorem 2.1 that there exists a unique function $\boldsymbol{v}_\eta:[0,T]\to V$ which satisfies

$$\boldsymbol{v}_\eta \in L_2(0,T;V)\cap C([0,T];H),\ \ \dot{\boldsymbol{v}}_\eta\in L_2(0,T;V'), \tag{4.14}$$

$$\dot{\boldsymbol{v}}_\eta(t)+A\boldsymbol{v}_\eta(t)+\boldsymbol{\eta}(t)=\boldsymbol{f}(t) \quad \text{a.e. } t\in(0,T), \tag{4.15}$$

$$\boldsymbol{v}_\eta(0)=\boldsymbol{u}_1. \tag{4.16}$$

Let $\boldsymbol{u}_\eta:[0,T]\to V$ be the function defined by

$$\boldsymbol{u}_\eta(t)=\int_0^t \boldsymbol{v}_\eta(s)\,ds+\boldsymbol{u}_0 \quad \forall\, t\in[0,T]. \tag{4.17}$$

It follows from (4.13)–(4.17) that $\boldsymbol{u}_\eta$ is a solution of the variational problem $\mathscr{P}_1^{\eta-disp}$ and it satisfies the regularity expressed in (4.8). This concludes the existence part of Lemma 4.1. The uniqueness of the solution follows from the uniqueness of the solution to problem (4.14)–(4.16), guaranteed by Theorem 2.1.

Consider now $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2\in L_2(0,T;V')$ and denote $\boldsymbol{u}_i=\boldsymbol{u}_{\eta_i}$, $\boldsymbol{v}_i=\boldsymbol{v}_{\eta_i}=\dot{\boldsymbol{u}}_{\eta_i}$ for $i=1,2$. We obtain from (4.10)

$$\begin{aligned}&\langle\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle_{V'\times V}+(\mathscr{A}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_1)-\mathscr{A}\,\boldsymbol{\varepsilon}(\boldsymbol{v}_2),\boldsymbol{\varepsilon}(\boldsymbol{v}_1)-\boldsymbol{\varepsilon}(\boldsymbol{v}_2))_{\mathcal{H}}\\ &\quad+\langle\boldsymbol{\eta}_1-\boldsymbol{\eta}_2,\boldsymbol{v}_1-\boldsymbol{v}_2\rangle_{V'\times V}=0,\end{aligned}$$

a.e. on $(0,T)$. Let $t\in[0,T]$. We integrate the previous equality with respect to time and use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{u}_1$ and the properties of the operator $\mathscr{A}$ to find

$$\begin{aligned}&\frac{1}{2}\,\|\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)\|_H^2+a_0\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2\,ds\\ &\quad\le-\int_0^t\langle\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\rangle_{V'\times V}\,ds+a_0\int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_H^2\,ds.\end{aligned}$$

Now,

$$
\begin{aligned}
-\int_0^t \langle \boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\rangle_{V'\times V}\, ds \\
\le \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'} \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V\, ds \\
\le \frac{1}{2a_0}\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds + \frac{a_0}{2}\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2\, ds.
\end{aligned}
$$

The previous two inequalities lead to

$$
\begin{aligned}
\frac{1}{2}\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_H^2 + \frac{a_0}{2}\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2\, ds \\
\le \frac{1}{2a_0}\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds + a_0\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_H^2\, ds.
\end{aligned} \tag{4.18}
$$

We use a Gronwall argument in (4.18) and find

$$
\|\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)\|_H^2 \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds \tag{4.19}
$$

then we use (4.18) and (4.19) and obtain

$$
\int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V^2\, ds \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{V'}^2 ds,
$$

which implies (4.12). ◇

We use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.1 to construct the following Cauchy problem for the stress field.

**Problem** $\mathscr{P}_1^{\eta-st}$ *Find a stress field* $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ *such that*

$$
\boldsymbol{\sigma}_\eta(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds \tag{4.20}
$$

for all $t \in [0,T]$.

In the study of Problem $\mathscr{P}_1^{\eta-st}$ we have the following result.

**Lemma 4.2** *There exists a unique solution of Problem* $\mathscr{P}_1^{\eta-st}$ *and it satisfies* $\boldsymbol{\sigma}_\eta \in W_2^1(0,T;\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i$ *and* $\boldsymbol{u}_i$ *represent the solutions of problem* $\mathscr{P}_1^{\eta_i-st}$ *and* $\mathscr{P}_1^{\eta_i-disp}$, *respectively, for* $\boldsymbol{\eta}_i \in L_2(0,T;V')$, $i = 1,2$, *there exists* $c > 0$ *such that*

$$
\begin{aligned}
\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V + \\
\int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\Big) \qquad \forall\, t \in [0,T].
\end{aligned} \tag{4.21}
$$

**Proof.** Let $\Lambda_\eta : L_2(0,T;\mathcal{H}) \to L_2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta \boldsymbol{\sigma}(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds \tag{4.22}$$

for all $\boldsymbol{\sigma} \in L_2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in L_2(0,T;\mathcal{H})$ we use (4.22) and (3.25) to obtain

$$\|\Lambda_\eta\, \boldsymbol{\sigma}_1(t) - \Lambda_\eta\, \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le L_{\mathscr{G}} \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds$$

for all $t \in [0,T]$. It follows from this inequality that for $m$ large enough, a power $\Lambda_\eta^m$ of the operator $\Lambda_\eta$ is a contraction on the Banach space $L_2(0,T;V)$ and therefore there exists a unique element $\boldsymbol{\sigma}_\eta \in L_2(0,T;\mathcal{H})$ such that $\Lambda_\eta \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem $\mathscr{P}_1^{\eta-st}$ and, using (4.20), the regularity of $\boldsymbol{u}_\eta$ and the properties of the operators $\mathscr{E}$ and $\mathscr{G}$, it follows that $\boldsymbol{\sigma}_\eta \in W_2^1(0,T;\mathcal{H})$.

