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ON THE SOLVABILITY OF VON KÁRMÁN EQUATIONS

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The nonlinear operator equation connected with general boundary value problem for von Kármán equation is studied. In the paper there is proved the coerciveness of corresponding operator and the properties which are sufficient for the existence of the solution. The main idea is due to Knightly [3] who used it in case of Dirichlet problem. The different approach to the same boundary value problem based on Berger's idea [1] is developed in the paper [2] of Hlaváček and Naumann. Using the technique of Knightly we are able to weaken in some way the restrictions put on the behavior of boundary functions.

To avoid technical difficulties we restrict ourselves to consider the domains with infinitely smooth boundary.

1. NOTATION AND PRELIMINARIES

Let $w : \Theta \to E_1$, $\Theta \subset E_2$. Denote $w_x = \partial w/\partial x$, $w_y = \partial w/\partial y$. Let Ω be a simply connected bounded domain in E_2 with its boundary $\partial\Omega$ infinitely smooth. (See section 4.) Let $\partial\Omega$ be divided into three pairwise disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3$, so that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$.

Denote

(1.1) $$w_n = w_x n_x + w_y n_y$$

the outward normal derivative,

(1.2) $$w_\tau = -w_x n_y + w_y n_x$$

the tangential derivative at the generic point of boundary $\partial\Omega$.

Denote further

(1.3) $$\Delta^2 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$$

(1.4) $$[w, f] = w_{xx} f_{yy} + w_{yy} f_{xx} - 2w_{xy} f_{xy}$$

The boundary operators M, T are defined by

$$(1.5) \qquad Mw = \nu \Delta w + (1 - \nu)(w_{xx}n_x^2 + 2w_{xy}n_xn_y + w_{yy}n_y^2)$$

$$(1.6) \qquad Tw = -(\Delta w)_n + (1 - \nu)(w_{xx}n_xn_y - w_{xy}(n_x^2 - n_y^2) - w_{yy}n_xn_y)_\tau$$

where ν (the Poisson constant) is from the interval $\langle 0, \frac{1}{2})$.

We deal with the bilinear forms

$$(1.7) \qquad (u, v)_{W_0{}^{2,2}} = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\,\mathrm{d}x\,\mathrm{d}y\,,$$

$$(1.8) \qquad (u, v)_V = (u, v)_{W_0{}^{2,2}} + \nu \int_\Omega [u, v]\,\mathrm{d}x\,\mathrm{d}y$$

and with the expression

$$(1.9) \qquad B(v; u, \varphi) = \int_\Omega (v_{xy}\,u_x\,\varphi_y + v_{xy}u_y\varphi_x - v_{xx}u_y\varphi_y - v_{yy}u_x\varphi_x)\,\mathrm{d}x\,\mathrm{d}y\,.$$

If $v, u \in W^{2,2}$, $\varphi \in W_0^{2,2}$ we obtain (using the integration by parts)

$$(1.10) \qquad B(v; u, \varphi) = B(v; \varphi, u) = B(\varphi; u, v)$$

Let the functions k_2, k_{31}, k_{32} have the following properties (with p-an arbitrary real number bigger than one):

$$(1.11) \qquad k_2 \in L_p(\Gamma_2)\,, \quad k_2 \geqq 0 \quad \text{on} \quad \Gamma_2 \quad \text{a.e.}$$

$$(1.12) \qquad k_{31} \in L_p(\Gamma_3)\,, \quad k_{31} \geqq 0 \quad \text{on} \quad \Gamma_3 \quad \text{a.e.}$$

$$(1.13) \qquad k_{32} \in L_1(\Gamma_3)\,, \quad k_{32} \geqq 0 \quad \text{on} \quad \Gamma_3 \quad \text{a.e.}$$

The right-hand sides of the equations and the boundary conditions of the problem formulated in Section 2 are submitted to the conditions

$$(1.14) \qquad m_2 \in L_p(\Gamma_2)\,, \quad m_3 \in L_p(\Gamma_2)\,, \quad r_3 \in L_1(\Gamma_3)\,, \quad p > 1$$

$$(1.15) \qquad P \in L_p(\Omega)\,, \quad p > 1\,,$$

$$(1.16) \qquad \Phi_0 \in W^{3-1/q,q}(\partial\Omega)\,, \quad \Phi_1 \in W^{2-1/q,q}(\partial\Omega) \quad \text{for some} \quad q > 2\,,$$

$$(1.16a) \qquad \Phi_0 = \Phi_1 = 0 \quad \text{on} \quad \Gamma_3\,.$$

We enlist here two assertions used in the following

Proposition 1. (*Hardy's inequality.*) *Let* $\alpha > 0$, $p > 1$, $f \in C^1(\langle 0, \alpha\rangle)$, $f(0) = 0$.

Then

$$\int_0^\alpha \left|\frac{f(x)}{x}\right|^p \mathrm{d}x \leqq \left(\frac{p}{p-1}\right)^p \int_0^\alpha |f'(x)|^p \,\mathrm{d}x\,. \tag{1.17}$$

(This is a corollary of a more general inequality due to Hardy – see e.g. [4], Chapitre 2, Lemma 5.1).

