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Published on: 25 Oct 2013 - Journal of Mathematical Physics (American Institute of Physics)

Topics: Hilbert space, Differential equation, Algebraic equation and Entire function

AIP Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **54**, 102104 (2013); doi: 10.1063/1.4826356
View online: http://dx.doi.org/10.1063/1.4826356
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/54/10?ver=pdfcov
Published by the AIP Publishing

JOURNAL OF MATHEMATICAL PHYSICS **54**, 102104 (2013)

On the solvability of the quantum Rabi model and its 2-photon and two-mode generalizations

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(Received 17 July 2013; accepted 8 October 2013; published online 25 October 2013)

We study the solvability of the time-independent matrix Schrödinger differential equations of the quantum Rabi model and its 2-photon and two-mode generalizations in Bargmann Hilbert spaces of entire functions. We show that the Rabi model and its 2-photon and two-mode analogs are quasi-exactly solvable. We derive the exact, closed-form expressions for the energies and the allowed model parameters for all the three cases in the solvable subspaces. Up to a normalization factor, the eigenfunctions for these models are given by polynomials whose roots are determined by systems of algebraic equations. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4826356]

I. INTRODUCTION

The quantum Rabi model describes the interaction of a two-level atom with a single harmonic mode of electromagnetic field. It is perhaps the simplest system for modeling the ubiquitous matter-light interactions in modern physics, and has applications in a variety of physical fields, including quantum optics,[1] cavity and circuit quantum electrodynamics,[2,3] solid state semiconductor systems,[4] and trapped ions.[5]

Recently, Braak[6] presented a transcendental function defined as an infinite power series with coefficients satisfying a three-term recursive relation, and argued that the spectrum of the Rabi model is given by the zeros of the transcendental function. This theoretical progress has renewed the interest in the Rabi and related models.[7–12] However, since Braak's transcendental function is given as an infinite power series, unless the model parameters satisfy certain constraints for which the infinite series truncates, its exact zeros and, therefore, closed-form expressions for the energies of the Rabi model cannot be obtained even for those corresponding to the low-lying spectrum.

This is not surprising because, as pointed out in Refs. 8 and 12, the Rabi model is not exactly solvable but quasi-exactly solvable.[13–15] A typical feature of a quasi-exactly solvable system is that only a finite part of the spectrum can be obtained in closed form and the remaining part of the spectrum is not algebraically accessible (i.e., can only be determined by numerical means).

In this paper, we examine the solvability of the quantum Rabi model and its multi-mode and multi-quantum generalizations within the framework of Bargmann Hilbert spaces of entire functions.[16] We show that the Rabi model and its 2-photon and two-mode generalizations are quasi-exactly solvable. We derive the explicit, closed-form expressions for the energies, the allowed model parameters as well as the wavefunctions for all the three cases once and for all in terms of the roots of the systems of algebraic equations. We note that the energies for the Rabi and 2-photon Rabi models were obtained previously[17–20] by different methods.

The work is organized as follows. In Sec. II, we recall the widely accepted characterization of solvability of a linear differential operator in terms of invariants subspaces. In Secs. III–V, we obtain the exact solutions of the Rabi model and its 2-photon and 2-mode generalizations in their respective solvable subspaces.

II. EXACT SOLVABILITY VERSUS QUASI-EXACT SOLVABILITY

Exact solvability and quasi-exact solvability are closely related.[22] A quantum mechanical system is exactly solvable in the Schrödinger picture if all the eigenvalues and the corresponding eigenfunctions of the system can be determined exactly. In contrast, a system is quasi-exactly solvable if only a finite number of exact eigenvalues and eigenfunctions can be obtained. Among various characterizations of solvability, the one about the existence of invariant polynomial subspaces is conceptually the simplest.

A linear differential operator H is called quasi-exactly solvable if it has a finite-dimensional invariant subspace $\mathcal{V}_\mathcal{N}$ with explicitly described basis, that is,

$$H\mathcal{V}_\mathcal{N} \subset \mathcal{V}_\mathcal{N},\, \dim\mathcal{V}_\mathcal{N} < \infty,\, \mathcal{V}_\mathcal{N} = \mathrm{span}\{\xi_1, \cdots, \xi_{\dim\mathcal{V}_\mathcal{N}}\}.$$

An immediate consequence of this characterization of quasi-exact solvability for the operator H is that it can be diagonalized *algebraically* and exact, closed-form expressions of the corresponding spectra can be obtained in the (solvable) subspace $\mathcal{V}_\mathcal{N}$. The remaining part of the spectrum is not analytically accessible and can only be computed through approximations (though sometimes rather accurately). If the space $\mathcal{V}_\mathcal{N}$ is a subspace of a Bargmann-Hilbert space of entire functions in which H is naturally defined, the solvable spectra and the corresponding vectors in $\mathcal{V}_\mathcal{N}$ give the exact eigenvalues and eigenfunctions of H, respectively.

A linear differential operator H is exactly solvable if it preserves an infinite flag of finite-dimensional functional spaces,

$$\mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_\mathcal{N} \subset \cdots,$$

whose bases admit explicit analytic forms that is there exists a sequence of finite-dimensional invariant subspaces $\mathcal{V}_\mathcal{N}, \mathcal{N} = 1, 2, 3, \cdots$,

$$H\mathcal{V}_\mathcal{N} \subset \mathcal{V}_\mathcal{N},\, \dim\mathcal{V}_\mathcal{N} < \infty,\, \mathcal{V}_\mathcal{N} = \mathrm{span}\{\xi_1, \cdots, \xi_{\dim\mathcal{V}_\mathcal{N}}\}.$$

As will be seen in Secs. III–V, based on the above characterization of solvability of a quantum system, the Rabi model and its 2-photon and two-mode analogs are not exactly solvable, contrary to the claims by Refs. 6, 7, 10, and 11. Rather they are quasi-exactly solvable, as was also noted by Moroz[12] for the Rabi model, because only a finite part of their spectra can be determined exactly.