Consider now $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L_2(0,T;V')$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_i(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)))\, ds \quad \forall\, t \in [0,T],$$

and, using the properties (3.24) and (3.25) of $\mathscr{E}$ and $\mathscr{G}$, we find

$$\begin{aligned} \|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} &\le c \Big( \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \\ &+ \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\, ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \Big) \qquad \forall\, t \in [0,T]. \end{aligned}$$

Using now a Gronwall argument in the previous inequality we deduce (4.21), which concludes the proof. ◇

We now introduce the operator $\Theta : L_2(0,T;V') \to L_2(0,T;V')$ which maps every element $\boldsymbol{\eta} \in L_2(0,T;V')$ to the element $\Theta\boldsymbol{\eta} \in L_2(0,T;V')$ defined by

$$\begin{aligned} \langle \Theta\boldsymbol{\eta}(t), \boldsymbol{w} \rangle_{V' \times V} &= (\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \\ &+ \Big( \int_0^t \mathscr{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{w}) \Big)_{\mathcal{H}} \\ &+ j(\boldsymbol{u}_\eta(t), \boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in V, \quad \forall\, t \in [0,T]. \end{aligned} \tag{4.23}$$

Here, for every $\boldsymbol{\eta} \in L_2(0,T;V')$, $\boldsymbol{u}_\eta$ and $\boldsymbol{\sigma}_\eta$ represent the displacement field and the stress field obtained in Lemmas 4.1 and 4.2, respectively. We have the following result.

**Lemma 4.3** *The operator $\Theta$ has a unique fixed point $\boldsymbol{\eta}^* \in L_2(0,T,V')$.*

**Proof.** Let $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in L_2(0,T;V')$, let $t \in [0,T]$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $i = 1,2$. We use (4.23), (3.24), (3.25) and elementary algebraic manipulations to obtain

$$\begin{aligned} |\langle \Theta\boldsymbol{\eta}_1(t) - \Theta\boldsymbol{\eta}_2(t), \boldsymbol{w} \rangle_{V' \times V}| &\le c \Big( \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_V \\ &+ \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q\, ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds \Big) \|\boldsymbol{w}\|_V \\ &+ |j(\boldsymbol{u}_1(t), \boldsymbol{w}) - j(\boldsymbol{u}_2(t), \boldsymbol{w})|. \end{aligned} \tag{4.24}$$

Now, it follows from (3.36) and (3.8) that

$$|j(\boldsymbol{u}_1(t),\boldsymbol{w})-j(\boldsymbol{u}_2(t),\boldsymbol{w})|\le c\int_{\Gamma_3}\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|\,\|\boldsymbol{w}\|\,dx$$
$$\le c\,\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_{\boldsymbol{L}_2(\Gamma_3)}\|\boldsymbol{w}\|_{\boldsymbol{L}_2(\Gamma_3)}$$

and, using (2.3), we find

$$|j(\boldsymbol{u}_1(t),\boldsymbol{w})-j(\boldsymbol{u}_2(t),\boldsymbol{w})|\le c\,\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\|\boldsymbol{w}\|_V. \tag{4.25}$$

We plug (4.25) in (4.24) and find

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_1(t)\|_{V'}\le c\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds$$
$$+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_{\mathcal{H}}\,ds\Big). \tag{4.26}$$

We use now (4.21) in (4.26) to obtain

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_1(t)\|_{V'}\le c\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds\Big) \tag{4.27}$$

and since $\boldsymbol{u}_1(0)=\boldsymbol{u}_2(0)=\boldsymbol{u}_0$, we have

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V\le\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds, \tag{4.28}$$

$$\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds\le c\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds. \tag{4.29}$$

It follows from (4.27)–(4.29) that

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_1(t)\|_{V'}\le c\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V\,ds,$$

which implies that

$$\|\Theta\boldsymbol{\eta}_1(t)-\Theta\boldsymbol{\eta}_1(t)\|_{V'}^2\le c\int_0^t\|\dot{\boldsymbol{u}}_1(s)-\dot{\boldsymbol{u}}_2(s)\|_V^2\,ds. \tag{4.30}$$

Lemma 4.3 is now a direct consequence of inequalities (4.30), (4.12) and Banach's fixed point theorem. ◇

We have now all the ingredients to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let $\boldsymbol{\eta}^*\in L_2(0,T;V')$ be the fixed point of the operator $\Theta$ defined by (4.23) and denote

$$\boldsymbol{u}^*=\boldsymbol{u}_{\eta^*},\qquad\boldsymbol{\sigma}^*=\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)+\boldsymbol{\sigma}_{\eta^*}. \tag{4.31}$$