To prove the existence of solution for the operator equation formulated in Section 3 we use the following result:

Proposition 2. *Let B be a reflexive separable Banach space. Let the mapping $\mathcal{T} : B \to B^*$ be*

(1.18) *demicontinuous* (*i.e.*, $\|x_m - x\|_B \to 0 \Rightarrow \mathcal{T}x_m \rightharpoonup \mathcal{T}x$,

(1.19) *bounded* (*i.e.*, $\mathcal{T}$ *maps bounded sets in B onto bounded sets in B^**)

(1.20) *coercive* (*i.e.*, $\lim_{\|x\|\to\infty} \dfrac{\langle \mathcal{T}x, x\rangle}{\|x\|} = +\infty$)

(1.21) *satisfying condition S i.e.*, $x_m \rightharpoonup x$ *and*

$$\langle \mathcal{T}x_m - \mathcal{T}x, x_m - x\rangle \to 0 \quad \textit{implies} \quad \|x_m - x\|_B \to 0.$$

Then $\mathcal{T}(B) = B^*$ and $\mathcal{T}^{-1}$ (which is in general multivalued mapping) is bounded.

(Here we denote by $\rightharpoonup$ the weak convergence in B, by $\langle \blacksquare, \blacksquare \rangle$ the pairing between B and B^*.)

This proposition follows immediately from [5], Theorem 2.1, which is due to F. E. Browder.

2. CLASSICAL AND VARIATIONAL FORMULATIONS OF THE PROBLEM

Definition 1. *The couple $/w, \Phi/$ of functions from $C^4(\bar{\Omega})$ is said to be a classical solution of the problem if*

$$\Delta^2 w = [w, \Phi] + P \quad \text{on} \quad \Omega\,, \tag{2.1}$$

$$\Delta^2 \Phi = -[w, w] \quad \text{on} \quad \Omega\,, \tag{2.2}$$

$$w = w_n = 0 \quad \text{on} \quad \Gamma_1\,, \tag{2.3}$$

$$w = 0\,, \quad Mw + k_2 w_n = m_2 \quad \text{on} \quad \Gamma_2\,, \tag{2.4}$$

$$Mw + k_{31} w_n = m_3\,, \quad Tw + (w_x \Phi_{y\tau} - w_y \Phi_{x\tau}) + k_{32} w = r_3 \quad \text{on} \quad \Gamma_3 \;^{1)} \tag{2.5}$$

$$\Phi = \Phi_0\,, \quad \Phi_n = \Phi_1 \quad \text{on} \quad \partial\Omega\,. \tag{2.6}$$

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[1]) We write here the nonlinear part $w_x \Phi_{y\tau} - w_y \Phi_{n\tau}$ obtained in course of the deduction of von Kármán equations. To get our existence result we must formulate further the conditions on Φ under which $w_x \Phi_{y\tau} - w_y \Phi_{n\tau} = 0$. (See condition (5.3)).

Let us denote

$$(2.7) \qquad \mathscr{V} = \{u \in C^\infty(\bar{\Omega});\ u = u_n = 0 \text{ on } \Gamma_1,\ u = 0 \text{ on } \Gamma_2\}\,.$$

If $/w, \Phi/$ is a classical solution of the problem and $\varphi \in \mathscr{V}$, $\psi \in C_0^\infty(\Omega)$ we can obtain – using the standard procedure of integration by parts and the relation (1.16a) – the identities

$$(2.8) \qquad (w, \varphi)_V + a(w, \varphi) = B(w; \Phi, \varphi) + \int_\Omega P\varphi \,\mathrm{d}x\,\mathrm{d}y + \\ + \int_{\Gamma_3} (r_3\varphi + m_3\varphi_n)\,\mathrm{d}S + \int_{\Gamma_2} m_2\varphi_n\,\mathrm{d}S\,,$$

$$(2.9) \qquad (\Phi, \psi)_{W_0^{2,2}} = -B(w; w, \psi)\,,$$

where

$$(2.10) \qquad a(w, \varphi) = \int_{\Gamma_2} k_2 w_n \varphi_n\,\mathrm{d}S + \int_{\Gamma_3} (k_{32} w\varphi + k_{31} w_n \varphi_n)\,\mathrm{d}S\,.$$

Definition 2. *Let us denote by V the closure of the set $\mathscr{V}$ in the norm of $W^{2,2}(\Omega)$.*

Definition 3. *Let (1,14)–(1,16a) be satisfied. The couple $/w, \Phi/ \in V \times W^{2,2}(\Omega)$ is said to be a variational solution of the problem if*

(i) *for each $\varphi \in V$, (2.8) holds,*
(ii) *for each $\psi \in W_0^{2,2}(\Omega)$, (2.9) holds,*
(iii) *Φ satisfies (2.6) in the sense of traces.*

3. THE IDEA OF KNIGHTLY

Let F be a function from $C^2(\bar{\Omega})$ which satisfies the conditions

$$(3.1) \qquad F = \Phi_0\,, \quad F_n = \Phi_1 \quad \text{on} \quad \partial\Omega\,.$$

(The assumptions on Φ_0, Φ_1 and $\partial\Omega$ formulated in Section 1 are sufficient for the existence of such a function – see e.g. [4], Chapitre 2 Théoréme 5.8 and Théoréme 3.8. In fact we could demand immediately that Φ_0, Φ_1 be such functions that there exists a function $F \in C^2(\bar{\Omega})$ satisfying (3.1)).