III. QUASI-EXACT SOLVABILITY OF THE QUANTUM RABI MODEL

Quasi-exact solvability of the Rabi model has recently been noted by Moroz.[12] Special exact spectrum of the model was obtained in Refs. 17, 18, and 20 by different methods. In this section, we re-examine this model within the framework of the Bargmann Hilbert space of entire functions. We will solve the time-independent Schrödinger matrix differential equations by means of the functional Bethe ansatz method.[23–25] In addition to the exact spectrum, we are also able to obtain the closed form expressions for the allowed model parameters and the polynomial wavefunctions in terms of the roots of a set of algebraic equations.

The Hamiltonian of the Rabi model is

$$H = \omega a^\dagger a + \Delta\sigma_z + g\,\sigma_x\left[a^\dagger + a\right], \tag{3.1}$$

where g is the interaction strength, σ_z, σ_x are the Pauli matrices describing the two atomic levels separated by energy difference 2Δ, and $a\dagger$ (a) are creation (annihilation) operators of a boson mode with frequency ω. In the Bargmann realization $a^\dagger \to z$, $a \to \frac{d}{dz}$, the Hamiltonian becomes a matrix differential operator

$$H = \omega z\frac{d}{dz} + \Delta\sigma_z + g\,\sigma_x\left(z + \frac{d}{dz}\right). \tag{3.2}$$

Working in a representation defined by σ_x diagonal and in terms of the two-component wavefunction $\psi(z) = \begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}$, the time-independent Schrödinger equation gives rise to a coupled system of two

1st-order differential equations[6]

$$(\omega z+g)\frac{d}{dz}\psi_+(z)+(gz-E)\psi_+(z)+\Delta\psi_-(z)=0,$$
$$(\omega z-g)\frac{d}{dz}\psi_-(z)-(gz+E)\psi_-(z)+\Delta\psi_+(z)=0. \tag{3.3}$$

If $\Delta=0$ these two equations decouple and reduce to the differential equations of two uncoupled displaced harmonic oscillators which can be exactly solved separately.[26] For this reason in the following we will concentrate on the $\Delta\neq 0$ case.

With the substitution $\psi_\pm(z)=e^{-gz/\omega}\phi_\pm(z)$, it follows[6]

$$\left[(\omega z+g)\frac{d}{dz}-\left(\frac{g^2}{\omega}+E\right)\right]\phi_+(z)=-\Delta\phi_-(z),$$
$$\left[(\omega z-g)\frac{d}{dz}-\left(2gz-\frac{g^2}{\omega}+E\right)\right]\phi_-(z)=-\Delta\phi_+(z). \tag{3.4}$$

The differential operator

$$\mathcal{L}_R^1\equiv(\omega z+g)\frac{d}{dz}-\left(\frac{g^2}{\omega}+E\right) \tag{3.5}$$

in the first equation is then exactly solvable. Eliminating $\phi_-(z)$ from the system we obtain the uncoupled differential equation for $\phi_+(z)$,

$$\left[(\omega z-g)\frac{d}{dz}-\left(2gz-\frac{g^2}{\omega}+E\right)\right]\left[(\omega z+g)\frac{d}{dz}-\left(\frac{g^2}{\omega}+E\right)\right]\phi_+=\Delta^2\phi_+. \tag{3.6}$$

This is a second-order differential equation of Fuchs' type. Explicitly,

$$(\omega z-g)(\omega z+g)\frac{d^2\phi_+}{dz^2}+\left[-2\omega g z^2+(\omega^2-2g^2-2E\omega)z+\frac{g}{\omega}(2g^2-\omega^2)\right]\frac{d\phi_+}{dz}$$
$$+\left[2g\left(\frac{g^2}{\omega}+E\right)z+E^2-\Delta^2-\frac{g^4}{\omega^2}\right]\phi_+\equiv\mathcal{L}\phi_+=0. \tag{3.7}$$

It is easy to see that for any positive integer n,

$$\mathcal{L}z^n=\left[-2n\omega g+2g\left(E+\frac{g^2}{\omega}\right)\right]z^{n+1}$$
$$+\left[n(n-1)\omega^2+(\omega^2-2g^2-2\omega E)n+E^2-\Delta^2-\frac{g^4}{\omega^2}\right]z^n$$
$$+\text{lower order terms}. \tag{3.8}$$

Due to the z^{n+1} term on the right-hand side of (3.8), the operator $\mathcal{L}$ is not exactly solvable. Only if $n=\mathcal{N}$ such that $-2\mathcal{N}\omega g+2g\left(E+\frac{g^2}{\omega}\right)=0$, the z^{n+1} term disappears from the right-hand side, and $\mathcal{L}$ preserves a finite dimensional subspace $\mathcal{V}_\mathcal{N}=\text{span}\{1,z,z^2,\cdots,z^\mathcal{N}\}$. Therefore, $\mathcal{L}$ is quasi-exactly solvable with invariant subspace $\mathcal{V}_\mathcal{N}=\text{span}\{1,z,z^2,\cdots,z^\mathcal{N}\}$.

We now seek exact solutions of the differential equation in the solvable subspace $\mathcal{V}_\mathcal{N}$. Obviously, they are polynomials in z of degree $\mathcal{N}$, which can be written as

$$\phi_+(z)=\prod_{i=1}^{\mathcal{N}}(z-z_i),\ \mathcal{N}=1,2,\cdots, \tag{3.9}$$

where z_i are the roots of the polynomial to be determined. The energies follow immediately from the equation in the 2nd line below (3.8),

$$E=\omega\left(\mathcal{N}-\frac{g^2}{\omega^2}\right). \tag{3.10}$$

The differential equation (3.7) is, in fact, a special case of the general equation studied in Refs. 25. Applying the results of this reference, we obtain the constraint for the system parameters

$$\Delta^2 + 2\mathcal{N}g^2 + 2\omega g \sum_{i=1}^{\mathcal{N}} z_i = 0. \tag{3.11}$$

Here, z_i satisfy the set of algebraic equations

$$\sum_{j\neq i}^{\mathcal{N}} \frac{2}{z_i - z_j} = \frac{2\omega g z_i^2 + (2\mathcal{N}-1)\omega^2 z_i + g(\omega^2 - 2g^2)/\omega}{(\omega z_i - g)(\omega z_i + g)}, i = 1, 2, \cdots, \mathcal{N}. \tag{3.12}$$

The corresponding wavefunction component $\psi_+(z)$ of the model is then given by

$$\psi_+(z) = e^{-\frac{g}{\omega}z} \prod_{i=1}^{\mathcal{N}} (z - z_i) \tag{3.13}$$

and the component $\psi_-(z) = e^{-gz/\omega}\phi_-(z)$ with $\phi_-(z)$ determined by the first equation of (3.4) for $\Delta \neq 0$, i.e., $\phi_-(z) = -\frac{1}{\Delta}\mathcal{L}_R^1\,\phi_+(z)$. Because $\mathcal{L}_R^1$ preserves $\mathcal{V}_\mathcal{N}$ for any system parameters, $\phi_-(z)$ automatically belongs to the same invariant subspace as $\phi_+(z)$.

