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ON THE SPACE OF RIEMANNIAN METRICS SATISFYING SURGERY STABLE CURVATURE CONDITIONS

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Abstract. We utilize a condition for algebraic curvature operators called *surgery stability* as suggested by the work of S. Hoelzel to investigate the space of riemannian metrics over closed manifolds satisfying these conditions. Our main result is a parametrized Gromov-Lawson construction with not necessarily trivial normal bundles and shows that the homotopy type of this space of metrics is invariant under surgeries of a suitable codimension. This is a generalization of a well-known theorem by V. Chernysh and M. Walsh for metrics of positive scalar curvature.

As an application of our method, we show that the space of metrics of positive scalar curvature on quaternionic projective spaces are homotopy equivalent to that on spheres.

§1. Introduction

In this work we aim to give a sufficient criterion for a curvature condition such that the space of such metrics is weakly homotopy equivalent to a subspace of prescribed form around a fixed submanifold and derive various conclusions.

A curvature condition is an open subset of the space of $\mathrm{O}(n)$-invariant operators $\bigwedge^2 \mathbb{E}^n \to \bigwedge^2 \mathbb{E}^n$ satisfying the Bianchi identity, where $\mathbb{E}^n$ denotes the euclidean space. We will define two notions that we call *deformable* and *codimension c surgery stable*, which allow to extend the Gromov-Lawson construction for curvature conditions developed by S. Hoelzel in [Hoe16] and will be explained in greater detail in §§ 2.2 and 2.3.

Let M^n be a closed, smooth manifold of dimension n and let N^k be a compact, k-dimensional submanifold in M. We denote by $\mathcal{R}_C(M)$ the space of riemannian metrics that satisfy the condition C, i.e. for any $p \in M$ and any linear isometry $i\colon \mathbb{E}^n \to (\mathrm{T}_p M, g_p)$ we have $i^* R_p \in C$, where R_p is the riemannian curvature operator of the riemannian metric g at p. Given a tubular map $\phi\colon \nu N \to M$ and a distinct metric on νN, we will define a space of metrics $R_C^{\mathrm{torp}}(M)$, which are *standard near* N.

Theorem A (parametrized Gromov-Lawson construction). *Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a deformable, codimension c surgery stable curvature condition with $n - k \geq c$. Then the inclusion of metrics, which are standard near N*

$$\mathcal{R}_C^{\mathrm{torp}}(M) \hookrightarrow \mathcal{R}_C(M)$$

is a weak homotopy equivalence.

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The condition positive scalar curvature, i.e. $C = (\mathrm{psc})$, is deformable, codimension 3 surgery stable and thus Theorem A generalizes a result of V. Chernysh [Che04] and M. Walsh [Wal13] for submanifolds with trivial normal bundle, which has recently been revisited by J. Ebert and G. Frenck [EF18]. Note that the normal bundle of N does not necessarily have to be trivial in Theorem A. We remark that the Gromov-Lawson constructions given in [SY79] and [Hoe16] also do not require a trivial normal bundle.

In the case of surgeries along embedded spheres with trivial normal bundles we immediately obtain from Theorem A:

Theorem B. *Let M_0 be a closed n-manifold and let M_1 be obtained from M_0 by surgery of codimension $n-k \geq c$ with $k \geq c-1$. Then $\mathcal{R}_C(M_0)$ is homotopy equivalent to $\mathcal{R}_C(M_1)$.*

Theorem A can be used to consider more general gluing constructions, namely surgeries along embedded spheres with nontrivial normal bundles. In particular, we can cut along the sphere bundle of the Hopf fibration $S^{4k+3} \to \mathbb{H}P^k$ within $\mathbb{H}P^{k+1}$ to obtain $S^{4(k+1)}$ from a generalized surgery of codimension 4.

Theorem C. *Let $k \geq 1$. Then*

$$\mathcal{R}_{\mathrm{psc}}(\mathbb{H}P^k) \simeq \mathcal{R}_{\mathrm{psc}}(S^{4k}) \quad \text{and} \quad \mathcal{R}_{1\text{-}curv>0}(\mathbb{H}P^k) \simeq \mathcal{R}_{1\text{-}curv>0}(S^{4k}).$$

By an application of the original Gromov-Lawson construction ([GL80] and independently [SY79]) we know that a closed, simply-connected manifold of dimension $n \geq 5$, which is oriented bordant (resp. spin bordant, if M is spin) to a manifold with positive scalar curvature admits a metric of positive scalar curvature itself. The proof of this result relies on the fact that one can arrange the bordism to be composed from traces of codimension ≥ 3 surgeries, which allow to push through positive scalar curvature. It is well-known that this procedure can be generalized to other tangential structures. Denoting by $\mathrm{B\,O}\langle l \rangle$ the lth stage in the Whitehead tower of $\mathrm{B\,O}$, a closed manifold M^n is called $\mathrm{B\,O}\langle l \rangle$*-manifold*, if the stable normal bundle $M \to \mathrm{B\,O}$ lifts to $M \to \mathrm{B\,O}\langle l \rangle$ through the tower. The corresponding bordism group will be denoted by $\Omega_n^{\langle l \rangle}$.

Proposition. * *Let M_0, M_1 be closed $\mathrm{B\,O}\langle l \rangle$-bordant n-manifolds and let $r \geq 1$ be such that $n \geq 2r+3$ and $l \geq r+2$. If M_1 is r-connected, then M_1 can be obtained from M_0 by surgeries of codimension at least $r+2$.*

Combined with the Gromov-Lawson construction for curvature conditions [Hoe16, Theorem A] we obtain:

Theorem D. *Let (M_0, g) be a closed riemannian manifold satisfying a codimension c surgery stable curvature condition, which is $\mathrm{B\,O}\langle l \rangle$-bordant to an r-connected, closed manifold M_1. If $n \geq 2r+3$, $l \geq r+2$ and $r+2 \geq c$, then M_1 admits a riemannian metric satisfying the same curvature condition as M_0.*

Moreover, using Theorem A, we can strengthen this statement for highly-connected manifolds.

Theorem E. *Let C be a deformable, codimension c surgery stable curvature condition. Let M_0, M_1 be $\mathrm{B\,O}\langle l \rangle$-bordant, r-connected, closed manifolds with $n \geq 2r+3$, $l \geq r+2$ and $r+2 \geq c$. Then $\mathcal{R}_C(M_0)$ is homotopy equivalent to $\mathcal{R}_C(M_1)$.*

*e.g. [BL14, Proposition 3.4]

Let us state the result for the following examplary case. A riemannian manifold (M, g) is is said to have *positive* 1*-curvature*[†], if $\operatorname{scal}_g - 2\operatorname{Ric}_g(v) > 0$ for all points $p \in M$ and unit vectors $v \in \mathrm{T}_p M$. It is not hard to see that every metric of positive 1-curvature has positive scalar curvature. We will see later that positive 1-curvature defines a codimension 4 surgery stable curvature condition $C \coloneqq (\text{1-curv} > 0)$. In particular, we can consider the space of metrics with positive 1-curvature $\mathcal{R}_{\text{1-curv}>0}(M)$ over a fixed manifold M. Noting that $\mathrm{B\,O}\langle 4\rangle = \mathrm{B\,Spin}$ and the spin bordism group in dimension 7 is trivial, we conclude the following.

Corollary F. *For every* 2*-connected,* 7*-dimensional, closed spin manifold* M *the space of riemannian metrics with positive* 1*-curvature* $\mathcal{R}_{\text{1-curv}>0}(M)$ *has the homotopy type of* $\mathcal{R}_{\text{1-curv}>0}(S^7)$*.*

This is quite a large class of manifolds (cf. the classification by topological invariants [CN18]), but in particular it holds for all exotic seven spheres. Typical interesting examples also include S^3-bundles over S^4, which contain a family of infinitely many mutually distinct homotopy types admitting metrics of non-negative sectional and positive Ricci curvature (cf. [GZ00, Proposition 3.3]).

It is a well-known consequence of the Atiyah-Singer index theorem that a homotopy sphere is spin-nullbordant if its α-invariant vanishes (cf. [LM89, p.144]).

Corollary G. *Let* Σ^n *be a homotopy sphere with* $\alpha(\Sigma^n) = 0$ *(e.g. in the case* $n \not\equiv 1, 2 \mod 8$*) and with* $n \geq 7$*. Then the space of riemannian metrics with positive* 1*-curvature* $\mathcal{R}_{\text{1-curv}>0}(\Sigma)$ *has the homotopy type of* $\mathcal{R}_{\text{1-curv}>0}(S^n)$*.*

The existence question for metrics of positive 1-curvature on 2-connected manifolds was already addressed in [Lab97b, Theorem I] and the homotopy equivalence of $\mathcal{R}_{\text{psc}}(\Sigma^n) \simeq \mathcal{R}_{\text{psc}}(S^n)$ is a corollary to the version of Theorem D for positive scalar curvature (psc) by Chernysh and Walsh (cf. [Wal13, Corollary 4.2]). Note however that the inclusion $\mathcal{R}_{\text{1-curv}>0}(M) \hookrightarrow \mathcal{R}_{\text{psc}}(M)$ is not understood.

The above corollaries hold in analogy for arbitrary curvature conditions, which are stable under surgery of codimension at least 4.

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[†]This condition is sometimes also called *positive Einstein* or the metric is said to have *positive Einstein tensor*.We will refrain from these terms to avoid confusion with *Einstein* metrics with positive Einstein constant.

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§2. Preliminaries

§2.1 Curvature conditions and spaces of riemannian metrics. We will briefly recall the definition of algebraic curvature operators and notions of curvature. These let us define what we understand under a curvature condition satisfied by a riemannian metric.