Instead of the variational solution $/w, \Phi/$ in sense of Definition 2 we consider the couple of functions $/w, g/$ where g and Φ are connected by the relation

$$(3.2) \qquad g = \Phi - \zeta F\,.$$

Here ζ is an auxiliary function of $C^\infty(\bar{\Omega})$ chosen in such a way that

$$\zeta = 1 \quad \text{on} \quad \partial\Omega\,, \quad \zeta_n = 0 \quad \text{on} \quad \partial\Omega \tag{3.3}$$

(hence $g \in W_0^{2,2}(\Omega)$) and that the "unpleasant" nonlinear term $B(w; \zeta F, w)$ is suppressed. (See formula (6.1).)

Substituting for Φ from (3.2) to (2.8), (2.9) we get

$$(w, \varphi)_V + a(w, \varphi) = B(w; g, \varphi) + B(w; \zeta F. \varphi) + \int_\Omega P\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_3} (r_3\varphi + m_3\varphi_n)\,\mathrm{d}S + \int_{\Gamma_2} m_2\varphi_n\,\mathrm{d}S\,, \tag{3.4}$$

$$(g, \psi)_{W_0{}^{2,2}} = -(\zeta F, \psi)_{W_0{}^{2,2}} - B(w; w, \psi)\,. \tag{3.5}$$

Definition 3, ζ.[1]) *The couple* $/w, g/ \in V \times W_0^{2,2}$ *is said to be a solution of the problem* $K(\zeta)$ *if*

(i) *for each* $\varphi \in V$, (3.4) *holds*

(ii) *for each* $\psi \in W_0^{2,2}(\Omega)$, (3.5) *holds.*

Remark. Let there exist a solution of the problem $K(\zeta)$ for some $\zeta \in C^\infty(\bar{\Omega})$ for which (3.3) holds. Then, thanks to (3.2), there exists a variational solution of the problem in sense of Definition 3.

Proposition 3. (*See* [2], *Lemma* 3,1) *Let the following implication hold*:

$$w \in V\,, \quad (w, w)_V + a(w, w) = 0 \Rightarrow w = 0\,. \tag{3.8}$$

Then $[(w, w)_V + a(w, w)]^{1/2}$ *is an equivalent norm to* $\|\cdot\|_{W^{2,2}}$ *in* V.

Remark. In the following we suppose that (3.8) holds. A wide class of conditions concerning k_2, k_{31}, k_{32} and the geometry of $\Gamma_1, \Gamma_2, \Gamma_3$ which are reasonable from the point of view of mechanics and which guarantee the validity of (3.8) is deduced in [2].

Definition 4. *Let* H *be the Hilbert space* $V \times W_0^{2,2}(\Omega)$ *with the norm generated by the scalar product* ((,)) *defined by*

$$U = /w, g/\,, \quad \Psi = /\varphi, \psi/\,, \tag{3.9}$$
$$((U, \Psi)) = (w, \varphi)_V + a(w, \varphi) + (g, \psi)_{W_0{}^{2,2}}\,.$$

[1]) Let (1.14)—(1.16a) be satisfied.

Adding now the relations (3.4), (3.5) and denoting

$$Q[\Psi] = \int_\Omega P\varphi \,dx\,dy + \int_{\Gamma_3} (r_3\varphi + m_3\varphi_n)\,dS + \int_{\Gamma_2} m_2\varphi_n\,dS \tag{3.10}$$

$$\mathscr{T}_\zeta(U)[\Psi] = ((U, \Psi)) - B(w; g, \varphi) + \tag{3.11}$$
$$+ B(w; w, \psi) - B(w\ \zeta F, \varphi) + (\zeta F, \psi)_{W_0^{2,2}}$$

we can write $\mathscr{T}_\zeta(U)[\Psi] = Q[\Psi]$.

It is easy to see that Q is a continuous linear functional on H given by the functions P, m_2, m_3 and r_3, and for each fixed $U \in H$, $\mathscr{T}_\zeta(U)$ is a continuous linear functional depending upon ζ, too. (See e.g. (5.17)).

Hence the solvability of the problem $K(\zeta)$ is equivalent to the solvability of the operator equation

$$\mathscr{T}_\zeta(U) = Q \tag{3.12}$$

in the space H.

4. DEFINITION OF $\Omega \in C^\infty$. THE AUXILIARY FUNCTION ζ

Definition 5. *Let $\Omega \subset E_2$ be a simply connected bounded domain with its boundary $\partial\Omega$ being a simple curve with a parametrization Θ. Θ is a one-to-one mapping of $\langle 0, R\rangle$ onto $\partial\Omega$ defined by*

$$\Theta : t \mapsto (\omega_1(t), \omega_2(t)) \tag{4.1}$$

with the properties

$$\omega_i \in C^\infty(\langle 0, R)), \quad i = 1, 2, \tag{4.2}$$

$$\omega_{i+}^{(k)}(0) = \lim_{t \to R-} \omega_i^{(k)}(t), \quad i = 1, 2, \quad k = 0, 1, \dots \tag{4.3}$$

The parameter t is the length of arc so that

$$(\omega_1'(t))^2 + (\omega_2'(t))^2 = 1, \quad t \in \langle 0, R). \tag{4.4}$$

Let the orientation be such that $(-\omega_2'(t), \omega_1'(t))$ is the unit vector of the inner normal to $\partial\Omega$.

Then we say that Ω is of the class C^∞.