As an example to the above general expressions, let us consider the $\mathcal{N} = 1$ solution. The energy is $E = \omega\left(1 - \frac{g^2}{\omega^2}\right)$. Equation (3.12) becomes

$$2\omega g z_1^2 + \omega^2 z_1 + \frac{g}{\omega}(\omega^2 - 2g^2) = 0, \tag{3.14}$$

which has two solutions

$$z_1 = -\frac{g}{\omega}, \frac{2g^2 - \omega^2}{2\omega g}. \tag{3.15}$$

Substituting into (3.11) gives the constraints $\Delta = 0$ and $\Delta^2 + 4g^2 = \omega^2$, respectively. The constraint $\Delta = 0$ corresponds to the case of degenerate atomic levels. The other constraint agrees with that obtained in Ref. 20 by a different approach. The corresponding wave function is

$$\psi_+(z) = e^{-\frac{g}{\omega}z}\left(z - \frac{2g^2 - \omega^2}{2\omega g}\right). \tag{3.16}$$

IV. QUASI-EXACT SOLVABILITY OF THE 2-PHOTON QUANTUM RABI MODEL

The Hamiltonian of the 2-photon Rabi model reads

$$H = \omega a^\dagger a + \Delta\sigma_z + g\,\sigma_x\left[(a^\dagger)^2 + a^2\right]. \tag{4.1}$$

Introduce the operators $K_\pm$, K_0 by[19–21]

$$K_+ = \frac{1}{2}(a^\dagger)^2, K_- = \frac{1}{2}a^2, K_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right). \tag{4.2}$$

Then the Hamiltonian (4.1) becomes[19,20]

$$H = 2\omega\left(K_0 - \frac{1}{4}\right) + \Delta\sigma_z + 2g\,\sigma_x(K_+ + K_-). \tag{4.3}$$

The operators $K_\pm$, K_0 form the usual $su(1, 1)$ Lie algebra,

$$[K_0, K_\pm] = \pm K_\pm, [K_+, K_-] = -2K_0. \tag{4.4}$$

The quadratic Casimir operator C of the algebra is given by

$$C = K_+K_- - K_0(K_0 - 1). \tag{4.5}$$

In what follows we shall use an infinite-dimensional unitary irreducible representation of $su(1, 1)$ known as the positive discrete series $\mathcal{D}^+(q)$. The parameter q is the so-called Bargmann index. In this representation the basis states $\{|q, n\rangle\}$ diagonalize the operator K_0,

$$K_0|q, n\rangle = (n + q)|q, n\rangle \tag{4.6}$$

for $q > 0$ and $n = 0, 1, 2, \cdots$, and the Casimir operator C has the eigenvalue $q(1 - q)$. The operators K_+ and K_- are hermitian to each other and act as raising and lowering operators, respectively, within $\mathcal{D}^+(q)$,

$$\begin{aligned} K_+|q, n\rangle &= \sqrt{(n+1)(n+2q)}\; |q, n+1\rangle, \\ K_-|q, n\rangle &= \sqrt{n(n+2q-1)}\; |q, n-1\rangle. \end{aligned} \tag{4.7}$$

For the single-mode bosonic realization (4.2), $C = \frac{3}{16}$ and q is equal to either $\frac{1}{4}$ or $\frac{3}{4}$. In terms of the original Bose operators the states $|q, n\rangle$ are given equivalently as

$$|q, n\rangle = \frac{(a^\dagger)^{2(n+q-\frac{1}{4})}}{\sqrt{\left[2\left(n+q-\frac{1}{4}\right)\right]!}}|0\rangle, q = 1/4, 3/4; n = 0, 1, 2, \cdots . \tag{4.8}$$

Thus by means of the $su(1, 1)$ representation, we have decomposed the Fock-Hilbert space of the boson field into the direct sum of two independent subspaces labeled by $q = 1/4, 3/4$, respectively.

Let us now derive a single variable differential operator realization of $K_\pm$, K_0 (4.2), i.e., differential realization of the infinite-dimensional unitary irreducible representation corresponding to $q = 1/4, 3/4$. Using the Fock-Bargmann correspondence $a^\dagger \to w, a \to \frac{d}{dw}, |0\rangle \to 1$, we can make the association

$$|q, n\rangle \longrightarrow \frac{w^{2(n+q-1/4)}}{\sqrt{\left[2\left(n+q-\frac{1}{4}\right)\right]!}} = \frac{w^{2q-1/2}\,(w^2)^n}{\sqrt{\left[2\left(n+q-\frac{1}{4}\right)\right]!}}.$$

Since q is constant for a given representation we can rewrite the above as a mapping of the Fock states $|q, n\rangle$ to the monomials in $z = w^2$,

$$\Psi_{q,n}(z) = \frac{z^n}{\sqrt{\left[2\left(n+q-\frac{1}{4}\right)\right]!}}, q = 1/4, 3/4; n = 0, 1, 2, \cdots . \tag{4.9}$$

Then in the Bargmann space with basis vectors $\Psi_{q,n}(z)$, the operators $K_\pm$, K_0 (4.2) are realized by single-variable 2nd differential operators as

$$K_0 = z\frac{d}{dz} + q, \quad K_+ = \frac{z}{2}, \quad K_- = 2z\frac{d^2}{dz^2} + 4q\frac{d}{dz}. \tag{4.10}$$

It can be checked that the differential operators (4.10) satisfy the $su(1, 1)$ commutation relations (4.4) and their action on $\Psi_{q,n}(z)$ gives the representation (4.6) and (4.7) corresponding to $q = 1/4$, 3/4. In checking, e.g., $K_-\Psi_{q,n}(z) = \sqrt{n(n+2q-1)}\;\Psi_{q,n-1}(z)$, it is useful to note that $(q - 1/4)(q - 3/4) \equiv 0$ for both $q = 1/4, 3/4$ and the differential operator K_- above can be expressed as $K_- = 2z^{-1}\left(z\frac{d}{dz} + q - \frac{1}{4}\right)\left(z\frac{d}{dz} + q - \frac{3}{4}\right)$.