We prove that the couple $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ satisfies (4.5)–(4.7). Indeed, we write (4.20) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ and use (4.31) to obtain that (4.5) is satisfied. Then we use (4.10) for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$ to find

$$\begin{aligned} \langle \ddot{\boldsymbol{u}}^*(t), \boldsymbol{w}\rangle_{V'\times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \langle \boldsymbol{\eta}^*(t), \boldsymbol{w}\rangle_{V'\times V} \\ = \langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} \quad \forall\, \boldsymbol{w} \in V, \text{ a.e. } t \in (0,T). \end{aligned} \tag{4.32}$$

Equality $\Theta\boldsymbol{\eta}^* = \boldsymbol{\eta}^*$ combined with (4.23) and (4.31) shows that

$$\begin{aligned} \langle \boldsymbol{\eta}^*(t), \boldsymbol{w}\rangle_{V'\times V} &= (\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^*(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} \\ &+ \Big(\int_0^t \mathscr{G}(\boldsymbol{\sigma}^*(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}^*(s)))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + \\ &+ j(\boldsymbol{u}(t), \boldsymbol{w}) \quad \forall\, \boldsymbol{w} \in V,\ t \in [0,T]. \end{aligned} \tag{4.33}$$

We plug now (4.33) in (4.32) and use (4.5), to see that $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*)$ satisfies (4.6). Next, (4.7) and (4.8) follow from Lemma 4.1 and the regularity $\boldsymbol{\sigma}^* \in L_2(0,T;\mathcal{H})$ follows from Lemmas 4.1, 4.2 and (4.31). Finally (4.6) implies that

$$\rho\ddot{\boldsymbol{u}}^*(t) = \operatorname{Div}\boldsymbol{\sigma}^*(t) + \boldsymbol{f}_0(t) \quad \text{in} \quad V', \quad \text{a.e.} \ \ t \in (0,T),$$

and therefore by (3.27) we find that $\operatorname{Div}\boldsymbol{\sigma}^* \in L_2(0,T;V')$, which concludes the existence part of the theorem.

The uniqueness part can be obtained by standard arguments. It follows from the uniqueness of the fixed point of the operator $\Theta$ defined by (4.23). $\diamond$

# 5 The problem with finite penetration and unilateral constraint

In this section we study the weak solvability of Problem $\mathscr{P}_2$. We start with a brief description of the steps in the derivation of a variational formulation for this mechanical problem. To this end, assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are smooth functions satisfying (3.11)–(3.17). We use the set of admissible displacements fields, (3.34), as well as the functional $j$, (3.36), defined on $K \times V$. Also, we introduce the set of test functions

$$\mathscr{K} = \{\ \boldsymbol{v} \in W_2^1(0,T;V) \mid \boldsymbol{v}(t) \in K \quad \forall\, t \in [0,T]\ \}. \tag{5.1}$$

Let $t \in [0,T]$ and let $\boldsymbol{w} \in \mathscr{K}$. We take the dot product of equation (3.12) with $\boldsymbol{w}(t) - \boldsymbol{u}(t)$, integrate the result over $\Omega$ and use Green's formula (2.1) to obtain

$$\begin{aligned} (\rho\, \ddot{\boldsymbol{u}}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t))_{\boldsymbol{L}_2(\Omega)} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ = \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{w}(t) - \boldsymbol{u}(t))\, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{w} - \boldsymbol{u}(t))\, da. \end{aligned} \tag{5.2}$$

Applying the boundary conditions (3.14) and (3.16) and noting that $\boldsymbol{w}(t) = \boldsymbol{0}$ on $\Gamma_1$, we have

$$\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu}\cdot(\boldsymbol{w}(t) - \boldsymbol{u}(t))\, da = \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{w}(t) - \boldsymbol{u}(t))\, da + \int_{\Gamma_3} \sigma_\nu(t)\,(w_\nu(t) - u_\nu(t))\, da. \tag{5.3}$$

Moreover, (3.15) yields

$$\int_{\Gamma_3} \sigma_\nu(t)\,(w_\nu(t) - u_\nu(t))\,da \geq \int_{\Gamma_3} p(u_\nu(t)(u_\nu(t) - w_\nu(t))\,da. \tag{5.4}$$

We combine now (5.2)–(5.4) and use (3.29), (3.31) and (3.32) to find

$$\begin{aligned} \langle \ddot{\boldsymbol{u}}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}} \\ + j(\boldsymbol{u}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{w} - \boldsymbol{u}(t)\rangle_{V'\times V}. \end{aligned} \tag{5.5}$$

Then, we integrate (5.5) on $[0, T]$, perform an integration by part, use the initial conditions (3.17) and combine the resulting inequality with the constitutive law (3.1) and with the unilateral constraint in (3.15). As a result we obtain the following variational formulation of Problem $\mathscr{P}_2$.