Definition 6. *Let $\delta > 0$. Let the mapping*

$$(x, y) : \langle 0, R) \times \langle 0, \delta\rangle \to E_2 \tag{4.5}$$

be defined by

$$(4.6)\qquad x : (t, s) \mapsto \omega_1(t) - s\omega_2'(t)\,,$$
$$y : (t, s) \mapsto \omega_2(t) + s\omega_1'(t)\,.$$

Denote by Ω_δ *the image of* $\langle 0, R) \times (0, \delta)$ *in this mapping.*

Lemma 1. *Let* $\Omega \in C^\infty$. *Then there exists* $\delta_0 > 0$ *such that the mapping* (x, y) *has the following properties*:

$$(4.7)\qquad \Omega_{\delta_0} \subset \Omega\,,$$

(4.8) (x, y) *is a one-to-one mapping of* $\langle 0, R) \times \langle 0, \delta_0\rangle$ *onto* $\bar{\Omega}_{\delta_0}$.

(4.9) *There exist two positive constants* K_1, K_2 *so that*

$$K_1 \leqq \frac{\partial(x, y)}{\partial(t, s)} \leqq K_2 \quad \text{on} \quad \langle 0, R) \times (0, \delta_0),$$

(4.10) $\partial\Omega$ *corresponds to* $s = 0$, $\partial(\Omega_{\delta_0}) \setminus \partial\Omega$ *corresponds to* $s = \delta_0$.[1])

Proof follows from the properties formulated in Definitions 5 and 6.

Remark. It is obvious that for each $\delta \in (0, \delta_0\rangle$ (4.7) –(4.10) hold with the same constants K_1, K_2 in (4.9).

Lemma 2. *Let* $\Omega \in C^\infty$, *let* δ_0 *be the number defined in Lemma* 1. *Then for each* $\delta \in (0, \delta_0\rangle$ *and each* $\varepsilon > 0$ *there exists a function* $\zeta \in C^\infty(\bar{\Omega})$ *for which*

$$(4.11)\qquad \operatorname{supp} \zeta \subset \Omega_\delta \cup \partial\Omega\,,$$

$$(4.12)\qquad \zeta = 1 \quad \text{on} \quad \partial\Omega\,, \quad \zeta_x = \zeta_y = 0 \quad \text{on} \quad \partial\Omega\,,$$

$$(4.13)\qquad |\zeta| \leqq 1 \quad \text{on} \quad \Omega\,,$$

$$(4.14)\qquad \zeta^*(t, s) = \zeta(x(t, s), y(t, s)) \quad \text{depends only on} \quad s\,,$$

$$(4.15)\qquad |\zeta_s^*(t, s)| \leqq \frac{\varepsilon}{s}\,.$$ [2])

Proof. Choose $\delta \in (0, \delta_0\rangle$, $\varepsilon > 0$. Fix $d \in (0, \min\{1, \delta\})$ and define the function $Z : E_1 \to E_1$ by

$$(4.16)\qquad \begin{aligned} Z(s) &= 1 && \text{for} \quad s \in (-\infty, de^{-2/\varepsilon})\,, \\ Z(s) &= \varepsilon/2 \log d/s && \text{for} \quad s \in \langle de^{-2/\varepsilon}, d\rangle\,, \\ Z(s) &= 0 && \text{for} \quad s \in (d, +\infty)\,. \end{aligned}$$

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[1]) From lemma 1 follows immediately that the inverse to the mapping (x, y) is infinitely smooth.

[2]) Here as well as in the following $\zeta_s = \partial\zeta/\partial s$, $\zeta_t = \partial\zeta/\partial t$.

For an arbitrary fixed number $h \in (0, \min\{\delta - d, de^{-2/\varepsilon}/2\})$ define

$$z : \langle 0, +\infty) \to E_1, \quad z(s) = Z_h(s) \tag{4.17}$$

(the regularized function Z restricted to $\langle 0, +\infty)$ – for the definition see e.g. [4]).

Function z has the following properties:

$$\begin{aligned} &\text{(i)} && \operatorname{supp} z \subset \langle 0, \delta), \\ (4.18)\quad &\text{(ii)} && z(0) = 1, \quad z'(0) = 0, \\ &\text{(iii)} && |z| \leqq 1 \quad \text{on} \quad \langle 0, +\infty), \\ &\text{(iv)} && |z'(s)| \leqq \varepsilon/s \quad \text{on} \quad \langle 0, +\infty). \end{aligned}$$

Defining now

$$\begin{aligned} (4.19)\qquad \zeta(x, y) &= z(s(x, y)) && \text{on} \quad \Omega_\delta \cup \partial\Omega \\ \zeta(x, y) &= 0 && \text{on} \quad \Omega \setminus \Omega_\delta \end{aligned}$$

we can see easily that this function satisfies (4.11)–(4.15).

5. THE MAIN RESULT

Let

$$\Omega \in C^\infty \tag{5.1}$$

and let the sets $\Gamma_1, \Gamma_2, \Gamma_3$ of (2.3)–(2.5) be expressed as

$$\Gamma_i = \Theta(\gamma_i), \quad i = 1, 2, 3 \tag{5.2}$$

where Θ is the mapping from Definition 5 and γ_i, $i = 1, 2, 3$ are pairwise disjoint measurable subsets of $\langle 0, R)$. (The situation that some γ_i or a pair of them are empty is not excluded provided the condition (3,8) still holds).