In terms of the differential realization (4.10), the 2-photon Rabi Hamiltonian becomes

$$H = 2\omega\left(z\frac{d}{dz} + q - \frac{1}{4}\right) + \Delta\sigma_z + 2g\,\sigma_x\left(\frac{z}{2} + 2z\frac{d^2}{dz^2} + 4q\frac{d}{dz}\right). \tag{4.11}$$

Working in a representation defined by σ_x diagonal and in terms of the two component wavefunction, the time-independent Schrödinger equation leads to two coupled 2nd-order differential equations,

$$
\begin{aligned}
&4gz\frac{d^2}{dz^2}\psi_+(z)+(2\omega z+8gq)\frac{d}{dz}\psi_+(z)+\left[gz+2\omega\left(q-\frac{1}{4}\right)-E\right]\psi_+(z)+\Delta\psi_-(z)=0,\\
&4gz\frac{d^2}{dz^2}\psi_-(z)+(-2\omega z+8gq)\frac{d}{dz}\psi_-(z)+\left[gz-2\omega\left(q-\frac{1}{4}\right)+E\right]\psi_-(z)-\Delta\psi_+(z)=0.
\end{aligned}
\tag{4.12}
$$

If $\Delta = 0$ these two equations decouple and reduce to the differential equations of two uncoupled single-mode squeezed harmonic oscillators which can be exactly solved separately.[26] For this reason in the following we will concentrate on the $\Delta \neq 0$ case.

With the substitution

$$
\psi_\pm(z)=e^{-\frac{\omega}{4g}(1-\Omega)z}\varphi_\pm(z),\ \Omega=\sqrt{1-\frac{4g^2}{\omega^2}},
\tag{4.13}
$$

where $\left|\frac{2g}{\omega}\right|<1$, it follows,

$$
\begin{aligned}
&\left\{4gz\frac{d^2}{dz^2}+[2\omega\Omega z+8gq]\frac{d}{dz}+2q\omega\Omega-\frac{1}{2}\omega-E\right\}\varphi_+=-\Delta\varphi_-,\\
&\left\{4gz\frac{d^2}{dz^2}+[2\omega(\Omega-2)z+8gq]\frac{d}{dz}+\frac{\omega^2}{g}(1-\Omega)z+2q\omega(\Omega-2)+\frac{1}{2}\omega+E\right\}\varphi_-=\Delta\varphi_+.
\end{aligned}
\tag{4.14}
$$

Then the differential operator

$$
\mathcal{L}_{2-p}\equiv 4gz\frac{d^2}{dz^2}+[2\omega\Omega z+8gq]\frac{d}{dz}+2q\omega\Omega-\frac{1}{2}\omega-E
\tag{4.15}
$$

appearing in the first equation is exactly solvable. Eliminating $\varphi_-(z)$ from the system, we obtain the uncoupled differential equation for $\varphi_+(z)$

$$
\begin{aligned}
&\left\{4gz\frac{d^2}{dz^2}+[2\omega(\Omega-2)z+8gq]\frac{d}{dz}+\frac{\omega^2}{g}(1-\Omega)z+2q\omega(\Omega-2)+\frac{1}{2}\omega+E\right\}\\
&\quad\times\left\{4gz\frac{d^2}{dz^2}+[2\omega\Omega z+8gq]\frac{d}{dz}+2q\omega\Omega-\frac{1}{2}\omega-E\right\}\varphi_+(z)=-\Delta^2\varphi_+(z).
\end{aligned}
\tag{4.16}
$$

This is a 4th-order differential equation of Fuchs' type. Explicitly,

$$
\begin{aligned}
&16g^2z^2\frac{d^4\varphi_+}{dz^4}+64g^2\left[\frac{\omega}{4g}(\Omega-1)z^2+\left(q+\frac{1}{2}\right)z\right]\frac{d^3\varphi_+}{dz^3}\\
&\quad+\left\{4\omega^2(\Omega^2-3\Omega+1)z^2+16\omega g\left[3\left(q+\frac{1}{2}\right)\Omega-3q-1\right]z+64g^2q\left(q+\frac{1}{2}\right)\right\}\frac{d^2\varphi_+}{dz^2}\\
&\quad+\left\{2\frac{\omega^3}{g}\Omega(1-\Omega)z^2+\left[8\omega^2q(1-\Omega)+8\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2\right.\right.\\
&\qquad\left.\left.+4\omega\left(E-2\omega\left(q+\frac{1}{4}\right)\right)\right]z+32\omega gq\left[\left(q+\frac{1}{2}\right)\Omega-q\right]\right\}\frac{d\varphi_+}{dz}\\
&\quad+\left\{\frac{\omega^2}{g}(1-\Omega)\left(2q\omega\Omega-\frac{1}{2}\omega-E\right)z\right.\\
&\qquad\left.+4\omega^2q^2(1-\Omega)^2-\left[E-2\omega\left(q-\frac{1}{4}\right)\right]^2+\Delta^2\right\}\varphi_+=0.
\end{aligned}
\tag{4.17}
$$