Definition 2.1. Let (M, g) be a riemannian manifold with riemann curvature tensor $R\colon \mathcal{V}(M) \times \mathcal{V}(M) \to \mathrm{End}(\mathcal{V}(M))$, where $\mathcal{V}(M)$ denotes the space of smooth vector fields on M. One obtains what is called *curvature operator* $\overline{R}\colon \bigwedge^2 \mathcal{V}(M) \to \bigwedge^2 \mathcal{V}(M)$ of M via the relation

$$g(\overline{R}(X \wedge Y), Z \wedge W) = g(R(X, Y)W, Z),$$

where g on the left hand side is the extension of g to $\Gamma(M, \bigwedge^2 \mathrm{T}\, M)$.
At every point $p \in M$ the curvature operator $\overline{R}$ defines a self-adjoint endomorphism

$$\overline{R}_p\colon \bigwedge^2 \mathrm{T}_p\, M \to \bigwedge^2 \mathrm{T}_p\, M.$$

We denote by $\mathbb{E}^n$ the euclidean space $\mathbb{R}^n$ endowed with the standard inner product $\langle \cdot, \cdot \rangle$. A self-adjoint endomorphism $R\colon \bigwedge^2 \mathbb{E}^n \to \bigwedge^2 \mathbb{E}^n$ is called *algebraic curvature operator*. From the inner product, we obtain the isomorphism

$$\eta\colon \bigwedge^2 \mathbb{E}^n \to \mathfrak{so}(n) \subset \mathrm{End}(\mathbb{R}^n), \quad x \wedge y \mapsto -\langle x, \cdot \rangle\, y + \langle y, \cdot \rangle\, x$$

and thus for every $x, y \in \mathbb{R}^n$, an algebraic curvature operator gives rise to a skew-symmetric endomorphism

$$R(x, y)\colon \mathbb{R}^n \to \mathbb{R}^n, \quad z \mapsto (\eta \circ R(x \wedge y))(z).$$

An algebraic curvature operator R is said to satisfy the *Bianchi indentity*, if

$$R(x, y)z + R(y, z)x + R(z, x)y = 0$$

for all $x, y, z \in \mathbb{R}^n$.
Exactly in the same manner as we define sectional, Ricci and scalar curvature in a tangent space, we let

(1) $\sec(R, E) \coloneqq \langle R(x, y)y, x\rangle$ for an orthonormal basis $\{x, y\}$ of a 2-plane $E \leq \mathbb{E}^n$,
(2) $\mathrm{Ric}(R, z) \coloneqq \sum_{i=1}^n \langle R(e_i, z)z, e_i\rangle$ for $z \in \mathbb{E}^n$ with $\|z\| = 1$,
(3) $\mathrm{scal}(R) \coloneqq \sum_{i,j=1}^n \langle R(e_i, e_j)e_j, e_i\rangle = 2\,\mathrm{tr}\, R$,

where $\{e_i\}$ is the standard basis of $\mathbb{E}^n$.‡

‡As common notation suggests, we will write relations such as $\sec(R) < \alpha$ to mean "$\sec(R, E) < \alpha$ for every plane $E < \mathbb{E}^n$".

Let (M, g) be a riemannian manifold and let $i\colon \mathbb{E}^n \to (\mathrm{T}_p M, g_p)$ be a linear isometry into the tangent space at some point p in M. This defines an algebraic curvature operator

$$i^*\overline{R}_p := (i^{\wedge 2})^{-1} \circ R_p \circ i^{\wedge 2}\colon \bigwedge^2 \mathbb{E}^n \to \bigwedge^2 \mathbb{E}^n$$

for $i^{\wedge 2}\colon \bigwedge^2 \mathbb{E}^n \to \bigwedge^2 \mathrm{T}_p M$ and $(i^{\wedge 2})^{-1}\colon \bigwedge^2 \mathrm{T}_p M \to \bigwedge^2 \mathbb{E}^n$ induced by i and its inverse. Clearly, $i^*\overline{R}_p$ satisfies the Bianchi identity and we have $\sec(i^*\overline{R}_p, E) = \sec^g_p(i(E))$, $\mathrm{Ric}(i^*R_p, z) = \mathrm{Ric}^g_p(i(z))$ and $\mathrm{scal}(i^*R_p) = \mathrm{scal}^g(p)$. Hence, we can use an algebraic curvature operator, which satisfies the Bianchi identity, to describe curvature properties of g at p up to the choice of an orthonormal basis in $\mathrm{T}_p M$.

Let $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ denote the vector space of algebraic curvature operators satisfying the Bianchi identity. Then $\mathrm{O}(n)$ acts on $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ by change of the orthonormal basis in $\mathbb{E}^n$, that is via

$$(A, R) \mapsto (A \wedge A)^{-1} \circ R \circ (A \wedge A).$$

Definition 2.2. A *curvature condition* is an open subset C of $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$, which is invariant under the action of $\mathrm{O}(n)$ on $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$. We say that a riemannian metric g on M *satisfies* C, if $i^*\overline{R}_p \in C$ for all linear isometries $i\colon \mathbb{E}^n \to (\mathrm{T}_p M, g_p)$ and $p \in M$, where $\overline{R}_p$ denotes the curvature operator with respect to g in p.

Thus, any curvature condition is a global statement on the structure of a riemannian manifold and obviously describes an isometry invariant property of a riemannian metric satisfying it.

Example 2.3. (i) Lower curvature bounds can be expressed as a curvature condition. For example, we can express (globally pointwise) positive sectional curvature as condition

$$(\sec > 0) := \{R \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \mid \sec(R) > 0\},$$

while positive scalar curvature is described by

$$\mathrm{psc} := (\mathrm{scal} > 0) := \{R \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \mid \mathrm{scal}(R) > 0\}.$$

We note that both sets are open cones in $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ with $(\sec > 0) \subsetneq \mathrm{psc}$ for $n \geq 3$, while $(\sec > 0) = \mathrm{psc}$ for $n = 2$.
We denote by $\mathbb{S}^n$ the n-sphere endowed with the round metric of radius 1. For any isometry $i\colon \mathbb{E}^n \to \mathrm{T}_p \mathbb{S}^n$, we have $i^*\overline{R}_p = \mathrm{id}_{\bigwedge^2 \mathbb{E}^n}$. In particular $\mathbb{S}^n$ satisfies $(\sec > 0)$.

(ii) Upper curvature bounds, such as

$$(\mathrm{scal} < \alpha) := \{R \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \mid \mathrm{scal}(R) < \alpha\}$$

for $\alpha \in \mathbb{R}$ are of course curvature conditions, although they will not play an important role in the following.

Definition 2.4. Let M^n be a smooth manifold and denote by $\mathcal{R}(M)$ the space of complete riemannian metrics on M equipped with the compact-open C^∞-topology. Define, the *space of riemannian metrics satisfying* C as

$$\mathcal{R}_C(M) := \{g \in \mathcal{R}(M) \mid g \text{ satisfies } C\} \subset \mathcal{R}(M).$$

As curvature conditions are preserved under isometries, $\mathrm{Diff}(M)$ acts on $\mathcal{R}_C(M)$ by pullback of riemannian metrics, i.e. via

$$\mathrm{Diff}(M) \times \mathcal{R}_C(M) \to \mathcal{R}_C(M), \quad (\psi, g) \mapsto \psi^* g.$$

The quotient $\mathcal{M}_C(M) := \mathcal{R}_C(M)/\mathrm{Diff}(M)$ under this action is called *moduli space of riemannian metrics satisfying* C.

Remark. These spaces are actually quite classical objects of interest in global riemannian geometry (cf. [Gro91, § 7³/4]). We refer to the survey [TW15] for further details and references regarding spaces of riemannian metrics.

Recently, there has been much progress showing that $\mathcal{R}_C(M)$ on certain closed manifolds is topologically non-trivial for $C = \mathrm{psc}$ and $C = (\mathrm{Ric} > 0)$ (cf. [CS13], [HSS14], [BERW17] and [Wra11]). For positive scalar curvature non-trivial homotopy elements can be exhibited factoring the Atiyah-Bott-Shapiro map of spectra $\mathcal{A}\colon \mathrm{M\,Spin} \to \mathrm{KO}$ through the space of metrics with positive scalar curvature carried out in [BERW17], which is a vast generalization of an idea by N. Hitchin. Incidentally, already Hitchin's original construction (cf. [Hit74, Section 4.4]) identifies non-trivial homotopy elements in $\mathcal{R}_C(M)$ on any spin manifold M of a certain dimension for every $C \subset \mathrm{psc}$ with $\mathcal{R}_C(M) \neq \varnothing$ that are produced using the orbit map of the above described $\mathrm{Diff}(M)$ action on $\mathcal{R}_C(M)$.

Convention. *By a* (compact) family of metrics *on a manifold M, we always understand the image of a continuous map $g\colon S \to \mathcal{R}(M)$ from a compact topological space S and denote it by $\{g_\xi\}_{\xi\in S}$.*

§2.2 Surgery stable curvature conditions.

Definition 2.5. Let M be a smooth manifold of dimension n and further let $\phi \in \mathrm{Emb}(S^k, M)$ be an embedding with trivial normal bundle. Thus ϕ extends to a tubular embedding $\phi \in \mathrm{Emb}(S^k \times D^{n-k}, M)$. Recall that a *surgery* on M along ϕ is a procedure, which endows the push-out

$$\begin{array}{ccc} S^k \times D^{n-k}(\tfrac{1}{2},1) & \xrightarrow{\;\;i\;\;} & D^{k+1}(\tfrac{1}{2},1) \times S^{n-k-1} \\ \Big\downarrow{\scriptstyle \phi(\frac{1}{2},1)} & & \Big\downarrow \\ M \setminus \mathrm{Im}(S^k \times \mathring{D}^{n-k}(\tfrac{1}{2})) & \longrightarrow & \chi(M,\phi) \end{array}$$

where $\phi(\tfrac{1}{2},1) := \phi|_{S^k\times D^{n-k}(\frac{1}{2},1)}$, $i\colon (\theta_1, \lambda\theta_2) \mapsto ((1-\lambda)\theta_1, \theta_2)$ and where we denote $D^k(a,b) := \{x \in D^k \mid a \le |x| \le b\}$, with a differentiable structure compatible with the given differentiable structures on the individual parts. The number $n-k$ is called *codimension of the surgery.*

Clearly, one can also equip M with a riemannian metric and it is not hard to see that there exists a metric on $\chi(M,\phi)$, which coincides with the original metric away from the embedding ϕ. But since cutting and gluing of riemannian metrics along arbitrary submanifolds requires to smoothen the metric, it is not clear if this process keeps the curvature controlled. It was only realized by Gromov and Lawson [GL80] and independently by R. Schoen and S. T. Yau [SY79] that positive scalar curvature can be preserved under surgeries of codimension greater or equal three.