**Problem** $\mathscr{P}_2^V$ *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$ *such that* $\boldsymbol{u} \in \mathscr{K}$,

$$\boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds \quad \text{a.e. } t \in (0, T), \tag{5.6}$$

$$\begin{aligned} &\int_0^T (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\,dt - \int_0^T (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t))_H\,dt \\ &\quad + \int_0^T j(\boldsymbol{u}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t))\,dt + (\dot{\boldsymbol{u}}(T), \boldsymbol{w}(T) - \boldsymbol{u}(T))_H \\ &\qquad \geq \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)\rangle_{V'\times V}\,dt + (\boldsymbol{u}_1, \boldsymbol{w}(0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{w} \in \mathscr{K}. \end{aligned} \tag{5.7}$$

The main result of this section concerns the solvability of Problem $\mathscr{P}_2^V$ and can be stated as follows.

**Theorem 5.1** *Assume that conditions* (3.20), (3.23)–(3.28) *and* (3.35) *hold. Then Problem* $\mathscr{P}_2^V$ *has at least a solution. Moreover, the solution satisfies*

$$\dot{\boldsymbol{u}} \in \boldsymbol{H}^{1,1/2}(Q) \equiv L_2(0, T; \boldsymbol{H}^1(\Omega)) \cap H^{1/2}(0, T; \boldsymbol{L}_2(\Omega)), \tag{5.8}$$

$$\boldsymbol{\sigma} \in L_2(0, T; \mathcal{H}). \tag{5.9}$$

We conclude by Theorem 5.1 that the frictionless contact problem with normal compliance, finite penetration and unilateral constraint, (3.11)–(3.17), has at least a *weak solution* and it satisfies (5.8)–(5.9). The question of the uniqueness of the solution is left open.

We turn now to the proof of Theorem 5.1 which will be carried out in several steps and it is based on a limit procedure on estimates for the solutions of a sequence of regularized problems, similar to that used in [11, Ch. 4]. Since the modifications are straightforward, sometimes we omit the details. Everywhere below we assume that (3.18), (3.23)–(3.28) and (3.35) hold.

We start with the construction of the regularized problems. To this end, for every $\lambda > 0$ we consider the function $p_\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$p_\lambda(r) = \begin{cases} p(r) & \text{if } r \le g, \\ \frac{1}{\lambda}(r-g) + p(g) & \text{if } r > g, \end{cases} \tag{5.10}$$

and let $P_\lambda : \mathbb{R} \to \mathbb{R}$ be the function defined by $P_\lambda(r) = \int_0^r p_\lambda(s)\, ds$, i.e.

$$P_\lambda(r) = \begin{cases} \int_0^r p(s)\, ds & \text{if } r \le g, \\ \frac{1}{2\lambda}(r-g)^2 + p(g)(r-g) + \int_0^g p(s)\, ds & \text{if } r > g, \end{cases} \tag{5.11}$$

We also consider the functional $j_\lambda : V \times V \to R$ given by

$$j_\lambda(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\lambda(u_\nu) v_\nu \, da. \tag{5.12}$$

We use the notation above to define the following regularized frictionless contact problems.

**Problem** $\mathscr{P}_{2\lambda}^V$ *Find a displacement field* $\boldsymbol{u}_\lambda : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}_\lambda : [0,T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}_\lambda(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(t)) + \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_\lambda(s) - \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\lambda(s)))\, ds, \quad \text{a.e. } t \in (0,T), \tag{5.13}$$

$$\langle \ddot{\boldsymbol{u}}_\lambda(t), \boldsymbol{w} \rangle_{V' \times V} + (\boldsymbol{\sigma}_\lambda(t), \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{H}} + j_\lambda(\boldsymbol{u}_\lambda(t), \boldsymbol{w}) = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V}, \quad \forall\, \boldsymbol{w} \in V, \ \text{a.e. } t \in (0,T), \tag{5.14}$$

$$\boldsymbol{u}_\lambda(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\lambda(0) = \boldsymbol{u}_1. \tag{5.15}$$

Clearly, problem $\mathscr{P}_{2\lambda}^V$ represents the variational formulation of a contact problem of the form $\mathscr{P}_1$, in which the contact condition (3.5) is defined with the function $p = p_\lambda$. In this problem the penetration is allowed and unlimited. However, keeping in mind the definition of the function $p_\lambda$, we formally recover condition (3.15) in the limit as $\lambda \to 0$. For this reason we refer to Problem $\mathscr{P}_{2\lambda}^V$ as a *regularization* of the original frictionless contact problem $\mathscr{P}_2^V$.

Note that the function $p_\lambda$ defined in (5.10) satisfies assumptions (3.8). Therefore, using Theorem 4.1 it follows that Problem $\mathscr{P}^V_{2\lambda}$ has a unique solution which satisfies

$$\boldsymbol{u}_\lambda \in W^{1,2}(0,T;V) \cap C^1([0,T];H), \quad \ddot{\boldsymbol{u}}_\lambda \in L_2(0,T;V'), \tag{5.16}$$

$$\boldsymbol{\sigma}_\lambda \in L_2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma}_\lambda \in L_2(0,T;V'). \tag{5.17}$$

We now proceed to *a priori* estimates. Below in this section $c$ will represent a generic positive constant which may depend on the problem data but does not depend on $\lambda$ or $T$, nor on the positive numbers $k$ and $T_0$ which will be specified later; also its value may change from line to line.