Theorem. *Let* (5.1), (5.2) *hold. Let*

$$\Phi_1 = \Phi_0 = 0 \quad \text{on} \quad \Gamma_3 \quad (\text{i.e. } (1.16\text{a}) \text{ holds}) \tag{5.3}$$

$$s_{xx}(s_y)^2 + s_{yy}(s_x)^2 - 2s_{xy}s_xs_y = 0 \quad \text{on} \quad \Gamma_2 . \tag{5.4}$$

Then there exists $\zeta \in C^\infty(\bar{\Omega})$ *satisfying* (3.3) *such that the equation* (3.12) *has a solution.*

Proof. Lemmas 3–7 assert that there exists a function $\zeta \in C^\infty(\bar{\Omega})$ satisfying (3.3) such that the operator $\mathscr{T}_\zeta$ satisfies the assumptions (1.18)–(1.21). Hence the existence of a solution of (3.12) follows from Proposition 2.

Lemma 3. *For each $\zeta \in C^{\infty}(\bar{\Omega})$ the operator $\mathcal{T}_{\zeta}$ is demicontinuous.*

Proof. Denote

$$(5.5) \qquad U^n = |w^n, g^n| \in H\,, \quad n = 1, 2, 3, \ldots\,,$$
$$U = |w, g| \in H\,, \quad \Psi = |\varphi, \psi| \in H\,,$$

and suppose

$$(5.6) \qquad U^n \to U \quad \text{in} \quad H\,.$$

It is obvious from (5.6) that

$$(5.7) \qquad ((U^n - U, \Psi)) \to 0\,, \quad B(w^n; \zeta F, \varphi) - B(w; \zeta F, \varphi) \to 0\,.$$

Thus, to establish the relation

$$(5.8) \qquad \lim_{n \to \infty} \{\mathcal{T}_{\zeta}(U^n)[\Psi] - \mathcal{T}_{\zeta}(U)[\Psi]\} = 0\,, \quad \forall \Psi \in H$$

we need to prove

$$(5.9) \qquad \lim_{n \to \infty} [B(w^n; g^n, \varphi) - B(w; g, \varphi)] = 0\,, \quad \forall \varphi \in V$$

$$(5.10) \qquad \lim_{n \to \infty} [B(w^n; w^n, \psi) - B(w; w, \psi)] = 0\,, \quad \forall \psi \in W_0^{2,2}(\Omega).$$

It follows from the inequality $|B(w^n, g^n, \varphi) - B(w; g, \varphi)| \leqq |B(w^n; g^n, \varphi) - B(w^n; g, \varphi)| + |B(w^n; g, \varphi) - B(w; g, \varphi)|$ and the definition (1.9) that we must estimate eight expressions of the type

$$(5.11) \qquad I_1^n = \int_{\Omega} |w_{xy}^n| \cdot |(g^n - g)_y| \cdot |\varphi_x| \,\mathrm{d}x\,\mathrm{d}y$$

$$I_2^n = \int_{\Omega} |(w^n - w)_{xy}| \cdot |g_y| \cdot |\varphi_x| \,\mathrm{d}x\,\mathrm{d}y\,.$$

Using the Sobolev immersion theorem (see e.g. [4], Chapitre 2, Théoréme 3.8) we have

$$(5.12) \qquad (g^n - g)_y \in L_4(\Omega)\,, \quad \|(g^n - g)_y\|_{L_4} \leqq c\|g^n - g\|_{W^{2,2}}$$

and the same estimate for φ_x. Hence

$$(5.13) \qquad I_1^n \leqq c_1 \|w^n\|_{W^{2,2}} \|g^n - g\|_{W^{2,2}} \|\varphi\|_{W^{2,2}} \leqq$$
$$\leqq c_2 \|U^n\|_H \|U^n - U\|_H \|\varphi\|_{W^{2,2}}\,.$$

The integral I_2^n from (5.11) can be estimated analogously. (5.6), (5.13) imply $\lim_{n\to\infty} I_1^n = \lim_{n\to\infty} I_2^n = 0$.

Thus we get (5.9). The proof of (5.10) is quite similar.

Lemma 4. *For each $\zeta \in C^\infty(\bar{\Omega})$ the operator $\mathcal{T}_\zeta$ is bounded on H. The following estimation holds*

$$\|\mathcal{T}_\zeta(U)\|_{H^*} \leqq c\{\|\zeta F\|_{C^2} + (\|\zeta F\|_{C^1} + 1)\|U\|_H + \|U\|_H^2\}. \tag{5.14}$$

Proof. From (1.7) Definition 4 and Proposition 3 we obtain immediately

$$|((U, \Psi))| \leqq \|U\|_H \|\Psi\|_H, \tag{5.15}$$

$$|(\zeta F, \psi)_{W_0^{2,2}}| \leqq c\|\zeta F\|_{C^2}\|\Psi\|_H. \tag{5.16}$$

According to (1.9) all estimations of "B-terms" are reduced to the estimation of integrals of the type $\int_\Omega u_{xy} v_x w_y \, dx\, dy$. Thus we have

$$\begin{aligned}|B(w; \zeta F, \varphi)| &\leqq \ldots \int_\Omega |w_{xy}| |(\zeta F)_y| |\varphi_x| \, dx\, dy \ldots \leqq \\ &\leqq c\|\zeta F\|_{C^1} \|w\|_{W^{2,2}} \|\varphi\|_{W^{2,2}} \leqq \bar{c}\|\zeta F\|_{C^1} \|U\|_H \|\Psi\|_H\end{aligned} \tag{5.17}$$