This theorem has been generalized in several directions and in particular to curvature conditions as defined above by Hoelzel.

Definition 2.6. A curvature condition $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ is said to *satisfy an inner cone condition with respect to* $S \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \setminus \{0\}$, if there exists a continuous function $\rho\colon C \to (0,\infty)$ and for every $\rho = \rho(R)$ an open, convex $\mathrm{O}(n)$-invariant cone C_ρ containing $B_\rho(S)$ such that

$$R + C_\rho = \{R + E \mid E \in C_\rho\} \subset C.$$

Remark 2.7. (i) If $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ is a curvature condition given by an open, convex cone and $S \in C$, then C satisfies an inner cone condition with respect to S.

(ii) If $C \neq \varnothing$ satisfies an inner cone condition with respect to $S \neq 0$, then there exists a $\lambda_0 > 0$ such that $\lambda S \in C$ for all $\lambda > \lambda_0$.
The argument goes as follows. Let $R \in C$ be arbitrary and conclude from the inner cone condition that $R + C_\rho \in C$ for some open, convex $\mathrm{O}(n)$-invariant cone with $B_\rho(S) \subset C_\rho$. Clearly, this implies $B_{\mu\rho}(R + \mu S) \subset C$ and the cone $\bigcup_{\mu>0} B_{\mu\rho}(R + \mu S)$ intersects the line λS for some μ and λ large enough.

Remark 2.8. Since the curvature operator of the standard sphere $S^q(1)$ of radius 1, which we also denote by $\mathbb{S}^q$, is given by the identity, the curvature operator of $\mathbb{E}^{n-q} \times \mathbb{S}^q$ with the canonical product metric is precisely the projection map $R_{\mathbb{E}^{n-q} \times \mathbb{S}^q} \coloneqq \pi_{\bigwedge^2 \mathbb{E}^q} \colon \bigwedge^2 \mathbb{E}^n \to \bigwedge^2 \mathbb{E}^n$ induced by the projection $\mathbb{E}^n = \mathbb{E}^{n-q} \times \mathbb{E}^q \to \mathbb{E}^q$ on the last q coordinates.

Definition 2.9. Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a curvature condition satisfying an inner cone condition with respect to $R_{\mathbb{E}^{n-c+1} \times \mathbb{S}^{c-1}}$ for some $c \in \{3, \dots, n\}$. Then C is said to *admit codimension c surgeries*.

Proposition 2.10 ([Hoe16, Proposition 2.2])**.** *If $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ is a curvature condition satisfying an inner cone condition with respect to $R_{\mathbb{E}^{n-c+1} \times \mathbb{S}^{c-1}}$ for $3 \leq c \leq n$, then C satisfies an inner cone condition with respect to $R_{\mathbb{E}^{n-c} \times \mathbb{S}^c}$.*

Corollary 2.11. *Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a curvature condition admitting codimension c surgeries for some $c \in \{3, \dots, n-1\}$. Then C admits codimension $c+1$ surgeries.*

Corollary 2.12. *If C admits codimension c surgeries, then C satisfies an inner cone condition with respect to $R_{\mathbb{E}^{n-q} \times \mathbb{S}^q}$ for all $c - 1 \leq q \leq n$ and by Remark 2.7 (ii) there exists a $\lambda_0 > 0$ such that $\lambda R_{\mathbb{E}^{n-q} \times \mathbb{S}^q} \in C$ for all $\lambda > \lambda_0$. In particular, $S^n(\frac{1}{\sqrt{\lambda}})$ with the round metric satisfies C.*

As the name suggests, admittance of surgery is precisely the assumption needed to obtain a new metric again satisfying this curvature condition on a surgery result.

Theorem 2.13 ([Hoe16, Theorem A])**.** *Let C be a curvature condition admitting codimension c surgeries and let (M, g) be a riemannian manifold with g satisfying C. If $\chi(M, \phi)$ is obtained from M by surgery of codimension $\geq c$, then $\chi(M, \phi)$ admits a metric satisfying C.*

Thus we are only interested in the smallest value for which C admits surgeries of such codimension leading us to the following definition.

Definition 2.14. A curvature condition C admitting codimension $\tilde{c}$ surgeries is said to be *codimension c surgery stable*, if c is minimal among all $\tilde{c}$.

Example 2.15. (i) By Remark 2.7 (i), a curvature condition C given by an open, convex cone containing $R_{\mathbb{E}^{n-q+1} \times \mathbb{S}^{q-1}}$ for all $q \geq c$ (i.e. $\mathbb{E}^{n-q+1} \times \mathbb{S}^{q-1}$ with the standard product metric satisfies C) admits codimension c surgeries.
(ii) Positive scalar curvature $C = \mathrm{psc}$ is codimension 3 surgery stable in this sense. Clearly, c is minimal in the allowed range $\{3, \dots n\}$ for c, but this conceptually makes sense as the standard metric on $\mathbb{S}^{2-1} \times \mathbb{E}^{n-2+1} = \mathbb{S}^1 \times \mathbb{E}^{n-1}$ is flat, i.e. does not have positive scalar curvature.

Example 2.16. Another interesting curvature condition is *positive p-curvature*, which interpolates between positive scalar (for $p = 0$) and positive sectional curvature (for $p = n - 2$). It has been proposed by Gromov and was studied extensively by M.-L. Labbi (cf. [Lab95, Lab97b, Lab97a, Lab06]).

It can be defined for $0 \leq p \leq n - 2$ as an open convex cone

$$(p\text{-curv} > 0) \coloneqq \{R \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \mid s_p(R)(P) > 0 \quad \forall P \leq \mathbb{R}^n \ \text{ with } \dim P = p\},$$

where $s_p(R)\colon G_p(\mathbb{R}^n) \to \mathbb{R}$ is the map $P \mapsto \sum_{i,j=1}^{n-p} \sec(R)(E_i, E_j)$ for an orthonormal basis $\{E_i\}$ of $P^\perp$ and $G_p(\mathbb{R}^n)$ is the real p-Grassmannian.

By definition, positive p-curvature implies positive $(p-1)$-curvature and for every fixed dimension n there is a sequence of cones

$$(\text{psc}) = (0\text{-curv}) \supset (1\text{-curv}) \supset \cdots \supset ((n-3)\text{-curv}) \supset ((n-2)\text{-curv}) = (\sec > 0).$$

Labbi showed in [Lab97b] that positive p-curvature is preserved under codimension $p+3$ surgeries. This is recovered by Hoelzel's theorem, as the condition is codimension $p+3$ surgery stable.

Riemannian metrics satisfying this condition for values of p, which are small relative to n, exist in abundance. The product of every compact manifold with a $(p+3)$-sphere admits a metric of positive p-curvature. More specifically, every compact, connected Lie group with a bi-invariant metric (which is not the torus), every Einstein manifold with positive Einstein constant, as well as every Kähler manifold with positive Ricci curvature has positive 1-curvature. On the other hand, there exist examples, which have positive 1-curvature, while they do not admit any metric of positive Ricci curvature. Botvinnik and Labbi in [BL14] investigate obstructions to positive p-curvature for $p = 2, 3$. For example, they show that a 3-connected non-string manifold M of dimension at least 9 admit a metric with positive 2-curvature if and only if Hitchin's KO-theoretic α-invariant vanishes for M. Moreover, they find dimensions in which every 3-connected string manifold admits a metric with positive 2-curvature. Note however, that the situation is less clear in lower connectivity.

Example 2.17. Similarly, one can consider a curvature condition, which interpolates between positive scalar (for $k = n$) and positive Ricci curvature (for $k = 1$) called *k-positive Ricci curvature*. It was introduced by J. Wolfson in [Wol09] and can be defined for $1 \leq k \leq n$ as an open convex cone

$$(k\text{-pos Ric}) \coloneqq \{R \in \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n) \mid \sum_{i=1}^{k} \operatorname{Ric}(R; e_i) > 0 \quad \forall \{e_1, \cdots, e_k\} \text{ orthonormal}\}.$$

As in the previous examples, there are successive inclusions $(k\text{-pos Ric}) \subset ((k+1)\text{-pos Ric})$ and in particular every metric with k-positive Ricci curvature for some k has positive scalar curvature. Moreover, Wolfson showed that k-positive Ricci curvature for $2 \leq k \leq n-1$ is preserved under codimension $n-k+2$ surgeries. Surprisingly, n-positive Ricci curvature (positive scalar curvature), as well as $(n-1)$-positive Ricci curvature are both stable under surgeries of codimension 3. It is an open question of Wolfson, if there exists a riemannian manifold of positive scalar curvature, which does not admit a metric of $(n-1)$-positive Ricci curvature.

Both of the curvature conditions decribed in the above examples can be regarded as *intermediate curvature notions*, by which one might hope to understand the differences between both extremes in greater detail.

§2.3 Torpedo metrics. It is well-known that one can use warped products to describe rotationally symmetric metrics (cf. [Pet16, p.18ff]). To do so consider a smooth function $\beta\colon [0,\infty) \to [0,\infty)$ and endow $(0,\infty) \times S^{q-1}$ with the metric $g^\beta \coloneqq \mathrm{d}r^2 + \beta^2(r) g_{\mathbb{S}^{q-1}}$, where $g_{\mathbb{S}^{q-1}}$ is the round metric on the $q-1$ sphere. If we assume that

(i) $\beta(0) = 0$, $\beta'(0) = 1$, $\beta^{(2l)}(0) = 0$ for $l \in \mathbb{N}$ and
(ii) $\beta|_{(0,\infty)} > 0$,

the metric g^β uniquely extends to a smooth rotationally symmetric metric on $\mathbb{R}^q$. The pull-back of the curvature operator of g^β is given by (cf. [Pet16, p.121])

$$R^{\beta,q}(\theta,r) = \frac{1-\beta'(r)^2}{\beta(r)^2} R_{\mathbb{E}\times\mathbb{S}^{q-1}} - \frac{\beta''(r)}{\beta(r)} L_q, \tag{2.18}$$

where $L_q(e_i \wedge e_j) = \begin{cases} e_i \wedge e_j & \text{if } i=1, j>1 \\ 0 & \text{otherwise} \end{cases}$.

If the dimension n is fixed, we will abuse notation denoting by $R^{\beta,q}$ and L_q the curvature operators of $\mathbb{R}^n$ which are zero on combinations of the first $n-q$ coordinate directions.