i) ***A priori* estimates.** Let $\lambda > 0$. We put $\boldsymbol{w} = \dot{\boldsymbol{u}}_\lambda(t)$ in (5.14) to obtain

$$\langle \ddot{\boldsymbol{u}}_\lambda(t), \dot{\boldsymbol{u}}_\lambda(t)\rangle_{V'\times V} + (\boldsymbol{\sigma}_\lambda(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\lambda(t)))_{\mathcal{H}} + j_\lambda(\boldsymbol{u}(t), \dot{\boldsymbol{u}}_\lambda(t)) = \langle \boldsymbol{f}(t), \dot{\boldsymbol{u}}_\lambda(t)\rangle_{V'\times V}, \quad \text{a.e. } t \in (0,T). \tag{5.18}$$

We integrate equation (5.18) with respect to time, use (5.13), the properties (3.23)–(3.25) of the operators $\mathscr{A}$, $\mathscr{E}$ and $\mathscr{G}$, the definition (5.11) of the function $P_\lambda$, and the regularity (3.35) of the initial data $\boldsymbol{u}_0$. After some calculation we obtain that there exists $T_0 \in (0,T]$ such that,

$$\|\dot{\boldsymbol{u}}_\lambda\|^2_{L_\infty(0,T_0;\boldsymbol{L}_2(\Omega))} + \|\dot{\boldsymbol{u}}_\lambda\|^2_{L_2(0,T_0;V)} + \|\boldsymbol{u}_\lambda\|^2_{L_\infty(0,T_0;V)} + \|P_\lambda(u_{\lambda\nu})\|_{L_\infty(0,T_0;L_1(\Gamma_3))} \le c. \tag{5.19}$$

Here and below $u_{\lambda\nu}$ and $\sigma_{\lambda\nu}$ represent the normal trace of $\boldsymbol{u}_\lambda$ and $\boldsymbol{\sigma}_\lambda$, respectively. Also, note that the restriction of the length of the interval of time arise from the need to obtain a convenient estimate involving the integral term in (5.13); a similar argument will be used in the step v) of the proof which we present below.

ii) ***Dual* estimate.** To obtain the *dual* estimate we test in (5.14) with an arbitrary element $\boldsymbol{w} \in L_2(0,T_0;\mathring{\boldsymbol{H}}^1(\Omega))$. This together with (5.19) yields

$$\|\ddot{\boldsymbol{u}}_\lambda\|^2_{L_2(0,T_0;\boldsymbol{H}^{-1}(\Omega))} \le c. \tag{5.20}$$

Interpolating (5.19) and (5.20) we finally arrive at

$$\|\dot{\boldsymbol{u}}_\lambda\|^2_{\boldsymbol{H}^{1,\frac{1}{2}}(Q_{T_0})} + \|\dot{\boldsymbol{u}}_\lambda\|^2_{L_\infty(0,T_0;\boldsymbol{L}_2(\Omega))} + \|P_\lambda(u_{\lambda\nu})\|_{L_\infty(0,T_0,L_1(\Gamma_3))} \le c. \tag{5.21}$$

Moreover, since $-\sigma_{\lambda\nu} = p(u_{\lambda\nu})$ on $S_3$, using standard trace estimates we have

$$\|p_\lambda(u_{\lambda\nu})\|_{H^{-1/4,-1/2}((0,T_0)\times\Gamma_3)} \le c. \tag{5.22}$$

iii) **First convergence results as $\lambda \to 0$.** We prove now some convergence results involving the approximate solution $(\boldsymbol{u}_\lambda, \boldsymbol{\sigma}_\lambda)$. To this end, consider a sequence of positive numbers $\{\lambda_n\}$ converging to zero as $n \to \infty$. The validity of (5.19)–(5.21) shows that there exists an element $\boldsymbol{u}$ such that

$$\dot{\boldsymbol{u}} \in \boldsymbol{H}^{1/2,1}(Q_{T_0}) \cap L_\infty(0,T_0;\boldsymbol{L}_2(\Omega)) \tag{5.23}$$

and, for a subsequence $\{\lambda_{n_k}\} \subset \{\lambda_n\}$, the following convergences hold as $k \to \infty$:

$$\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k) \rightharpoonup \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) \text{ in } L_2(0,T_0;\mathcal{H}), \tag{5.24}$$

$$\ddot{\boldsymbol{u}}_k \rightharpoonup \ddot{\boldsymbol{u}} \quad \text{in } L_2(0,T_0;\boldsymbol{H}^{-1}(\Omega)), \tag{5.25}$$

$$\dot{\boldsymbol{u}}_k \rightharpoonup \dot{\boldsymbol{u}} \quad \text{in } \boldsymbol{H}^{1/2,1}(Q_{T_0}), \tag{5.26}$$

$$\dot{\boldsymbol{u}}_k \to \dot{\boldsymbol{u}} \quad \text{in } \boldsymbol{L}_2(Q_{T_0}), \tag{5.27}$$

$$\boldsymbol{u}_k \to \boldsymbol{u} \quad \text{in } \boldsymbol{L}_2(S_{T_0}), \tag{5.28}$$

Here and below we use the notation $\boldsymbol{u}_k = \boldsymbol{u}_{\lambda_{n_k}}$ and $\lambda_k = \lambda_{n_k}$. Indeed, (5.27) follows from (5.26) by the standard compact imbedding theorem. An analogous argument works also for (5.28), and it is based on the convergence in the space $H^1(0,T_0;\boldsymbol{L}_2(\Omega)) \cap L_2(0,T_0;\boldsymbol{H}^{1/2}(\Gamma))$.