$$\begin{aligned}|B(w; g, \varphi)| &\leqq \ldots \int_\Omega |w_{xy}| |g_y| |\varphi_x| \, dx\, dy \ldots \leqq \\ &\leqq [\text{the same estimation as in the proof of (5.9)}] \leqq \\ &\leqq \ldots \left(\int_\Omega |w_{xy}|^2\right)^{1/2} \left(\int_\Omega |g_y|^4\right)^{1/4} \left(\int_\Omega |\varphi_x|^4\right)^{1/4} \ldots \leqq \\ &\leqq c\|w\|_{W^{2,2}} \|g\|_{W^{2,2}} \|\varphi\|_{W^{2,2}} \leqq \tilde{c}\|U\|_H^2 \|\Psi\|_H.\end{aligned} \tag{5.18}$$

Analogously

$$|B(w; w, \psi)| \leqq \tilde{c}\|U\|_H^2 \|\Psi\|_H. \tag{5.19}$$

From (5.15)–(5.19) the inequality (5.14) follows.

6. THE COERCIVENESS OF OPERATOR $\mathcal{T}_\zeta$

Lemma 5. *Let the conditions (5.3), (5.4) be satisfied. Then there exists $\zeta \in C^\infty(\bar{\Omega})$ such that the operator $\mathcal{T}_\zeta$ is coercive on H.*

Proof. Let $U = [w, g] \in H$. From (1.10), (3.11) we have

$$\mathcal{T}_\zeta(U)[U] = \|U\|_H^2 - B(w; \zeta F, w) + (\zeta F, g)_{W_0^{2,2}}. \tag{6.1}$$

According to the idea described in Section 3 we find the function $\zeta \in C^\infty(\bar{\Omega})$ (among the functions defined by Lemma 2) such that

$$(6.2) \qquad |B(w; \zeta F, w)| \leqq \tfrac{1}{4} \|U\|_H^2, \quad \forall U \in H .$$

After that the remaining term in (6.1) can be estimated as

$$|(\zeta F, g)_{W_0^{2,2}}| \leqq c\|\zeta F\|_{C^2} \|g\|_{W_0^{2,2}} \leqq c\|\zeta F\|_{C^2} \|U\|_H .$$

From here, using the well-known inequality $|ab| \leqq (1/2\varepsilon) a^2 + (\varepsilon/2) b^2$, $\varepsilon > 0$ we obtain

$$(6.3) \qquad |(\zeta F, g)_{W_0^{2,2}}| \leqq \tilde{c}\|\zeta F\|_{C^2}^2 + \tfrac{1}{4}\|U\|_H^2 .$$

(6.1)–(6.3) give then

$$(6.4) \qquad \mathscr{T}_\zeta(U)[U] \geqq \tfrac{1}{2}\|U\|_H^2 - \tilde{c}\|\zeta F\|_{C^2}^2$$

so that

$$(6.5) \qquad \lim_{\|U\|_H \to \infty} \frac{\mathscr{T}_\zeta(U)[U]}{\|U\|_H} = +\infty .$$

Consider a function ζ described by Lemma 2. Its parameters δ and ε will be specified later. We can restrict ourselves to the functions $w \in \mathscr{V}$ (see (2.7)). According to (4.11), (1.9) we have

$$(6.6) \qquad B(w; \zeta F, w) = \int_{\Omega_\delta} b^{xy}(w; \zeta F, w) \, \mathrm{d}x \, \mathrm{d}y$$

where

$$(6.7) \qquad b^{xy}(\varphi; \psi, \eta) = \varphi_{xy}\psi_x\eta_y + \varphi_{xy}\psi_y\eta_x - \varphi_{xx}\psi_y\eta_y - \varphi_{yy}\psi_x\eta_x .$$

Denote

$$(6.8) \qquad w^*(t, s) = w(x(t, s), y(t, s)), \ (t, s) \in \langle 0, R) \times \langle 0, \delta \rangle \text{ etc.}$$

Rewriting $b^{xy}(w; \zeta F, w)$ in terms of w^*, ζ^*, F^* and using the substitution theorem in the right-hand side integral in (6.6) with the transformation (4.6) we obtain

$$(6.9) \qquad B(w; \zeta F, w) = \int_0^R \int_0^\delta b^{ts}(w^*; \zeta^* F^*, w^*) \left[\frac{\partial(x, y)}{\partial(t, s)}\right]^{-1} \mathrm{d}s \, \mathrm{d}t + \\ + \sum \int_0^R \int_0^\delta \left\{ w_{z_1}^* (\zeta^* F^*)_{z_2} \, w_{z_3}^* \, b^{xy}(z_1; z_2, z_3) \frac{\partial(x, y)}{\partial(t, s)} \, \mathrm{d}s \, \mathrm{d}t , \right.$$

where $z_i = t$ or s for $i = 1, 2, 3$ and we summarize over all such triplets.