Definition 2.19. Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a codimension c surgery stable curvature condition. It is called *deformable*, if

(1) $0 \notin C$,
(2) it satisfies an inner ray condition with respect to L_q for all $q \geq c$, i.e. $R \in C$ implies $R + \lambda L_q \in C$ for all $\lambda \geq 0$,
(3) and $\mu R_{\mathbb{E}^{n-q+1}\times\mathbb{S}^{q-1}} \in C$ for all $\mu > 0$ and $q \geq c$.

Example 2.20. (i) By Remark 2.7 (i), a curvature condition C given by an open convex cone with $R_{\mathbb{E}^{n-c+1}\times\mathbb{S}^{c-1}} \in C$ and $L_q \in C$ for all $n \geq q \geq c$ is deformable. This is the case for positive scalar curvature.
(ii) Positive p-curvature for $0 \leq p < n-2$ is deformable (even though it does not satisfy an inner cone condition with respect to L_q for $1 \leq p \leq n-2$).
(iii) The condition k-positive Ricci curvature for $2 \leq k \leq n$ is deformable.

Remark 2.21. Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a deformable, codimension c surgery stable curvature condition. Then C satisfies an inner cone condition with respect to every $R_{\mathbb{E}^{n-q+1}\times\mathbb{S}^{q-1}} + \lambda L_q$ for all $q \geq c$ and $\lambda \geq 0$.

Proposition 2.22. *Suppose C is a deformable, codimension c surgery stable curvature condition. Then $R^{\beta,q} \in C$ for all $q \geq c$, if in $(0,\infty)$*

$$1-\beta'^2 > 0, \quad \beta'' \leq 0 \quad \textit{and} \quad \beta'''(0) < 0. \tag{2.23}$$

Proof. From (2.18) we conclude that, if

$$\mu := \frac{1-\beta'^2}{\beta^2} > 0 \quad \text{and} \quad \lambda := -\frac{\beta''}{\beta} \geq 0$$

then by definition of deformability $R^{\beta,q} = \mu R_{\mathbb{E}^{n-q}\times\mathbb{S}^{n-1}} + \lambda L_q \in C$. By L'Hôpital, we have $\lim_{r\to 0}\mu = \lim_{r\to 0}\lambda = -\beta'''(0) > 0$ and thus $\lim_{r\to 0} R^{\beta,q} \in C$. □

Definition 2.24. One function of particular interest satisfying these assumptions is given by a smoothing of

$$\beta_\delta\colon [0,\infty) \to [0,\infty), \quad r \mapsto \begin{cases} \delta \sin(r/\delta) & \text{for } r \leq \frac{\delta\pi}{2} \\ \delta & \text{otherwise} \end{cases}$$

where $\delta > 0$. The result, which we will also denote by β_δ is called a *torpedo function of radius δ*.

Remark 2.25. Observing that $R_{\mathbb{E}^{n-q}\times\mathbb{S}^q} = R_{\mathbb{E}^{n-q}\times\mathbb{E}\times\mathbb{S}^{q-1}} + L_q$, we see that

$$R^{\beta_\delta,q}(\theta,r) = \begin{cases} R_{\mathbb{E}^{n-q}\times\mathbb{S}^q} & \text{if } r = 0 \\ R_{\mathbb{E}^{n-q}\times\mathbb{E}\times\mathbb{S}^{q-1}} & \text{if } r \geq \frac{\delta\pi}{2} \end{cases}$$

i.e. the metric induced by this function agrees on the last q coordinates with a round metric near zero and has a cylindrical shape for large radii.

Definition 2.26. For every $q \geq 3$, the torpedo function β_δ gives rise to a metric g^{β_δ} on $\mathbb{R}^q$ and consequently on $D^q(r)$ (where we assume $r \geq \frac{\delta\pi}{2}$), which we will refer to as a *torpedo metric* and denote by g^δ_{torp}.

Remark 2.27. If $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ is a codimension c surgery stable curvature condition, and $n \geq q \geq c$, then $(\mathbb{R}^{n-q} \times \mathbb{R}^q, g_{\mathrm{eucl}} + g^\delta_{\mathrm{torp}})$ satisfies C.

§2.4 Connection metrics and riemannian submersions. Using the associated bundle construction, it is well-known that we can endow the total space of a vector bundle with a metric rotationally symmetric around the zero section.

Proposition 2.28 (cf. [GW09, Proposition 2.7.1, p.97])**.** *Let (N, g_N) be a closed riemannian manifold and let $\pi\colon E \to N$ be a riemannian vector bundle of rank q equipped with a metric connection ω. Let $g_{\mathrm{rot}} = \mathrm{d}r^2 + \beta^2(r)g_{\mathbb{S}^{q-1}}$ be a complete rotationally symmetric metric on $\mathbb{R}^q$. Then there exists a unique complete riemannian metric h^∇ on E such that $\pi\colon (E, h^\nabla) \to (N, g_N)$ is a riemannian submersion with totally geodesic fibres isometric to $(\mathbb{R}^q, g_{\mathrm{rot}})$ and with horizontal distribution determined by ω.*

Definition 2.29. We refer to h^∇ as a *connection metric* on E and write

$$h^\nabla = g_N \oplus_\omega g_{\mathrm{rot}}.$$

Recall that for a riemannian submersion $\pi\colon (E^n, g_E) \to (N^{n-q}, g_N)$ we can deform the metric g_E by *shrinking the fibre.* We obtain a continuous path of metrics $\{g^t_E\}_{t\in(0,1]} \subset \mathcal{R}(E)$ given by

$$g^t_E(X, Y) := t^2 g_{F(\pi(p))}(X^{\mathcal{V}}, Y^{\mathcal{V}}) + \pi^* g_N(X, Y) \qquad \text{for } X, Y \in \mathrm{T}_p E,$$

where $X^{\mathcal{V}}, Y^{\mathcal{V}} \in \mathrm{T}_{\pi(p)} F(\pi(p))$ are the orthogonal projections onto the tangent space at the fibre. Clearly, we have $g^1_E = g_E$.

Because riemannian submersions from a complete riemannian manifold into any other riemannian manifold are fibre bundles (cf. [Bes87, Theorem 9.42 p.245]), we can talk about the fibre of a riemannian submersion. As a minor adaptation of [Hoe16, Theorem 3.1], we obtain the following result.

Proposition 2.30. *Let (E^n, g_E) is a complete riemannian manifold and let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a curvature condition. Further, let $\pi\colon (E^n, g_E) \to (N^{n-q}, g_N)$ be a riemannian submersion with fibres $\mathbb{R}^q$ into a closed manifold N. If C satisfies an inner cone condition with respect to each curvature operator corresponding to*

$$R_{b,p} := R_{(\mathbb{R}^{n-q} \times F(b), g_{\mathbb{E}^k} + g_E|_{F(b)})}(p),$$

for all $b \in N$, $p \in F(b) := \pi^{-1}(b) \cong \mathbb{R}^q$ and $R_{b,p}$ is constant on $\{p \in E \mid d_{g_E}(p, N) > R\}$ for some $R > 0$, then there exists a $t_* > 0$ such that $g^t_E \in \mathcal{R}_C(E)$ for all $t \in (0, t_*)$.

Now consider the case that g_E is a connection metric obtained from a rotationally symmetric metric $g_{\mathrm{rot}} = \mathrm{d}r^2 + \beta^2(r)g_{\mathbb{S}^{q-1}}$ on $\mathbb{R}^q$. Then shrinking the fibre amounts to shrinking the warping function, since at $b \in N$

$$g^t_E|_{F(b)} = \mathrm{d}r^2 + t^2\beta^2(\tfrac{r}{t})g_{\mathbb{S}^{q-1}}.$$

If β is a torpedo function, the new warping function $r \mapsto t\beta(\frac{r}{t})$ can easily seen to be a torpedo function again.

Corollary 2.31. *Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a deformable, codimension c surgery stable curvature condition. Let (N^{n-q}, g_N) be a riemannian manifold, let $\pi\colon E \to N$ be a riemannian vector bundle of rank q for $q \geq c$ equipped with a metric connection ω and let $g_{\mathrm{rot}} = \mathrm{d}r^2 + \beta^2 g_{\mathbb{S}^{q-1}}$ be a rotationally symmetric metric on $\mathbb{R}^q$, which* *satisfies the conditions eq.* (2.23) *and* $\beta|_{[R,\infty)}$ *is constant for some* $R > 0$*. Then there exists a* $t_* > 0$ *such that* $(h^{\nabla})^t = g_N \oplus_\omega (\mathrm{d}r^2 + t^2\beta^2(\frac{r}{t})g_{\mathbb{S}^{q-1}})$ *satisfies* C *for all* $t \in (0, t_*)$.

Proof. Since g_{rot} satisfies the conditions eq. (2.23), we conclude that the curvature operators $R^{\beta,q}$ are contained in C and by deformability, C satisfies an inner cone condition with respect to them. These are precisely the curvature operators of the metric $g_{\mathbb{E}^{n-q}} + g_E|_{F(b)}$, where $F(b)$ denotes the fibre at $b \in N$. Applying Proposition 2.30 finishes the proof, because shrinking the metric is a deformation through connection metrics, as mentioned above. □

In particular, we can construct a connection metric on the total space of a vector bundle of suitable rank, which satisfies C, e.g. by considering the connection metric obtained from $g^{\delta}_{\mathrm{torp}}$ for δ small enough.

§2.5 Rotational symmetry around a submanifold. Applying this to the normal bundle of a submanifold N in M we can produce a metric satisfying C in a tubular neighbourhood of N from a rotationally symmetric one. If we start with an arbitrary metric in M, however, clearly it will not necessarily be a connection metric with respect to a rotationally symmetric metric on the normal bundle. Nevertheless, it was observed by Gromov and Lawson that every metric does actually look almost rotationally symmetric close to N.

Throughout this section, let M^n be a smooth manifold of dimension n, $i\colon N^k \hookrightarrow M^n$ be a closed submanifold with normal bundle $\pi\colon \nu N \to N$ equipped with a bundle metric $h^{\nu N}$. Moreover, let $\phi\colon \nu N \to N$ be a tubular map, i.e. ϕ is an embedding with $\phi|_0 \equiv i \circ \pi$ and $\phi \circ s_0 \simeq i$ (where $s_0\colon N \to \nu N$ is the zero section).