iv) **$\boldsymbol{u}$ is an locally admissible displacement field.** We use the notation $p_k = p_{\lambda_{n_k}}$ and we denote by $u_{k\nu}$ the normal trace of $\boldsymbol{u}_k$. Let $k \in \mathbb{N}$. It follows from (5.22) that

$$\int_0^{T_0}\int_{\Gamma_3} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c \tag{5.29}$$

which implies that

$$\int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}\le g\}} p_k(u_{k\nu})u_{k\nu}\, da\, dt - \int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}\le g\}} p_k(u_{k\nu})g\, da\, dt + \int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}> g\}} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c.$$

We neglect the first term in the left hand side of the previous inequality and note that $p_k(u_{k\nu}) \le p(g)$ on $\Gamma_3 \cap \{u_{k\nu} \le g\}$. As a result we obtain

$$\int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}> g\}} p_k(u_{k\nu})(u_{k\nu} - g)\, da\, dt \le c. \tag{5.30}$$

We use in (5.30) the definition of the function $p_k$, (5.10), and elementary manipulations to see that

$$\frac{1}{\lambda_k}\int_0^{T_0}\int_{\Gamma_3\cap\{u_{k\nu}> g\}} (u_{k\nu} - g)^2\, da\, dt \le c.$$

This last inequality shows that

$$\int_0^{T_0}\int_{\Gamma_3} \big[(u_{k\nu} - g)_+\big]^2\, da\, dt \le c\lambda_k. \tag{5.31}$$

We pass now to the limit in (5.31) as $k \to \infty$ and use (5.28) to see that

$$\int_0^{T_0} \int_{\Gamma_3} \big[(u_\nu - g)_+\big]^2 \, da \, dt \le 0,$$

which shows that $(u_\nu(t) - g)_+ = 0$ a.e. on $\Gamma_3$, for all $t \in [0, T_0]$. We conclude that

$$\boldsymbol{u}(t) \in K \qquad \forall\, t \in [0, T_0], \tag{5.32}$$

i.e. $\boldsymbol{u}$ is an locally admissible displacement field.

v) **A strong convergence result.** Let $k \in \mathbb{N}$. Consider the functions $\boldsymbol{\sigma}_k^I$ and $\boldsymbol{\sigma}^I$ defined by the equalities

$$\boldsymbol{\sigma}_k^I(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}_k^I(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)))\, ds, \tag{5.33}$$

$$\boldsymbol{\sigma}^I(t) = \mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathscr{G}(\boldsymbol{\sigma}^I(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\, ds, \tag{5.34}$$

for all $t \in [0, T_0]$. The definition of these functions is based on arguments similar to those used in Lemma 4.2, which show that the integral equations (5.33) and (5.34) have a unique solution.

We write (5.14) for $\lambda = \lambda_k$, take $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_k$ and use (5.33) to obtain

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} \\
&+ \int_{\Gamma_3} p_k(u_{k\nu})(u_\nu - u_{k\nu})\, da = \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} \quad \text{a.e. on } (0, T).
\end{aligned}$$

Next, using the monotonicity of the function $p_k$ and (5.32) we obtain

$$\begin{aligned}
&\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} p(u_\nu)(u_\nu - u_{k\nu})\, da \ge \langle \boldsymbol{f}, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} \quad \text{a.e. on } (0, T),
\end{aligned}$$

which shows that

$$\begin{aligned}
&(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k - \dot{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I - \boldsymbol{\sigma}^I, \boldsymbol{\varepsilon}(\boldsymbol{u}_k - \boldsymbol{u}))_{\mathcal{H}} \le \\
&\quad (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}), \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}^I, \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_k))_{\mathcal{H}} + \langle \ddot{\boldsymbol{u}}_k, \boldsymbol{u} - \boldsymbol{u}_k \rangle_{V' \times V} \\
&\qquad + \int_{\Gamma_3} p(u_\nu)(u_\nu - u_{k\nu})\, da + \langle \boldsymbol{f}, \boldsymbol{u}_k - \boldsymbol{u} \rangle_{V' \times V} \quad \text{a.e. on } (0, T).
\end{aligned}$$

Let $t \in [0, T_0]$. We integrate the previous inequality over $[0, t]$, use standard integration by parts and the initial conditions to find that

$$
\begin{aligned}
(5.35)\quad & (\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)))_{\mathcal{H}} \\
& +\int_0^t(\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds\le \\
& \int_0^t(\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)))_{\mathcal{H}}\,ds+\int_0^t(\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds \\
& +\int_0^t\langle\dot{\boldsymbol{u}}_k(s),\dot{\boldsymbol{u}}_k(s)-\dot{\boldsymbol{u}}(s)\rangle_{V'\times V}\,ds-(\dot{\boldsymbol{u}}_k(t),\boldsymbol{u}_k(t)-\boldsymbol{u}(t))_H+ \\
& +\int_0^t\int_{\Gamma_3}p(u_\nu)(u_\nu-u_{k\nu})\,da\,dt+\int_0^t\langle\boldsymbol{f},\boldsymbol{u}_k-\boldsymbol{u}\rangle_{V'\times V}\equiv C_t(k).
\end{aligned}
$$