By the arguments similar to those in Sec. III, this equation is quasi-exactly solvable provided that the system parameters Δ, ω, and g satisfy certain constraints, and exact solutions are polynomials in z in the solvable sector. We thus seek polynomial solutions of the form to the above differential equation,

$$\varphi_+(z) = \prod_{i=1}^{\mathcal{M}}(z - z_i), \mathcal{M} = 1, 2, \cdots, \tag{4.18}$$

where $\mathcal{M}$ is the degree of the polynomial solution and z_i are the roots of the polynomial to be determined. Substituting into (4.17) and dividing both sides by $\phi_+(z)$ yield

$$\begin{aligned}
&\left[E - 2\omega\left(q - \frac{1}{4}\right)\right]^2 - \Delta^2 - 4\omega^2 q^2(1-\Omega)^2 \\
&\quad = 16g^2 z^2 \sum_{i=1}^{\mathcal{M}} \frac{1}{z - z_i} \sum_{p\neq l\neq j\neq i}^{\mathcal{M}} \frac{4}{(z_i - z_p)(z_i - z_l)(z_i - z_j)} \\
&\quad +64g^2\left[\frac{\omega}{4g}(\Omega - 1)z^2 + \left(q + \frac{1}{2}\right)z\right]\sum_{i=1}^{\mathcal{M}} \frac{1}{z - z_i} \sum_{l\neq j\neq i}^{\mathcal{M}} \frac{3}{(z_i - z_l)(z_i - z_j)} \\
&\quad + \left\{4\omega^2(\Omega^2 - 3\Omega + 1)z^2 + 16\omega g\left[3\left(q + \frac{1}{2}\right)\Omega - 3q - 1\right]z\right. \\
&\qquad \left. +64g^2 q\left(q + \frac{1}{2}\right)\right\}\sum_{i=1}^{\mathcal{M}} \frac{1}{z - z_i} \sum_{j\neq i}^{\mathcal{M}} \frac{2}{z_i - z_j} \\
&\quad + \left\{2\frac{\omega^3}{g}\Omega(1-\Omega)z^2 + \left[8\omega^2 q(1-\Omega) + 8\omega^2\left(q + \frac{1}{2}\right)(1-\Omega)^2\right.\right. \\
&\qquad \left.\left. +4\omega\left(E - 2\omega\left(q + \frac{1}{4}\right)\right)\right]z + 32\omega g q\left[\left(q + \frac{1}{2}\right)\Omega - q\right]\right\}\sum_{i=1}^{\mathcal{M}} \frac{1}{z - z_i} \\
&\quad + \frac{\omega^2}{g}(1-\Omega)\left(2q\omega\Omega - \frac{1}{2}\omega - E\right)z.
\end{aligned} \tag{4.19}$$

The left-hand side is a constant and the right-hand side is a meromorphic function with simple poles at $z = z_i$ and singularity at $z = \infty$. The residues of the right-hand side at the simple poles $z = z_i$ are

$$\begin{aligned}
\mathrm{Res}_{z=z_i} &= 16g^2 z_i^2 \sum_{p\neq l\neq j\neq i}^{\mathcal{M}} \frac{4}{(z_i - z_p)(z_i - z_l)(z_i - z_j)} \\
&\quad +64g^2\left[\frac{\omega}{4g}(\Omega - 1)z_i^2 + \left(q + \frac{1}{2}\right)z_i\right]\sum_{l\neq j\neq i}^{\mathcal{M}} \frac{3}{(z_i - z_l)(z_i - z_j)} \\
&\quad + \left\{4\omega^2(\Omega^2 - 3\Omega + 1)z_i^2 + 16\omega g\left[3\left(q + \frac{1}{2}\right)\Omega - 3q - 1\right]z_i\right. \\
&\qquad \left. +64g^2 q\left(q + \frac{1}{2}\right)\right\}\sum_{j\neq i}^{\mathcal{M}} \frac{2}{z_i - z_j} \\
&\quad +2\frac{\omega^3}{g}\Omega(1-\Omega)z_i^2 + \left[8\omega^2 q(1-\Omega) + 8\omega^2\left(q + \frac{1}{2}\right)(1-\Omega)^2\right. \\
&\qquad \left. +4\omega\left(E - 2\omega\left(q + \frac{1}{4}\right)\right)\right]z_i + 32\omega g q\left[\left(q + \frac{1}{2}\right)\Omega - q\right].
\end{aligned} \tag{4.20}$$

Using the identities

$$\sum_{i=1}^{\mathcal{M}}\sum_{j\neq i}^{\mathcal{M}}\frac{1}{z_i-z_j}=0,\ \sum_{i=1}^{\mathcal{M}}\sum_{j\neq i}^{\mathcal{M}}\frac{z_i}{z_i-z_j}=\frac{1}{2}\mathcal{M}(\mathcal{M}-1),$$
$$\sum_{i=1}^{\mathcal{M}}\sum_{l\neq j\neq i}^{\mathcal{M}}\frac{1}{(z_i-z_l)(z_i-z_j)}=0,\ \sum_{i=1}^{\mathcal{M}}\sum_{l\neq j\neq i}^{\mathcal{M}}\frac{z_i}{(z_i-z_l)(z_i-z_j)}=0,$$
$$\sum_{i=1}^{\mathcal{M}}\sum_{p\neq l\neq j\neq i}^{\mathcal{M}}\frac{1}{(z_i-z_p)(z_i-z_l)(z_i-z_j)}=0,$$
$$\sum_{i=1}^{\mathcal{M}}\sum_{p\neq l\neq j\neq i}^{\mathcal{M}}\frac{z_i}{(z_i-z_p)(z_i-z_l)(z_i-z_j)}=0, \tag{4.21}$$

we can show that

$$\begin{aligned}&\left[E-2\omega\left(q-\frac{1}{4}\right)\right]^2-\Delta^2-4\omega^2q^2(1-\Omega)^2\\&=\sum_{i=1}^{\mathcal{M}}\frac{\mathrm{Res}_{z=z_i}}{z-z_i}+4\omega^2(\Omega^2-3\Omega+1)\mathcal{M}(\mathcal{M}-1)+2\frac{\omega^3}{g}\Omega(1-\Omega)\sum_{i=1}^{\mathcal{M}}z_i\\&+\left[8\omega^2q(1-\Omega)+8\omega^2\left(q+\frac{1}{2}\right)(1-\Omega)^2+4\omega\left(E-2\omega\left(q+\frac{1}{4}\right)\right)\right]\mathcal{M}\\&+\frac{\omega^2}{g}(1-\Omega)\left[(2\mathcal{M}+2q)\omega\Omega-\frac{1}{2}\omega-E\right]z.\end{aligned} \tag{4.22}$$