Definition 2.32. A riemannian metric g on M is called *adjusted to the tubular map* ϕ *on the* r*-tube*, if $[0, r] \to M$, $r \mapsto \phi(r\nu_p)$ is a unit speed geodesic w.r.t. g, where $\nu_p \in \nu M$ with $\|\nu_p\|_{h^{\nu N}} = 1$ and $\|\cdot\|_{h^{\nu N}}$ denotes the norm given by the riemannian vector bundle structure on νN.

From now on fix a metric connection ω on νN and we denote by $\nu^{\leq r}N$ the radius r disc bundle w.r.t. $h^{\nu N}$.

We can adjust an entire family of metrics to the tubular map ϕ.

Proposition 2.33 ([EF18, adapted from Proposition 3.4])**.** *Let* $\{g_\xi\}_{\xi \in D^l}$ *be a continuous family of metrics such that* g_ξ *for* $\xi \in S^{l-1}$ *is adjusted to* ϕ *on the* r*-tube. Then there exists an* $r_0 \in (0, r]$ *and a continuous map* $F\colon [0,1] \times D^l \to \mathrm{Diff}(M)$ *such that*

(i) $F|_{\{0\}\times D^l \cup [0,1]\times S^{l-1}} \equiv \mathrm{id}_M$,
(ii) $F(t,x)|_N \equiv \mathrm{id}_N$ *for all* $(t,x) \in [0,1] \times D^l$,
(iii) $(F(1,x))^* g_\xi$ *is adjusted to* ϕ *on the* r_0*-tube.*

Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a deformable, codimension c surgery stable curvature condition and fix an arbitrary riemannian metric g_N on the submanifold N. By Corollary 2.31, we know that there exists a connection metric $h^{\mathrm{torp}} := g_N \oplus_\omega g^{\delta}_{\mathrm{torp}}$, which satisfies C and which we will also fix from now on.

Definition 2.34. Let $R > 0$ be a fixed radius. We call a riemannian metric $g \in \mathcal{R}_C(M)$ *rotationally symmetric around* N, if

(1) g is adjusted to ϕ on the R-tube,
(2) $\phi|^*_{\nu^{\leq R}N} g = (g_N \oplus_\omega g_{\mathrm{rot}})|_{\nu^{\leq R}N}$ for some rotationally symmetric metric g_{rot} on $\mathbb{R}^{n-k}$ for which $(\mathbb{R}^n, g_{\mathrm{eucl}} + g_{\mathrm{rot}})$ satisfies C.

Now define the *space of rotationally symmetric metrics around* N as

$$\begin{aligned}\mathcal{R}_C^{\mathrm{rot}}(M) &:= \mathcal{R}_C^{\mathrm{rot}}(M;\phi, g_N, h^{\nu N}, \omega, R)\\ &:= \{g \in \mathcal{R}_C(M) \text{ is rotationally symmetric around } N\}.\end{aligned}$$

Moreover, we denote by

$$\begin{aligned}\mathcal{R}_C^{\mathrm{torp}}(M) &:= \mathcal{R}_C^{\mathrm{torp}}(M;\phi, h^{\nu N}, h^{\mathrm{torp}}, R)\\ &:= \{g \in \mathcal{R}_C(M) \mid \phi|_{\nu^{\leq R}N}^{*} g = h^{\mathrm{torp}}|_{\nu^{\leq R}N}\}.\end{aligned}$$

the *space of riemannian metrics satisfying* C*, which are standard near* N.

Remark 2.35. (i) The adjustment in item (1) simply amounts to g_{rot} in item (2) being of the form $\mathrm{d}r^2 + \beta g_{\mathbb{S}^{n-1}}$ in terms of Fermi coordinates around N.
(ii) Note that metrics in $\mathcal{R}_C^{\mathrm{torp}}(M)$ are adjusted to ϕ on the R-tube by definition and thus are rotationally symmetric. In particular, we have an inclusion $\mathcal{R}_C^{\mathrm{torp}}(M) \hookrightarrow \mathcal{R}_C^{\mathrm{rot}}(M)$.

§3. Main Results

§3.1 Main technical result. We will now state the main technical result of this work, which shows that the space of all metrics satisfying curvature conditions of a certain type is weakly homotopy equivalent to the space of metrics, which take a particular prescribed form around an initially fixed submanifold. The proof of this theorem is heavily built on the techniques and terminology developed in [Che04, Wal11] and essentially follows Chernysh's presentation implementing some improvements introduced by [EF18].

Theorem 3.1. *Let* $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ *be a deformable, codimension* c *surgery stable curvature condition.*

(1) Let M^n *be a smooth manifold of dimension* n *and let* N^k *be a compact,* k*-dimensional submanifold in* M *with* $n - k \geq c$.
(2) Let $\phi\colon \nu N \to M$ *be tubular map and let* $h^{\nu N}$ *be a bundle metric on the normal bundle* νN *equipped with a metric connection* ω.
(3) Further let g_N *be an arbitrary riemannian metric on* N *and let* $h^{\mathrm{torp}} = g_N \oplus_\omega g^{\delta}_{\mathrm{torp}}$ *be a connection metric on* νN*, which satisfies* C *(as constructed in § 2.4), obtained from* g_N*,* $h^{\nu N}, \omega$ *and a torpedo metric* $g^{\delta}_{\mathrm{torp}}$ *on* $\mathbb{R}^{n-k}$.

Then the inclusion of metrics, which are standard near N

$$i\colon \mathcal{R}_C^{\mathrm{torp}}(M) \hookrightarrow \mathcal{R}_C(M)$$

is a weak homotopy equivalence.

The proof of Theorem 3.1 is based on the following two propositions whose proofs will occupy the rest of this paper.

Proposition 3.2. *Let* $g\colon S \to \mathcal{R}_C(M)$*,* $\xi \mapsto g_\xi$ *be a family of metrics. There exists a continuous map*

$$\Pi\colon [0,1] \times S \to \mathcal{R}_C(M)$$

with the following properties:

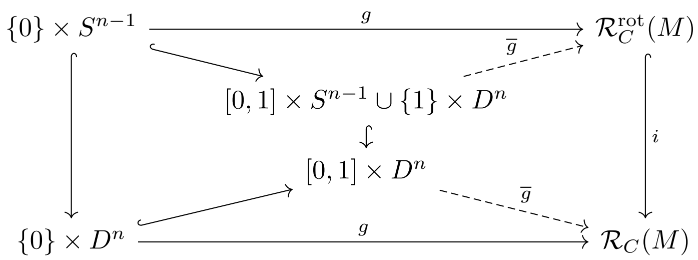
(i) $\Pi(0, \cdot\,) \equiv g$,
(ii) $\Pi(\{1\} \times S) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$,
(iii) If $g_\xi \in \mathcal{R}_C^{\mathrm{rot}}(M)$*, then* $\Pi(t, \xi) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$ *for all* $t \in [0,1]$.

The proof will be carried out in § 4.

Proposition 3.3. *The space* $\mathcal{R}_C^{\mathrm{torp}}(M)$ *is a weak deformation retract of* $\mathcal{R}_C^{\mathrm{rot}}(M)$*.*

The proof will be carried out in § 5.

Proof of Theorem 3.1. By Proposition 3.3, it is enough to show that the inclusion $i\colon \mathcal{R}_C^{\mathrm{rot}}(M) \hookrightarrow \mathcal{R}_C(M)$ is a weak homotopy equivalence.
It is well-known that i is a weak homotopy equivalence, if for every $n \in \mathbb{N}_0$ and map $g\colon \{0\} \times D^n \to \mathcal{R}_C(M)$ with $g(\{0\} \times S^{n-1}) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$ there exists a homotopy $\overline{g}\colon [0,1] \times D^n \to \mathcal{R}_C(M)$ such that the following diagram commutes



Applying Proposition 3.2 to g, we let $\overline{g} \coloneqq \Pi$. By item (ii), $\overline{g}(\{1\} \times D^n) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$ and since $\overline{g}(\{0\} \times S^{n-1}) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$, we conclude with item (iii) that $\overline{g}([0,1] \times S^{n-1}) \subset \mathcal{R}_C^{\mathrm{rot}}(M)$. □

§3.2 Applications. The following is a theorem of Chernysh [Che04, Theorem 1.2] and Walsh [Wal13, Theorem 4.1] for the case of $C = \mathrm{psc}$. It is the technical version of and clearly implies Theorem B.

Corollary 3.4. *Let* $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ *be a deformable, codimension* c *surgery stable curvature condition and let* M *be a closed smooth manifold of dimension* n*. If* $\chi(M,\phi)$ *is obtained from* M *by surgery along* $\phi|_{S^k \times \{0\}}$ *for* $\phi \in \mathrm{Emb}(S^k \times D^{n-k}, M)$ *for* $k \geq c-1$ *and* $n-k \geq c$*, then*

$$\mathcal{R}_C(M) \simeq \mathcal{R}_C(\chi(M,\phi)).$$

It results easily from Theorem 3.1 applied to a surgery and its reversal, combined with the observation that the spaces $\mathcal{R}_C(M)$ are open subsets of a Fréchet manifold. Thus, by work of Palais [Pal66, Theorem 14], they are dominated by CW-complexes and we obtain the homotopy equivalence in Corollary 3.4 from Whitehead's theorem.

Remark 3.5. The conditions $k \geq c-1$ and $n-k \geq c$ are needed to reverse the surgery and again have a sufficiently high codimension. Combined we see that $n-c \geq k \geq c-1$ and conclude that the surgery stability needs to be roughly "below the middle dimension" if we want to apply Corollary 3.4.

We note that we can slightly generalize the definition of a surgery as follows. Let W_1 and W_2 be manifolds with boundary with diffeomorphic and connected boundaries $\partial W_1 \cong \partial W_2$. If $\phi\colon W_1 \hookrightarrow M$ is an embedding, we can consider

$$\tilde{\chi}(M;\phi) \coloneqq M \setminus \mathrm{Int}(\phi(W_1)) \cup_{\partial W_1} W_2.$$

We can apply this procedure to prove Theorem C, which follows from the following corollary since both positive scalar curvature and positive 1-curvature are deformable and admit codimension 4 surgeries (cf. Examples 2.15, 2.16 and 2.20).
Denote by g^δ for $\delta \in (0,1]$ the metric on S^{4k+3} obtained via shrinking the fibres of the riemannian submersion given by the Hopf fibration $S^3 \hookrightarrow \mathbb{S}^{4k+3} \to \mathbb{H}P^k$.