To deal with the integrals in (6.9) we group them in the following way:

1st group. The subintegral function contains the expression

$$(\zeta^* F^*)_t \,. \tag{6.10}$$

As a representant of this group we estimate

$$J_1 = \int_0^R \int_0^\delta w_{st}^* (\zeta^* F^*)_t \, w_s^* \left[\frac{\partial(x, y)}{\partial(t, s)} \right]^{-1} \mathrm{d}s \, \mathrm{d}t \,. \tag{6.11}$$

According to (4.9), (4.13) and (4.14) we have

$$|J_1| \leqq \frac{1}{K_1} \int_0^R \int_0^\delta |w_{st}^*| \, |F_t^*| \, |w_s^*| \, \mathrm{d}s \, \mathrm{d}t \leqq [\text{Hölder inequality}] \leqq \tag{6.12}$$

$$\leqq \frac{1}{K_1} \left(\int_0^R \int_0^\delta (w_{st}^*)^2 \right)^{1/2} \left(\int_0^R \int_0^\delta (F_t^*)^4 \right)^{1/4} \left(\int_0^R \int_0^\delta (w_s^*)^4 \right)^{1/4} \leqq$$

$\leqq$ [after the resubstitution in integrals containing w^* and using (4.9), (4.7)] $\leqq$
$\leqq c\delta^{1/4} \|F\|_{C^1} \, \|w\|_{W^{2,2}}^2$.

All other integrals from (6.9) belonging to the 1st group can be estimated in the same way.

2nd group. The subintegral function contains the expression

$$(\zeta^* F^*)_s \, w_t^* \,. \tag{6.13}$$

Concerning the sum $\zeta^* F_s^* w_t^* + \zeta_s^* F^* w_t^*$ we notice that the first term can be estimated in the same way as (6.12). Substantially different approach is required for the second term. As a pattern we consider the integral

$$J_2 = \int_0^R \int_0^\delta w_s^* \zeta_s^* F^* w_t^* \, b^{xy}(s; s, t) \frac{\partial(x, y)}{\partial(t, s)} \, \mathrm{d}s \, \mathrm{d}t \,. \tag{6.14}$$

According to (4.9), (4.6), (4.8), (4.15) we have

$$|J_2| \leqq c\varepsilon \int_0^R \int_0^\delta |w_s^*| \left| \frac{F^* w_t^*}{s} \right| \mathrm{d}s \, \mathrm{d}t = c\varepsilon \int_{\gamma_1 + \gamma_2} \int_0^\delta |w_s^*| \frac{|w_t^*|}{s} |F^*| \, \mathrm{d}s \, \mathrm{d}t + \tag{6.15}$$

$$+ c\varepsilon \int_{\gamma_3} \int_0^\delta |w_s^*| \, |w_t^*| \frac{|F^*|}{s} \, \mathrm{d}s \, \mathrm{d}t = c\varepsilon (J_3 + J_4) \,.$$

From (2.7) we have $w^*(t, o) = w_t^*(t, 0) = 0$ on $\gamma_1 \cup \gamma_2$. Hence in case of J_3 we can use Proposition 1:

$$(6.16) \qquad J_3 \leqq \|F\|_C \left(\int_{\gamma_1 \cup \gamma_2} \int_0^\delta (w_s^*)^2\right)^{1/2} \left(\int_{\gamma_1 \cup \gamma_2} \int_0^\delta \left(\frac{(w_t^*)}{s}\right)^2\right)^{1/2} \leqq$$

$$\leqq 2\|F\|_C \left(\int_{\gamma_1 \cup \gamma_2} \int_0^\delta (w_s^*)^2\right)^{1/2} \left(\int_{\gamma_1 \cup \gamma_2} \int_0^\delta (w_{ts}^*)^2\right)^{1/2} \leqq c\|F\|_C \|w\|_{W^{2,2}}^2 .$$

According to (5.3) $F^*(t, 0) = F_t^*(t, 0) = 0$ on γ_3. Hence we have $|F^*(t, s)/s| \leqq \leqq \|F\|_{C^1}$ for $s \in (0, \delta\rangle$ and

$$(6.17) \qquad J_4 \leqq \left(\int_{\gamma_3} \int_0^\delta (w_s^*)^2\right)^{1/2} \left(\int_{\gamma_3} \int_0^\delta (w_t^*)^2\right)^{1/2} \|F\|_{C^1} \leqq c\|F\|_{C^1} \|w\|_{W^{2,2}}^2 .$$

From (6.15), (6.16) and (6.17) we obtain

$$(6.18) \qquad |J_2| \leqq c\varepsilon \|F\|_{C^1} \|w\|_{W^{2,2}}^2 .$$

Finally we estimate the two remaining terms of (6.9) which belong neither to the 1st nor to the 2nd group. The integral

$$(6.19) \qquad \int_0^R \int_0^\delta w_{tt}^* (\zeta^* F^*)_s \, w_s^* \left[\frac{\partial(x, y)}{\partial(t, s)}\right]^{-1} \mathrm{d}s \, \mathrm{d}t$$

can be transformed – by integrating by parts in t – to

$$(6.20) \qquad - \int_0^R \int_0^\delta w_t^* \left[(\zeta^* F^*)_s \, w_s^* \left(\frac{\partial(x, y)}{\partial(t, s)}\right)^{-1}\right]_t \mathrm{d}s \, \mathrm{d}t .$$

Thanks to the independence of function ζ of t the integrals obtained after the differentiation of the expression in squared brackets contain either the factor $w_t^*(\zeta^* F_t^*)_s$ or $w_t^*(\zeta^* F^*)_s$. In both cases it can be estimated as J_2 (see (6.18)) because $w_t^*(t, 0) = 0$ on $\gamma_1 \cup \gamma_2$ and (5.3) implies $F_t^*(t, 0) = 0$ on γ_3.