The right-hand side of (4.22) is a constant if and only if the coefficient of z as well as all the residues at the simple poles are vanishing. We thus obtain the energy eigenvalues,

$$E=-\frac{1}{2}\omega+\left[2\mathcal{M}+2\left(q-\frac{1}{4}\right)+\frac{1}{2}\right]\omega\Omega \tag{4.23}$$

and the constraint for the model parameters Δ, ω and g,

$$\Delta^2+4\omega^2(1-\Omega)\left[\mathcal{M}(\mathcal{M}+2q-1)+\frac{\omega}{2g}\Omega\sum_{i=1}^{\mathcal{M}}z_i\right]=0. \tag{4.24}$$

Here, the roots z_i satisfy the following system of algebraic equations,

$$\begin{aligned}&g^2z_i^2\sum_{p\neq l\neq j\neq i}^{\mathcal{M}}\frac{4}{(z_i-z_p)(z_i-z_l)(z_i-z_j)}\\&+g\left[\omega(\Omega-1)z_i^2+4g\left(q+\frac{1}{2}\right)z_i\right]\sum_{l\neq j\neq i}^{\mathcal{M}}\frac{3}{(z_i-z_l)(z_i-z_j)}\\&+\left\{\frac{\omega^2}{4}(\Omega^2-3\Omega+1)z_i^2+\omega g\left[3\left(q+\frac{1}{2}\right)\Omega-3q-1\right]z_i+4g^2q\left(q+\frac{1}{2}\right)\right\}\sum_{j\neq i}^{\mathcal{M}}\frac{2}{z_i-z_j}\\&+\frac{\omega^3}{8g}\Omega(1-\Omega)z_i^2+\frac{\omega^2}{2}\left[\mathcal{M}\Omega+\left(q+\frac{1}{2}\right)\Omega(\Omega-2)+q\right]z_i\\&+2\omega gq\left[\left(q+\frac{1}{2}\right)\Omega-q\right]=0,\ i=1,2,\cdots\mathcal{M}.\end{aligned} \tag{4.25}$$

Here, we have also used the relation (4.23) in obtaining (4.24) and (4.25). The corresponding wavefunction component $\psi_+(z)$ of the system is given by

$$\psi_+(z) = e^{-\frac{\omega}{4g}(1-\Omega)z} \prod_{i=1}^{\mathcal{M}} (z - z_i) \tag{4.26}$$

and the component $\psi_-(z) = e^{-\frac{\omega}{4g}(1-\Omega)z}\phi_-(z)$ with $\phi_-(z)$ computed from (4.14) for $\Delta \neq 0$, $\phi_-(z) = -\frac{1}{\Delta}\mathcal{L}_{2-p}\,\phi_+(z)$. Because for any positive integer n, $\mathcal{L}_{2-p}z^n \sim z^n +$ lower order terms, $\phi_-(z)$ automatically belongs to the same invariant subspace as $\phi_+(z)$.

Some remarks are in order. The results obtained in Ref. 27 on the solution of the 2-photon Rabi model are incorrect. The reason is simple. As the author himself noted in the paper, the solution for the 2nd component of the two-component wavefunction of the 2-photon Rabi model does not belong to the same invariant subspace as the 1st component. Therefore, his two-component wavefunction does not satisfy the matrix Schrödinger differential equation of the model and thus is not a solution to the coupled equations.

As an example to the above general expressions, let us consider the $\mathcal{M} = 1$ case. The energy is

$$E = -\frac{1}{2}\omega + (2q + 2)\,\omega\Omega. \tag{4.27}$$

Equation (4.25) becomes

$$\omega^2\Omega(1-\Omega)z_1^2 + 4\omega g\left[\Omega + q + \left(q + \frac{1}{2}\right)\Omega(\Omega - 2)\right] z_i + 16g^2 q\left[\left(q + \frac{1}{2}\right)\Omega - q\right] = 0. \tag{4.28}$$

The solutions to this equation are

$$z_1 = -\frac{4gq}{\omega\Omega}, \quad \frac{4gq(1-\Omega) - 2g\Omega}{\omega(1-\Omega)}. \tag{4.29}$$

Substituting into (4.24) gives the constraints $\Delta = 0$ and

$$\Delta^2 + 8q\omega^2 = 8\left(q + \frac{1}{2}\right)\omega^2\Omega^2, \tag{4.30}$$

respectively. The constraint $\Delta = 0$ corresponds to the case of degenerate atomic levels. The constraint (4.30) agrees with those obtained in Ref. 20 by the Bogoliubov transformation method. The corresponding wavefunction $\psi_+(z)$ is given by

$$\psi_+(z) = e^{-\frac{\omega}{4g}(1-\Omega)z}\left(z - \frac{4gq(1-\Omega) - 2g\Omega}{\omega(1-\Omega)}\right). \tag{4.31}$$

V. QUASI-EXACT SOLVABILITY OF THE TWO-MODE QUANTUM RABI MODEL

The Hamiltonian of the two-mode quantum Rabi model reads

$$H = \omega(a_1^\dagger a_1 + a_2^\dagger a_2) + \Delta\sigma_z + g\,\sigma_x(a_1^\dagger a_2^\dagger + a_1 a_2), \tag{5.1}$$

where we assume that the boson modes are degenerate with the same frequency ω. Introduce the operators $K_\pm$, K_0,

$$K_+ = a_1^\dagger a_2^\dagger,\ K_- = a_1 a_2,\ K_0 = \frac{1}{2}(a_1^\dagger a_1 + a_2^\dagger a_2 + 1). \tag{5.2}$$

Then the Hamiltonian (5.1) becomes

$$H = 2\omega\left(K_0 - \frac{1}{2}\right) + \Delta\sigma_z + g\sigma_x(K_+ + K_-). \tag{5.3}$$

The operators $K_\pm$, K_0 form the $su(1, 1)$ algebra (4.4). As in Sec. IV, we shall use the unitary irreducible representation (i.e., the positive discrete series). However, to avoid confusion in this section we shall use κ to denote the Bargmann index of the representation. Using this notation the action of the operators $K_\pm$, K_0 and the Casimir C (4.5) on the basis states $|\kappa, n\rangle$ of the representation reads

$$\begin{aligned} K_0|\kappa, n\rangle &= (n+\kappa)\,|\kappa, n\rangle, \\ K_+|\kappa, n\rangle &= \sqrt{(n+2\kappa)(n+1)}\,|\kappa, n+1\rangle, \\ K_-|\kappa, n\rangle &= \sqrt{(n+2\kappa-1)n}\,|\kappa, n-1\rangle, \\ C|\kappa, n\rangle &= \kappa(1-\kappa)\,|\kappa, n\rangle, \end{aligned} \tag{5.4}$$

for $\kappa > 0$ and $n = 0, 1, 2, \cdots$.