Corollary 3.6. *Let $k \geq 1$, let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^{4(k+1)})$ be a deformable curvature condition, which admits codimension 4 surgeries and such that $g^\delta + \mathrm{d}t^2$ satisfies C for all $\delta \in (0,1]$. Then spaces of metrics satisfying C on $S^{4(k+1)}$ and $\mathbb{H}P^{k+1}$ have the same homotopy type.*

A key ingredient in the proof of this statement is Gajer's Lemma, which was originally stated for positive scalar curvature, but holds in greater generality.

Lemma 3.7 ([Gaj87, p.184])**.** *Let $C \subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$ be a curvature condition, let N^{n-1} be a closed manifold and let $\{g_t\}_{t\in[0,1]} \subset \mathcal{R}(N)$ be a smooth path of riemannian metrics. If for every $t \in [0,1]$ the riemannian product metrics $(N, g_t) \times \mathbb{E}$ satisfy C, then there exists a $0 < \Lambda \leq 1$ such that for every smooth function $f\colon \mathbb{R} \to [0,1]$ with $|f'|, |f''| \leq \Lambda$ the metric $g_{f(t)} + \mathrm{d}t^2$ on $N \times \mathbb{R}$ satisfies C.*

Proof. The proof follows directly from calculations analogous to [Gaj87, p.185], which yield

$$R_{(N\times\mathbb{R}, g_{f(t)}+\mathrm{d}t^2)} = R_{(N, g_{f(t_0)})\times\mathbb{E}} + O(|f'|)E_1 + O(|f'|^2)E_2 + O(|f''|)E_3,$$

where E_1, E_2, E_3 only depend on the family of metrics $\{g_t\}$ and we note that $\operatorname{tr} E_1 = 0$. Since C is an open subset in $\mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$, we find a suitable choice for Λ. ☐

Proof of Corollary 3.6. Consider the inclusion $i\colon \mathbb{H}P^k \hookrightarrow \mathbb{H}P^{k+1}$. It is well-known that the normal bundle $\nu\mathbb{H}P^k$ can be written as an associated bundle $S^{4k+3} \times_{S^3} \mathbb{R}^4$ to the Hopf fibration, i.e. to the S^3-principal bundle $S^{4k+3} \to \mathbb{H}P^k$.

Now choose a bundle metric $h^{\nu\mathbb{H}P^k}$ and $\overline{r} > 0$ small enough such that the inclusion of the disc bundle $\phi\colon \nu^{\leq\overline{r}}\mathbb{H}P^k \hookrightarrow \mathbb{H}P^{k+1}$ is an embedding. Then we have $\partial(\nu^{\leq\overline{r}}\mathbb{H}P^k) = \nu^{\overline{r}}\mathbb{H}P^k \cong S^{4k+3}$ and it is well-known that the complement of the disc bundle is diffeomorphic to D^{4k}. Thus we conclude that

$$\big(\mathbb{H}P^{k+1} \setminus (\nu^{\leq\overline{r}}\mathbb{H}P^k)\big) \cup_{S^{4k+3}} D^{4(k+1)} \cong S^{4(k+1)}$$

In the same vain, we obtain

$$\big(S^{4(k+1)} \setminus D^{4(k+1)}\big) \cup_{S^{4k+3}} \big(S^{4k+3} \times_{S^3} D^4\big) \cong \mathbb{H}P^{k+1}.$$

In both cases we remove a tubular neighbourhood of a codimension 4 submanifold and thus we are in a position to apply Theorem 3.1 to both situations. It remains to check that the spaces of metrics which are standard near the submanifolds $\mathcal{R}_C^{\mathrm{torp}}(\mathbb{H}P^{k+1})$ and $\mathcal{R}_C^{\mathrm{torp}}(S^{4(k+1)})$ are weakly homotopy equivalent for suitable choice of torpedo- and connection metrics. We can strengthen the resulting weak homotopy equivalence between $\mathcal{R}_C(\mathbb{H}P^{k+1})$ and $\mathcal{R}_C(S^{4(k+1)})$ to a homotopy equivalence using Palais and Whitehead, as mentioned after Corollary 3.4.

Let g^δ_{torp} be a torpedo metric. Then S^3 acts isometrically on $\mathbb{S}^{4k+3}$ from the right via the Hopf fibration and isometrically on $(\mathbb{R}^4, g^\delta_{\mathrm{torp}})$ from the left, by the torpedo metric's rotational symmetry. Now we obtain a riemannian metric on the quotient $S^{4k+3} \times_{S^3} \mathbb{R}^4$, induced from the product metric $g_{\mathbb{S}^{4k+3}} + g_{\mathbb{E}^4}$, such that the projection map to $\mathbb{H}P^k$ is a riemannian submersion. Because the Hopf fibration has totally geodesic fibres and the action of S^3 is by isometries, the riemannian submersion $S^{4k+3} \times_{S^3} \mathbb{R}^4 \to \mathbb{H}P^k$ has totally geodesic fibres (cf. [GW09, p.98]). But such metrics are already connection metrics [GW09, Theorem 2.7.2, p.98].

By Corollary 2.31, we can choose $\delta > 0$ small enough such that the connection metric on $S^{4k+3} \times_{S^3} \mathbb{R}^4$ satisfies C. Moreover, $(\mathbb{R}^4 \setminus D^4(\frac{\delta\pi}{2}), g^\delta_{\mathrm{torp}})$ is isometric to a riemannian product $\mathbb{S}^3(\delta) \times (\frac{\delta\pi}{2}, \infty)$ and thus we obtain isometries

$$\begin{aligned}\big(S^{4k+3} \times_{S^3} \mathbb{R}^4\,\big) \setminus \big(S^{4k+3} \times_{S^3} D^4(\delta\pi/2)\big) &= S^{4k+3} \times_{S^3} \big(\mathbb{S}^3(\delta) \times (\delta\pi/2, \infty)\big)\\ &= \big(S^{4k+3} \times_{S^3} \mathbb{S}^3(\delta)\big) \times (\delta\pi/2, \infty).\end{aligned}$$

We denote the metric obtained from this on $S^{4k+3} \cong S^{4k+3} \times_{S^3} \mathbb{S}^3(\delta)$ by g^δ.

For any $0 < \delta \leq 1$ the metric $g^\delta + \mathrm{d}t^2$ on $S^{4k+3} \times \mathbb{R}$ satisfies C by assumption.

Now choose a smooth function $s\colon [0,1] \to \mathbb{R}$, which is constant near $\{0,1\}$ and satisfies $s(0) = 1$, $s(1) = \delta$ and $s' \leq 0$. We obtain a smooth path of metrics $\{g^{s(t)}\}_{t\in[0,1]}$ on S^{4k+3}, which satisfies the assumptions of Lemma 3.7.

If we choose the inclusion $i\colon \mathrm{pt} \hookrightarrow S^{4(k+1)}$ to a point in the sphere, the space $\mathcal{R}_C^{\mathrm{torp}}(S^{4(k+1)})$ is the space of metrics, which are fixed and of torpedo shape on one hemisphere, i.e. $\phi|^*_{\nu^{\leq R}\,\mathrm{pt}} g = h^{\mathrm{torp}}_{\nu^{\leq R}\,\mathrm{pt}}$. In particular, there exist $0 < R' \leq R$ and $b > 0$ such that for every $g \in \mathcal{R}_C^{\mathrm{torp}}(S^{4(k+1)})$ we have that $\phi|^*_{\nu^{R'\leq R}\,\mathrm{pt}} g$ is isometric to $(S^{4k+3} \times [0, 2b], g_{\mathbb{S}^{4k+3}} + \mathrm{d}t^2)$. Possibly by passing to another torpedo metric h^{torp}, we obtain b large enough such that $g^{\frac{1}{b}s(t)} + \mathrm{d}t^2$ is a metric satisfying C on $S^{4k+3} \times [0, b]$, by Lemma 3.7. Denote by $\tilde{R}$ the space of metrics satisfying C with $\phi|^*_{\nu^{\leq R'}\,\mathrm{pt}} g = h^{\mathrm{torp}}_{\nu^{\leq R'}\,\mathrm{pt}}$ and $\phi|^*_{\nu^{R'\leq R}\,\mathrm{pt}} g$ is isometric to

$$(S^{4k+3} \times [0, b], g_{\mathbb{S}^{4k+3}} + \mathrm{d}t^2) \cup (S^{4k+3} \times [0, b], g^{\frac{1}{b}s(t)} + \mathrm{d}t^2).$$

The spaces $\mathcal{R}_C^{\mathrm{torp}}(S^{4(k+1)})$ and $\tilde{R}$ are weakly homotopy equivalent, since we can use the family $\{g^{s(t)}\}_{t\in[0,1]}$ and Gajer's lemma to interchange the cylindrical pieces on $S^{4k+3} \times [0, 2b]$.

The spaces $\mathcal{R}_C^{\mathrm{torp}}(\mathbb{H}P^{k+1})$ and $\tilde{R}$ are homeomorphic. □

§4. Proof of Proposition 3.2

Before starting the actual proof, we will recall the classical graph deformation procedure introduced by Gromov and Lawson. The construction of Π then proceeds in two steps. First, the graph deformation is applied to a family of riemannian metrics to split a tubular neighbourhood around a submanifold N into three regions with particular properties. Then one deforms the metric on these three regions to obtain a metric, which is rotationally symmetric around N.

§4.1 Preliminaries and Chernysh's trick. Here we will recall a construction for a metric deformation and a few technical results of Hoelzel.

Before, let us recall elementary facts about curves in $\mathbb{R}^2$, which will be used to control the metric deformations. We will deal with arc-length parametrized curves $\gamma\colon \mathbb{R} \to \mathbb{R}^2, s \mapsto (r(s), t(s))$, which satisfy a number of properties.