With the bound

$$(\mathscr{A}\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(t)-\boldsymbol{u}(t)))_{\mathcal{H}}\ge 0,$$

inequality (5.35) leads to

$$(5.36)\qquad \int_0^t(\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}}\,ds\le C_t(k).$$

On the other hand, it follows from (5.33) and (5.34) that

$$
\begin{aligned}
(5.37)\quad & (\boldsymbol{\sigma}_k^I(s)-\boldsymbol{\sigma}^I(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}} \\
& =(\mathscr{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)-\boldsymbol{u}_k(s)))_{\mathcal{H}} \\
& +\Big(\int_0^s\big[\mathscr{G}(\boldsymbol{\sigma}_k^I(r),\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r))-\mathscr{G}(\boldsymbol{\sigma}^I(r),\boldsymbol{\varepsilon}(\boldsymbol{u}(r))\big]\,dr,\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))_{\mathcal{H}} \\
& \forall s\in[0,T_0].
\end{aligned}
$$

We combine (5.36) and (5.37) and use assumption (3.24) and (3.25) on the operators $\mathscr{E}$ and $\mathscr{G}$ to obtain

$$
\begin{aligned}
(5.38)\quad & e_0\int_0^t\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s)))\|_{\mathcal{H}}^2\,ds\le C_t(k)+ \\
& c\,T_0\Big(\int_0^t\big[\|\boldsymbol{\sigma}_k^I(r)-\boldsymbol{\sigma}^I(r)\|_{\mathcal{H}}+\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r)-\boldsymbol{u}(r))\|_{\mathcal{H}}\big]\,dr\Big)\int_0^t\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s)-\boldsymbol{u}(s))\|_{\mathcal{H}}\,ds.
\end{aligned}
$$

We use again (5.33), (5.34), (3.25) and Gronwall's inequality to see that

$$
\begin{aligned}
(5.39)\qquad & \|\boldsymbol{\sigma}_k^I(r)-\boldsymbol{\sigma}^I(r)\|_{\mathcal{H}}\le c\Big(\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(r)-\boldsymbol{u}(r))\|_{\mathcal{H}} \\
& +\int_0^r\|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(\xi)-\boldsymbol{u}(\xi))\|_{\mathcal{H}}\,d\xi\Big)\quad\forall\, r\in[0,T_0],
\end{aligned}
$$

and using this inequality in (5.38) we obtain

$$e_0 \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2 \, ds \leq C_t(k) + c\,T_0(1+T_0)\Big( \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}} \, ds \Big)^2. \tag{5.40}$$

Since

$$\Big( \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}} \, ds \Big)^2 \leq T_0 \int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2 \, ds,$$

it follows from (5.40) that for $T_0$ small enough we have

$$\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}_k(s) - \boldsymbol{u}(s))\|_{\mathcal{H}}^2 \, ds \leq c\,C_t(k). \tag{5.41}$$

We use now the convergences (5.24)–(5.28) and the definition of $C_t(k)$ in (5.35) to see that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_k) \to \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad L^2(0,T_0,\mathcal{H}), \quad \text{as } k \to \infty. \tag{5.42}$$

This convergence combined with inequality (5.39) shows that

$$\boldsymbol{\sigma}_k^I \to \boldsymbol{\sigma}^I \quad \text{in} \quad L^2(0,T_0,\mathcal{H}), \quad \text{as } k \to \infty. \tag{5.43}$$

vi) **Existence of the solution.** Let $k \in \mathbb{N}$. We write (5.14) for $\lambda = \lambda_k$, take $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_k$ where $\boldsymbol{v} \in \mathscr{K}$ is an arbitrary test function and use (5.33) to obtain

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{v} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_k))_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} p_k(u_{k\nu})(v_\nu - u_{k\nu}) \, da = \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_k \rangle_{V' \times V} \quad \text{a.e. on } (0,T). \end{aligned} \tag{5.44}$$

Now, since the function $p_k$ is increasing and $\boldsymbol{v} \in \mathscr{K}$ we find

$$\int_{\Gamma_3} p_k(u_{k\nu})(v_\nu - u_{k\nu}) \, da \leq \int_{\Gamma_3} p_k(v_\nu)(v_\nu - u_{k\nu}) \, da = \int_{\Gamma_3} p(v_\nu)(v_\nu - u_{k\nu}) \, da$$

and, using this inequality in (5.44), yields

$$\begin{aligned} &\langle \ddot{\boldsymbol{u}}_k, \boldsymbol{v} - \boldsymbol{u}_k \rangle_{V' \times V} + (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_k), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_k))_{\mathcal{H}} + (\boldsymbol{\sigma}_k^I, \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}_k))_{\mathcal{H}} \\ &\quad + \int_{\Gamma_3} p(v_\nu)(v_\nu - u_{k\nu}) \, da \geq \langle \boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_k \rangle_{V' \times V} \quad \text{a.e. on } (0,T). \end{aligned}$$