For the two-mode bosonic realization (5.2) of $su(1, 1)$ that we require here the Bargmann index κ can take any positive integers or half-integers, i.e., $\kappa = 1/2, 1, 3/2, \cdots$. In terms of the original Bose operators the states $|\kappa, n\rangle$ are given equivalently as

$$|\kappa, n\rangle = \frac{(a_1^\dagger)^{n+2\kappa-1}(a_2^\dagger)^n}{\sqrt{(n+2\kappa-1)!n!}}|0\rangle, \kappa = 1/2, 1, 3/2, \cdots; n = 0, 1, 2, \cdots. \tag{5.5}$$

Thus, by means of the $su(1, 1)$ representation we have decomposed the Fock-Hilbert space into the direct sum of infinite number of subspaces labeled by $\kappa = 1/2, 1, 3/2, \cdots$, respectively.

As in Sec. IV, using the Fock-Bargmann correspondence $a_i^\dagger \to w_i$, $a_i \to \frac{d}{dw_i}$, $|0\rangle \to 1$, we can make the association

$$|\kappa, n\rangle \longrightarrow \frac{w_1^{n+2\kappa-1}w_2^n}{\sqrt{(n+2\kappa-1)!n!}} = \frac{w_1^{2\kappa-1}\,(w_1w_2)^n}{\sqrt{(n+2\kappa-1)!n!}}.$$

Then, with κ being constant in a given representation, we can show that the set of monomials in $z = w_1w_2$,

$$\Psi_{\kappa,n}(z) = \frac{z^n}{\sqrt{(n+2\kappa-1)!n!}}, \quad \kappa = 1/2, 1, 3/2, \cdots; \quad n = 0, 1, 2, \cdots, \tag{5.6}$$

forms the basis carrying the unitary irreducible representation (5.4) corresponding to $\kappa = 1/2, 1, 3/2, \cdots$. That is the operators $K_\pm$, K_0 (5.2) have the single-variable differential realization,

$$K_0 = z\frac{d}{dz} + \kappa, \quad K_+ = z, \quad K_- = z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz}. \tag{5.7}$$

It is straightforward to verify that these differential operators satisfy the $su(1, 1)$ commutation relations (4.4) and their action on $\Psi_{\kappa,n}(z)$ gives the unitary representation (5.4) corresponding to $\kappa = 1/2, 1, 3/2, \cdots$.

By means of the differential representation (5.7), we can express the Hamiltonian (5.3) as the 2nd-order matrix differential operator

$$H = 2\omega\left(z\frac{d}{dz} + \kappa - \frac{1}{2}\right) + \Delta\sigma_z + g\,\sigma_x\left(z + z\frac{d^2}{dz^2} + 2\kappa\frac{d}{dz}\right). \tag{5.8}$$

Working in a representation defined by σ_x diagonal and in terms of the two-component wavefunction $\psi(z) = \begin{pmatrix}\psi_+(z)\\ \psi_-(z)\end{pmatrix}$, we see that the time-independent Schrödinger equation yields the two coupled differential equations,

$$\begin{aligned} &gz\frac{d^2}{dz^2}\psi_+(z) + 2(\omega z + g\kappa)\frac{d}{dz}\psi_+(z) + \left[gz + 2\omega\left(\kappa - \frac{1}{2}\right) - E\right]\psi_+(z) + \Delta\psi_-(z) = 0, \\ &gz\frac{d^2}{dz^2}\psi_-(z) + 2(-\omega z + g\kappa)\frac{d}{dz}\psi_-(z) + \left[gz - 2\omega\left(\kappa - \frac{1}{2}\right) + E\right]\psi_-(z) - \Delta\psi_+(z) = 0. \end{aligned} \tag{5.9}$$

If $\Delta = 0$ these two equations decouple and reduce to the differential equations of two uncoupled two-mode squeezed harmonic oscillators which can be exactly solved separately.[26] For this reason in the following we will concentrate on the $\Delta \neq 0$ case.

With the substitution

$$\psi_{\pm}(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\varphi_{\pm}(z),\ \Lambda = \sqrt{1-\frac{g^2}{\omega^2}}, \tag{5.10}$$

where $\left|\frac{g}{\omega}\right| < 1$, it follows,

$$\begin{aligned}
&\left\{gz\frac{d^2}{dz^2} + 2[\omega\Lambda z + g\kappa]\frac{d}{dz} + 2\kappa\omega\Lambda - \omega - E\right\}\varphi_+ = -\Delta\varphi_-,\\
&\left\{gz\frac{d^2}{dz^2} + 2[\omega(\Lambda-2)z + g\kappa]\frac{d}{dz} + \frac{4\omega^2}{g}(1-\Lambda)z + 2\kappa\omega(\Lambda-2) + \omega + E\right\}\varphi_- = \Delta\varphi_+.
\end{aligned} \tag{5.11}$$

Then the differential operator

$$\mathcal{L}_{2-m} \equiv gz\frac{d^2}{dz^2} + 2[\omega\Lambda z + g\kappa]\frac{d}{dz} + 2\kappa\omega\Lambda - \omega - E \tag{5.12}$$

in the 1st equation is exactly solvable. Eliminating $\varphi_-(z)$ from the system, we obtain the uncoupled differential equation for $\varphi_+(z)$

$$\begin{aligned}
&\left\{gz\frac{d^2}{dz^2} + 2[\omega(\Lambda-2)z + g\kappa]\frac{d}{dz} + \frac{4\omega^2}{g}(1-\Lambda)z + 2\kappa\omega(\Lambda-2) + \omega + E\right\}\\
&\times\left\{gz\frac{d^2}{dz^2} + 2[\omega\Lambda z + g\kappa]\frac{d}{dz} + 2\kappa\omega\Lambda - \omega - E\right\}\varphi_+(z) = -\Delta^2\varphi_+(z).
\end{aligned} \tag{5.13}$$