Definition 4.1. Let $\bar{r} > 0$. We denote by

$\Gamma_b(\bar{r}) \coloneqq \{\gamma\colon \mathbb{R} \to \mathbb{R}^2$ arc-length parametrized curve satisfying (i) – (iv) below $\}$

(i) $\gamma(0) = (\bar{r}, 0)$ and $t|_{(-\infty,0]} \equiv 0$,
(ii) $t(s) \geq 0$ for all $s \in \mathbb{R}$,
(iii) γ intersects the t-axis $\{0\} \times \mathbb{R}$ precisely once following the arc of a circle (of possibly infinite radius) at $\gamma(b)$ and is symmetric about it,
(iv) r is non-increasing, while t is non-decreasing for $s \in (-\infty, b]$.

$$\tilde{\Gamma}_b(\bar{r}) \coloneqq \{\gamma \in \Gamma_b(\bar{r}) \mid \gamma \text{ satisfies (v) below } \}$$

(v) There exists a partition $0 = s_0 \leq s_1 \leq \cdots \leq s_6 = b$ such that

$$\kappa|_{[s_0,s_1]\cup[s_2,s_3]\cup[s_4,s_5]} \equiv 0 \quad \text{and} \quad r'|_{[s_3,s_4]} \equiv 0$$

where κ is the signed curvature function of γ.

We endow each of the above sets with the subspace topology from $C^\infty(\mathbb{R}, \mathbb{R}^2)$ and denote $\Gamma(\bar{r}) \coloneqq \bigcup_{b>0} \Gamma_b(\bar{r})$ and $\tilde{\Gamma}(\bar{r}) \coloneqq \bigcup_{b>0} \tilde{\Gamma}_b(\bar{r})$.

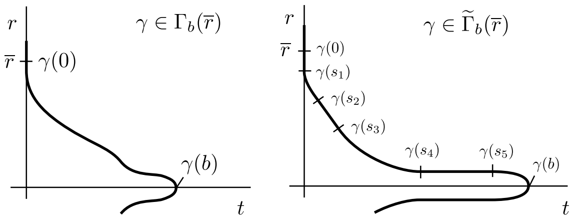


FIGURE 1. Examples for curves in $\Gamma_b(\overline{r})$ and $\tilde{\Gamma}_b(\overline{r})$.

Proposition 4.2. *Any curve γ in $\Gamma_b(\overline{r})$ is uniquely and continuously determined on the interval $[0, b]$ by each of the following:*

(1) $\gamma|_{[0,b]}$*,*
(2) its angular function $\theta \colon \mathbb{R} \to [0, \frac{\pi}{2}]$*,*
(3) its signed curvature function $\kappa \colon \mathbb{R} \to \mathbb{R}$*.*

Proof. The claim (1) is immidiately clear. For the remaining note that the curve γ is determined by its angular function $\theta \colon \mathbb{R} \to [0, \frac{\pi}{2}]$ as it gives rise to the following initial value problem

$$\begin{cases} \cos\theta = \langle \gamma', -\partial_r \rangle = -r', & r(0) = \overline{r} \\ (r')^2 + (t')^2 = 1 & t(0) = 0 \\ t \geq 0 \end{cases} .$$

The angular velocity θ' is precisely the signed curvature of γ and $\theta(0) = 0$ (by (i)). Moreover, we conclude that by the theory of ordinary differential equations γ continuously depends on θ or κ, respectively. □

Note that from this proof we can extract a description for $r(s)$ and $t(s)$ for $s \in [0, b]$ in terms of θ as follows

$$r(s) = \overline{r} - \int_0^s \cos\theta(u)\,\mathrm{d}u \text{ and } t(s) = \int_0^s \sin\theta(u)\,\mathrm{d}u. \tag{4.3}$$

If γ is determined by θ or κ, we will write $\gamma(\theta)$ or $\gamma(\kappa)$.

Next, let us revisit a construction of a metric on M altered in a tubular neighbourhood around a submanifold using a curve that we call *Chernysh's trick.*[§] It differs from the original construction by Gromov and Lawson in that is produces a metric on M again.

We consider the setup as in Theorem 3.1. Let g_M be a riemannian metric adjusted to ϕ on the $\overline{r}$-tube for an $\overline{r} > 0$ small enough such that $2\overline{r} < \mathrm{InjRad}_N^\perp(g_M)$, where $\mathrm{InjRad}_N^\perp(g_M) := \min_{p\in N} \sup\{r > 0 \mid \exp_p^{g_M,\perp}$ is injective on the r-ball$\}$ denotes the normal injectivity radius of g_M. Further, let $\gamma \in \Gamma_b(\overline{r})$.

Note that g_M being adjusted to ϕ on the $\overline{r}$-tube implies that $\phi|_{\nu^{\leq \overline{r}}N}$ coincides with the normal exponential map $\exp^\perp$ of the metric g_M.

[§]While there have been similar ideas prior to the work of Chernysh (e.g. [Gaj87, Car88]), the formalism as seen in the following is essentially laid out in [Che04].

From this we can construct the embedding $\psi_\gamma = \psi_\gamma(\phi, g_M)$ given by

$$\psi_\gamma\colon M \to M \times \mathbb{R},$$
$$p \mapsto \begin{cases} (p,0) & \text{if } \mathrm{d}_{g_M}(p,N) \geq \overline{r} \\ (\phi(r(s)\nu), t(s)) & \text{for } p = \phi(\overline{r}(1 - s/b)\nu) \text{ with } s \in (0,b), \nu \in \nu^1 N \\ (p, t(b)) & \text{if } p \in N. \end{cases}$$

whose image is denoted by

$$D_\gamma := \operatorname{Im} \psi_\gamma \subset M \times \mathbb{R},$$

and carries a metric g_D induced from $g_M + \mathrm{d}t^2$ on $M \times \mathbb{R}$. Its pull-back along ψ_γ to M will be denoted by g_γ.

In fact, we obtain a continuous map

$$\Gamma(\overline{r}) \to \operatorname{Emb}(M, M \times \mathbb{R}), \quad \gamma \mapsto \psi_\gamma.$$

The embeddings in the image all coincide outside the compact set $\phi(\nu^{\leq \overline{r}} N)$ with the inclusion $M \hookrightarrow M \times \{0\}$. Moreover, the embeddings ψ_γ coincide for all metrics adjusted to ϕ on the $\overline{r}$-tube.

It is well-known that the pull-back of riemannian metrics along embeddings of a compact manifold is continuous with respect to the C^∞-topology on the space of embeddings. From this we draw the following conclusion.

Let $\{g_\xi\}_{\xi \in S} \subset \mathcal{R}(M)$ be a family of metrics on M, which are adjusted to ϕ on the $\overline{r}$-tube for $\overline{r} > 0$ such that $2\overline{r} < \min_{\xi \in S} \operatorname{InjRad}^\perp_N(g_\xi)$. Then we have a continuous map

$$S \times \Gamma(\overline{r}) \to \mathcal{R}(M), \quad (\xi, \gamma) \mapsto \psi_\gamma^*(g_\xi + \mathrm{d}t^2) =: g_{\xi,\gamma}. \tag{4.4}$$

Remark 4.5. Moreover, if g_ξ is the pull-back of a connection metric on $\phi(\nu^{\leq \overline{r}} N)$ from νN, then $g_{\xi,\gamma}$ is a connection metric, as well.

The curvature operator of the induced metric g_γ can be connected to the curvature operator of the product metric on $M \times \mathbb{R}$ as demonstrated by Hoelzel.

By definition, a curve $\gamma \in \Gamma_b(\overline{r})$ induces a parametrization of $\operatorname{Im} \psi_\gamma(M \setminus N) \subset D_\gamma$ given by

$$\gamma\colon \nu^1 N \times [0,b] \to M \times \mathbb{R}, \quad (\nu, s) \mapsto (\phi(r(s)\nu), t(s)),$$

at whose image points the tangent space splits as

$$\mathrm{T}_{\gamma(\nu,s)}\, D_\gamma = \mathrm{T}_{\phi(r(s)\nu)}\, T(r(s)) \oplus \langle \gamma'(\nu,s) \rangle\,,$$

where $T(r) := \phi(\nu^r N)$ is the distance tube around N. For $(\nu_q, r) \in \nu^1 N \times (0, \overline{r}]$, denote by $\mathcal{H}(\nu_q, r)$ the parallel translation of $\mathrm{T}_q\, N$ into $\phi(r\nu_q)$. Further denote by $\mathcal{V}(\nu_q, r)$ the orthogonal complement to $\mathcal{H}(\nu_q, r) \oplus \langle \partial_r \rangle$, i.e.

$$\mathcal{H}(\nu_q, r) \oplus \mathcal{V}(\nu_q, r) \oplus \langle \partial_r \rangle = \mathrm{T}_{\phi(r\nu_q)}\, T(r) \oplus \langle \partial_r \rangle = \mathrm{T}_{\phi(r\nu_q)}\, M.$$

For every $(\nu, r) \in \nu^1 N \times (0, \overline{r}]$ choose an orthonormal basis of $\mathcal{H}(\nu, r) \oplus \mathcal{V}(\nu, r)$, i.e. an isometry $i_{\nu,r}\colon \mathbb{E}^{n-1} \to \mathcal{H}(\nu, r) \oplus \mathcal{V}(\nu, r)$.[¶]

With respect to this choice we introduce the following notation

$$\begin{aligned} \tilde{R}_D(\nu, s) &:= (i_{\nu, r(s)} \oplus (-\gamma'(\nu, s)))^* R_D, \\ \tilde{R}_M(\nu, r) &:= (i_{\nu,r} \oplus \partial_r)^* R_M, \\ \tilde{R}_T(\nu, r) &:= (i_{\nu,r} \oplus \partial_t)^* R_T, \end{aligned} \tag{4.6}$$

[¶]Note that in general this cannot be done a continuous way, if νN is not assumed to be trivial.

where R_T is the curvature operator of $T(r) \times \mathbb{R}$ endowed with the product metric $g_M|_{T(r)} + \mathrm{d}t^2$.

Proposition 4.7 ([Hoe16, Proposition 2.5])**.** *In the situation above, for every $(\nu, s) \in \nu^1 N \times I$, we have*

$$\tilde{R}_D(\nu, s) = \cos^2\theta(s)\tilde{R}_M(\nu, r(s)) + \sin^2\theta(s)\tilde{R}_T(\nu, r(s)) + E(\nu, s),$$

for $E(\nu, s)$ a curvature operator satisfying

$$\|E(\nu, s)\| \leq \cos\theta(s)(1-\cos\theta(s))C_1 + \frac{\theta'(s)\sin\theta(s)}{r(s)}C_2,$$

where C_1, C_2 are constants only depending on $D(\overline{r}) := \phi(\nu^{\leq \overline{r}}N)$, $g_M|_{D(\overline{r})}$ and N.