We integrate the last inequality on $(0,T_0)$, perform integration by parts and use the convergences (5.24)–(5.27), (5.42) and (5.43) to obtain

$$\begin{aligned} &\int_0^{T_0} (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_{\mathcal{H}} \, dt \\ &\quad + \int_0^{T_0} (\boldsymbol{\sigma}^I(t), \boldsymbol{\varepsilon}(\boldsymbol{v}(t) - \boldsymbol{u}(t)))_{\mathcal{H}} \, dt - \int_0^{T_0} (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{v}}(t) - \dot{\boldsymbol{u}}(t))_H \, dt \\ &\quad + \int_0^{T_0} \int_{\Gamma_3} p(v_\nu(t))(v_\nu(t) - u_\nu(t)) \, dt + (\dot{\boldsymbol{u}}(T_0), \boldsymbol{v}(T_0) - \boldsymbol{u}(T_0))_H \\ &\quad \geq \int_0^{T_0} \langle \boldsymbol{f}(t), \boldsymbol{v}(t) - \boldsymbol{u}(t) \rangle_{V' \times V} \, dt + (\boldsymbol{u}_1, \boldsymbol{v}(0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{v} \in \mathscr{K}. \end{aligned} \tag{5.45}$$

Next, we take $\boldsymbol{v} = \boldsymbol{u} + \theta(\boldsymbol{w} - \boldsymbol{u})$ in (5.45), where $\boldsymbol{w}$ is arbitrary in $\mathscr{K}$ and $\theta \in ]0,1[$, then we divide the resulting inequality by $\theta$. As a result we find

$$
\begin{aligned}
(5.46)\qquad & \int_0^{T_0} (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\, dt \\
& + \int_0^{T_0} (\boldsymbol{\sigma}^I(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\, dt - \int_0^{T_0} (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t))_H\, dt \\
& + \int_0^{T_0} \int_{\Gamma_3} p(u_\nu(t) + \theta(w_\nu(t) - u_\nu(t)))(w_\nu(t) - u_\nu(t))\, da\, dt \\
& + (\dot{\boldsymbol{u}}(T_0), \boldsymbol{w}(T_0) - \boldsymbol{u}(T_0))_H \\
& \geq \int_0^{T_0} \langle \boldsymbol{f}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)\rangle_{V'\times V}\, dt + (\boldsymbol{u}_1, \boldsymbol{w}(0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{w} \in \mathscr{K}.
\end{aligned}
$$

We now use the properties (3.18) of the function $p$ to see that

$$
\begin{aligned}
(5.47)\qquad & \int_0^{T_0} \int_{\Gamma_3} p(u_\nu(t) + \theta(w_\nu(t) - u_\nu(t)))(w_\nu(t) - u_\nu(t))\, da\, dt \to \\
& \int_0^{T_0} \int_{\Gamma_3} p(u_\nu(t))(w_\nu(t) - u_\nu(t))\, da\, dt \quad \text{as } \theta \to 0.
\end{aligned}
$$

Therefore, passing to the limit in (5.46) as $\theta \to 0$ and using (5.47) and the definition (3.36) of the functional $j$ we obtain

$$
\begin{aligned}
(5.48)\qquad & \int_0^{T_0} (\mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{u}(t)))_{\mathcal{H}}\, dt \\
& + \int_0^{T_0} (\boldsymbol{\sigma}^I(t), \boldsymbol{\varepsilon}(\boldsymbol{w}(t) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_{\mathcal{H}}\, dt - \int_0^{T_0} (\dot{\boldsymbol{u}}(t), \dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t))_H\, dt \\
& + \int_0^{T_0} j(\boldsymbol{u}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t))\, dt + (\dot{\boldsymbol{u}}(T_0), \boldsymbol{v}(T_0) - \boldsymbol{u}(T_0))_H \\
& \geq \int_0^{T_0} \langle \boldsymbol{f}(t), \boldsymbol{w}(t) - \boldsymbol{u}(t)\rangle_{V'\times V}\, dt + (\boldsymbol{u}_1, \boldsymbol{w}(T_0) - \boldsymbol{u}_0)_H \quad \forall\, \boldsymbol{w} \in \mathscr{K}.
\end{aligned}
$$

Let $\boldsymbol{\sigma} : [0, T_0] \to \mathcal{H}$ be the function given by

$$
(5.49)\qquad \boldsymbol{\sigma}(t) = \mathscr{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \boldsymbol{\sigma}^I(t) \qquad \text{a.e. } t \in (0, T_0).
$$

It follows from (5.48), (5.49) and (5.34) that $(\boldsymbol{u}, \boldsymbol{\sigma})$ satisfy (5.6), (5.7) on the interval $[0, T_0]$. Also, it follows from (5.23), (5.34) and (5.49) that the pair $(\boldsymbol{u}, \boldsymbol{\sigma})$ has the regularity expressed in (5.8), (5.9), on the time interval $[0, T_0]$. We conclude that $(\boldsymbol{u}, \boldsymbol{\sigma})$ is a local solution of the Problem $\mathscr{P}_2^V$. Using now the standard successive approximation argument we obtain a solution on the whole interval $[0, T]$, which concludes the proof.$\diamond$

**Acknowledgement**

*1) This work was partially supported by the Academy of Sciences of the Czech Republic under the grant IAA 1075402 and under the Institutional research plan AVOZ 10190503.*

*2) The authors express their gratitude to the referees for their careful examination of the manuscript as well as for their useful comments and remarks.*

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