This is a 4th-order differential equation of Fuchs' type. Explicitly,

$$\begin{aligned}
&g^2z^2\frac{d^4\varphi_+}{dz^4} + 4g^2\left[\frac{\omega}{g}(\Lambda-1)z^2 + \left(q+\frac{1}{2}\right)z\right]\frac{d^3\varphi_+}{dz^3}\\
&+\left\{4\omega^2(\Lambda^2-3\Lambda+1)z^2 + 4\omega g\left[3\left(\kappa+\frac{1}{2}\right)\Lambda - 3\kappa - 1\right]z + 4g^2\kappa\left(\kappa+\frac{1}{2}\right)\right\}\frac{d^2\varphi_+}{dz^2}\\
&+\left\{8\frac{\omega^3}{g}\Lambda(1-\Lambda)z^2 + \left[8\omega^2\kappa(1-\Lambda) + 8\omega^2\left(\kappa+\frac{1}{2}\right)(1-\Lambda)^2\right.\right.\\
&\left.\left. +4\omega(E-2\omega\kappa)\right]z + 8\omega g\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda - \kappa\right]\right\}\frac{d\varphi_+}{dz}\\
&+\left\{4\frac{\omega^2}{g}(1-\Lambda)(2\kappa\omega\Lambda - \omega - E)z\right.\\
&\left. +4\omega^2\kappa^2(1-\Lambda)^2 - \left[E - 2\omega\left(\kappa-\frac{1}{2}\right)\right]^2 + \Delta^2\right\}\varphi_+ = 0.
\end{aligned} \tag{5.14}$$

As we show below, this equation is quasi-exactly solvable provided that the system parameters Δ, ω, and g satisfy certain constraints, and exact solutions are polynomials in z. To this end, we seek polynomial solutions of the form to the differential equation (5.14),

$$\varphi_+(z) = \prod_{i=1}^{\mathcal{M}}(z - z_i),\ \mathcal{M} = 1, 2, \cdots, \tag{5.15}$$

where $\mathcal{M}$ is the degree of the polynomial and z_i are roots of the polynomial to be determined. Following the procedure similar to that in Sec. IV, we obtain the energies

$$E = -\omega + \left[2\mathcal{M} + 2\left(\kappa - \frac{1}{2}\right) + 1\right]\omega\Lambda \tag{5.16}$$

and the constraint for the model parameters Δ, ω, and g,

$$\Delta^2 + 4\omega^2(1-\Lambda)\left[\mathcal{M}(\mathcal{M}+2\kappa-1) + \frac{2\omega}{g}\Lambda\sum_{i=1}^{\mathcal{M}} z_i\right] = 0. \tag{5.17}$$

Here, the roots z_i satisfy the following system of algebraic equations,

$$\begin{aligned}
&g^2 z_i^2 \sum_{p\neq l\neq j\neq i}^{\mathcal{M}} \frac{4}{(z_i-z_p)(z_i-z_l)(z_i-z_j)} \\
&\quad +4g\left[\omega(\Lambda-1)z_i^2 + g\left(\kappa+\frac{1}{2}\right)z_i\right]\sum_{l\neq j\neq i}^{\mathcal{M}}\frac{3}{(z_i-z_l)(z_i-z_j)} \\
&\quad +\left\{4\omega^2(\Lambda^2-3\Lambda+1)z_i^2 + 4\omega g\left[3\left(\kappa+\frac{1}{2}\right)\Lambda - 3\kappa - 1\right]z_i + 4g^2\kappa\left(\kappa+\frac{1}{2}\right)\right\}\sum_{j\neq i}^{\mathcal{M}}\frac{2}{z_i-z_j} \\
&\quad +8\frac{\omega^3}{g}\Lambda(1-\Lambda)z_i^2 + 8\omega^2\left[\mathcal{M}\Lambda + \left(\kappa+\frac{1}{2}\right)\Lambda(\Lambda-2)+\kappa\right]z_i \\
&\qquad +8\omega g\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda - \kappa\right] = 0, i = 1, 2, \cdots \mathcal{M}.
\end{aligned} \tag{5.18}$$

Here, we have also used the relation (5.16) in obtaining (5.17) and (5.18). The corresponding wavefunction component $\psi_+(z)$ is given by

$$\psi_+(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\prod_{i=1}^{\mathcal{M}}(z-z_i) \tag{5.19}$$

and the other component is $\psi_-(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\varphi_-(z)$ with $\varphi_-(z)$ determined from the first equation of (5.11) for $\Delta \neq 0$, $\varphi_-(z) = -\frac{1}{\Delta}\mathcal{L}_{2-m}\,\varphi_+(z)$. Because for any positive integer n, $\mathcal{L}_{2-m}z^n \sim z^n$ + lower order terms, $\varphi_-(z)$ automatically belongs to the same invariant subspace as $\varphi_+(z)$.

As an example of the above general expressions, we consider the $\mathcal{M} = 1$ case. The energy is

$$E = -\omega + (2\kappa+2)\omega\Lambda. \tag{5.20}$$

Equation (5.18) becomes

$$\omega^2\Lambda(1-\Lambda)z_1^2 + \omega g\left[\Lambda + \kappa + \left(\kappa+\frac{1}{2}\right)\Lambda(\Lambda-2)\right]z_i + g^2\kappa\left[\left(\kappa+\frac{1}{2}\right)\Lambda - \kappa\right] = 0. \tag{5.21}$$

It has two solutions

$$z_1 = -\frac{\kappa g}{\omega\Lambda}, \quad \frac{g\kappa(1-\Lambda)-g\Lambda}{\omega(1-\Lambda)}. \tag{5.22}$$

Substituting into (5.17) gives the constraints $\Delta = 0$ and

$$\Delta^2 + 8\kappa\omega^2 = 8\left(\kappa+\frac{1}{2}\right)\omega^2\Lambda^2 = 0, \tag{5.23}$$

respectively. The constraint $\Delta = 0$ corresponds to the case of degenerate atomic levels. The constraint (5.23) is the non-trivial one. The corresponding wavefunction $\psi_+(z)$ is given by

$$\psi_+(z) = e^{-\frac{\omega}{g}(1-\Lambda)z}\left(z - \frac{g\kappa(1-\Lambda) - g\Lambda}{\omega(1-\Lambda)}\right). \tag{5.24}$$

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council through Discovery Projects Grant No. DP110103434.

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