The last term

$$(6.21) \qquad \int_0^R \int_0^\delta w_s^* (\zeta^* F^*)_s \, w_s^* \, b^{xy}(s; s, s) \frac{\partial(x, y)}{\partial(t, s)} \mathrm{d}s \, \mathrm{d}t$$

splits into two integrals. The first one with $\zeta^* F^*$ can be estimated as (6.11). The second one can be written in the form

$$(6.22) \qquad J_5 = \int_0^R \int_0^\delta \left| w_s^* \zeta_s^* F^* w_s^* \, b^{xy}(s; s, s) \frac{\partial(x, y)}{\partial(t, s)} \right| \mathrm{d}s \, \mathrm{d}t \leqq$$

$$\leqq c\varepsilon \left(\int_{\gamma_1} \int_0^\delta |w_s^*| \, |F^*| \left| \frac{w_s^*}{s} \right| \mathrm{d}s \, \mathrm{d}t + \right.$$

$$\left. + \int_{\gamma_2} \int_0^\delta |w_s^*|^2 \, |F^*| \left| \frac{b^{xy}(s; s, s)}{s} \right| \mathrm{d}s \, \mathrm{d}t + \int_{\gamma_3} \int_0^\delta (w_s^*)^2 \, \frac{|F^*|}{s} \, \mathrm{d}s \, \mathrm{d}t \right) =$$

$$= c\varepsilon(J_6 + J_7 + J_8) \, .$$

To estimate J_6 we apply Proposition 1 to w_s^*, knowing that $w_s^*(t, 0) = 0$ on γ_1 In J_7, using (5.4) we can estimate $|b^{xy}(s; s, s)|/s$ by a constant on γ_2. For J_8 we us again $|F^*|/s \leqq \|F\|_{C^1}$.

Putting all this into (6.9) we get the inequality

$$|B(w; \zeta F, w)| \leqq (\tilde{c}\varepsilon + \tilde{\tilde{c}}\delta^{1/4}) \, \|F\|_{C^2} \, \|w\|_{W^{2,2}} \tag{6.23}$$

with constants $\tilde{c}$, $\tilde{\tilde{c}}$ depending only on Ω. Hence, choosing adequately the numbers ε and δ and taking for ζ the corresponding function from Lemma 2 we obtain finally (6.2) which completes the proof.

Lemma 6. *Let the conditions* (5.3), (5.4) *be satisfied. Let* ζ *be the function from lemma* 5. *Then* $\mathscr{T}_\zeta$ *satisfies* (1.21) (*S-condition*)

Proof. For each $U^n = /w^n, g^n/$, $U = /w, g/ \in H$,

$$Q_n = \mathscr{T}_\zeta(U^n) \, [U^n - U] - \mathscr{T}_\zeta(U) \, [U^n - U] = \tag{6.24}$$

$$= \|U_n - U\|_H^2 - B(w^n; g^n, w^n - w) + B(w; g, w^n - w) +$$

$$+ B(w^n; w^n, g^n - g) - B(w; w, g^n - g) - B(w^n - w; \zeta F, w^n - w) \, .$$

Let ζ be a function for which (6.2) holds. Its existence was established in proof of Lemma 5. In this case we have

$$\begin{aligned} & Q_n + [B(w^n; g^n, w^n - w) - B(w; g, w^n - w) - \\ & - B(w^n; w^n, g^n - g) + B(w; w, g^n - g)] \geqq \tfrac{3}{4} \, \|U^n - U\|_H^2 \, . \end{aligned} \tag{6.25}$$

Let now $U^n \rightharpoonup U$ in H and $Q^n \to 0$. If we show that the weak convergence implies the convergence of the expression in squared brackets to zero then (6.25) implies $U^n \to U$ in H and (1.21) is valid.

However, $U^n \rightharpoonup U$ in H implies $w^n \rightharpoonup w$ in $W^{2,2}(\Omega)$ and both $\{g^n\}$ and $\{w^n\}$ are bounded in $W^{2,2}(\Omega)$. Because of the compactness of the immersion

$$E : W^{2,2}(\Omega) \to W^{1,4}(\Omega)$$

there exists a subsequence $\{w^{n_k}\}$, $w^{n_k} \to u$ in $W^{1,4}(\Omega)$. It is obvious that $u = w$. Hence $w^n \to w$ in $W^{1,4}(\Omega)$. Using the estimate

$$|B(w^n; g^n, w^n - w)| \leqq c\|w^n\|_{W^{2,2}} \|g^n\|_{W^{2,2}} \|w^n - w\|_{W^{1,4}}$$

we obtain finally $\lim_{n\to\infty} B(w^n; g^n, w^n - w) = 0$.

Similarly we prove that all the other terms in squared brackets in (6.25) tend to zero.

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Souhrn

ŘEŠITELNOST VON KÁRMÁNOVYCH ROVNIC

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V článku je zkoumána existence variačního řešení obecného okrajového problému pro von Kármánovu soustavu nelineárních rovnic. Úloha je převedena na otázku řešitelnosti jisté operátorové rovnice. Dokazuje se, že operátor je koercitivní a splňuje některé další podmínky, které dohromady zaručují existenci řešení.

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