If $\{g_\xi\}_{\xi\in S}$ is a family of metrics on M, we will choose isometries $i_{\nu,r}$ for every metric g_ξ. Moreover, we denote by $\tilde{R}_{D,\xi}$, $\tilde{R}_{M,\xi}$ and $\tilde{R}_{T,\xi}$ the corresponding entities defined in eq. (4.6).

§4.2 Constructing the deformation map Π**.** The first step will be to show that the construction of a new metric using a graph described above is actually a continuous deformation procedure within $\mathcal{R}_C(M)$ that can be applied to compact families of metrics simultaneously.

Throughout this section, we always consider the setup as in Theorem 3.1.

Proposition 4.8. *Let $\{g_\xi\}_{\xi\in S} \subset \mathcal{R}_C(M)$ be a family of metrics on M satisfying C, which are adjusted to ϕ on the r-tube for some $r > 0$. Then there exists an $\overline{r} \leq r$ and a curve $\gamma \in \tilde{\Gamma}(\overline{r})$ such that $(M, g_{\xi,\gamma})$ satisfies C, where $g_{\xi,\gamma}$ is obtained from g_ξ via eq.* (4.4)*.*

Moreover γ can be chosen such that according to the partition $r(s_4) = r(s_5)$ is arbitrarily small.

This follows from the constructive proof of [Hoe16, Theorem 2.1], which can easily be adapted to construct a curve γ as required for an entire family of metrics. We only need to make sure that during the bend of γ towards the t-axis C remains satisfied.

Lemma 4.9 (Initial bending, adapted from [Hoe16, Lemma 2.9])**.** *There exist $s_2 > 0$, $\theta_0 > 0$ and a smooth non-decreasing function $\theta\colon [0, s_2] \to [0, \theta_0]$ with $\theta'|_{[0,\varepsilon)\cup(s_2-\varepsilon,s_2]} \equiv 0$ for all $\varepsilon > 0$ small enough such that $\tilde{R}_{D,\xi}(\nu, s) \in C$ for $s \in [0, s_0]$ and all $\xi \in S$.*

Proof. The proof from [Hoe16] directly carries over to this case as all choices involved can be made in accordance with a compact family of metrics. □

Lemma 4.10 (Second bend, adapted from [Hoe16, Lemma 2.10])**.** *There exists an $r^* \in (0, \overline{r})$ such that for every $r \in (0, r^*)$ there is an extension of θ obtained from Lemma 4.9 to a smooth non-decreasing function $\theta\colon [0, s_5] \to [0, \frac{\pi}{2}]$ such that $\tilde{R}_{D,\xi}(\nu, s) \in C$ for all $\xi \in S$, $r(s) > 0$ and $\theta|_{[s_4,s_5]} \equiv \frac{\pi}{2}$ and $r|_{[s_4,s_5]} \equiv r$ for some s_4, s_5 large enough.*

Proof. We will only cover a part of the proof that we want to utilize later. Hoelzel shows in [Hoe16, p.29f] that to conclude that $\tilde{R}_{D,\xi}(\nu, s) \in C$, it is enough to ensure

$$\theta'(s) \leq \frac{\rho}{2C_2}\frac{\sin\theta(s)}{r(s)},$$

for $s \in [s_2, s_5]$, where $\rho = \rho(\tilde{R}_{M,\xi})$ and C_2 (as in Proposition 4.7) depend on the family of metrics. We let $C_3 := \min\{\frac{\rho}{C_2} \,|\, \xi \in S\}$ and conclude that while

$$\theta'(s) \leq \frac{C_3}{2} \frac{\sin\theta(s)}{r(s)}, \tag{4.11}$$

we have $\tilde{R}_{D,\xi}(\theta, s) \in C$ for all $\xi \in S$.
This can be used to inductively define the extension of θ. Assume θ is defined on $[0, s_l]$ with $\theta(s_l) < \frac{\pi}{2}$ and define $s_{l+1} := s_l + \frac{r(s_l)}{2}$. Now choose a bump function η_l with support in $[s_l + \frac{r(s_l)}{16}, s_{l+1} - \frac{r(s_l)}{16}]$ which is constantly $\frac{C_3}{4}\frac{\sin\theta(s_l)}{r(s_l)}$ on $[s_l + \frac{r(s_l)}{8}, s_{l+1} - \frac{r(s_l)}{8}]$. Setting

$$\theta(s) := \theta(s_l) + \int_{s_l}^{s} \eta_l(u)\,\mathrm{d}u$$

for $s \in (s_l, s_{l+1}]$ defines an extension of θ to $[0, s_{l+1}]$, which ensures that $r(s_l) \geq r(s) \geq \frac{r(s_l)}{2} > 0$ (cf. (4.3)) and thus satisfies

$$\theta'(s) \leq \frac{C_3}{4}\frac{\sin\theta(s_l)}{r(s_l)} < \frac{C_3}{2}\frac{\sin\theta(s)}{r(s)}. \tag{4.12}$$

Most importantly, θ increases at least by

$$\theta(s_{l+1}) - \theta(s_l) \geq \int_{s_l + \frac{r(s_l)}{8}}^{s_{l+1} - \frac{r(s_l)}{8}} \eta_l(u)\,\mathrm{d}u \geq \frac{C_3}{16}\sin\theta_0.$$

Now after finitely many steps we obtain a smooth non-increasing $\theta\colon [0, a] \to [0, \frac{\pi}{2} + \varepsilon]$, which we can adjust by a cutoff function to yield a smooth non-increasing $\theta\colon [0, a+1] \to [0, \frac{\pi}{2}]$ with $\theta|_{[a,a+1]} \equiv 1$ keeping (4.11) satisfied. W.l.o.g. we can assume that $r(a) = r$, since we can let θ follow a straight line after passing s_2 to arbitrarily increase the r-coordinate. We let $s_4 := a$ and $s_5 := a + 1$. □

Proof of Proposition 4.8. By Lemma 4.10 we obtain a curve $\gamma = \gamma(\theta)$ determined by its angular function $\theta\colon [0, s_5] \to [0, \frac{\pi}{2}]$ such that $\tilde{R}_{D,\xi}(\nu, s) \in C$ for all $\xi \in S, \nu \in \nu^1 N$ and $s \in [0, s_5]$. Now extend θ to $[0, s_6]$ by choosing a smooth, on $[s_5, s_6]$ non-increasing function with $\theta(s_6) = 0$ such that $\gamma(\theta)$ follows the arc of a circle (of possibly infinite radius) centered on the t-axis (cf. fig. 2). Since $\theta'|_{[s_5,s_6)} \leq 0$, (4.11). Finally, we let $s_1 := \inf\{s \geq 0 \mid \kappa(s) > 0\}$ and $s_3 := \inf\{s \geq s_2 \mid \kappa(s) > 0\}$ to see that $\gamma \in \tilde{\Gamma}(\overline{r})$. □

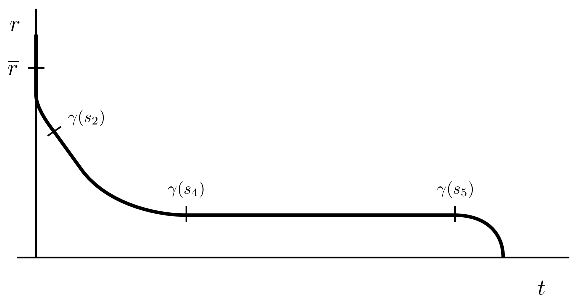


FIGURE 2. Bending γ to intersect the t-axis following the arc of a circle.

Proposition 4.13. *If γ is obtained from Proposition 4.8, there exists an isotopy $\alpha\colon [0,1] \to \Gamma(\overline{r})$ such that $\alpha(0) = \gamma$ and $\alpha(1) = \gamma^0$, where $\gamma^0\colon \mathbb{R} \to \mathbb{R}^2$, $s \mapsto (\overline{r} - s, 0)$ is the curve along the r-axis, such that $g_{\xi,\alpha(t)} \in \mathcal{R}_C(M)$ for all $\xi \in S$, $t \in [0,1]$.*

Remark. This proposition states that we can deform γ (cf. fig. 3) and thereby the corresponding metrics to the originial metric keeping the curvature condition C satisfied. Such an isotopy of curves (possibly satisfying additions assumptions) is often referred to as a *Gromov-Lawson curve* (cf. [EF18, Wal13]).

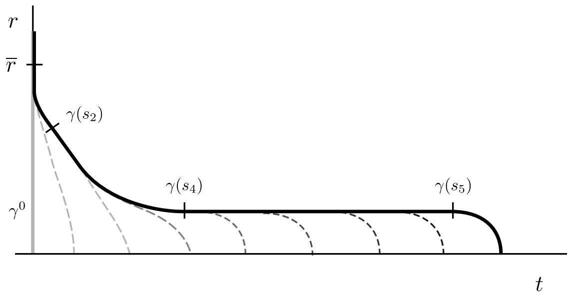


FIGURE 3. Isotopy between γ and γ^0.

Recall that during the so-called *second bend* (within $[s_3, s_4]$), γ satisfies (4.11) for all $s \in [s_3, s_4]$, where C_3 is a constant, which depends on the curvature condition C and the family of metrics. We will argue that we can modify γ via its curvature function. This will depend on the following Lemma adapted from [Che04, Proposition 2.3].

Lemma 4.14. *Let $\gamma \in \tilde{\Gamma}_b(\overline{r})$ and let $\kappa\colon [0,b] \to \mathbb{R}$ be its signed curvature function such that*

$$\theta'(s) \le \frac{C_3}{3}\frac{\sin\theta(s)}{r(s)} \tag{4.15}$$

is satisfied for $s \in [s_3, s_4]$. Then there exists a $\delta > 0$ such that for every $s_\bullet \in [s_3, s_4]$ the curve $\tilde{\gamma}$ determined by the curvature function $\delta_{s_\bullet}\kappa$ satisfies (4.11)*, where $\delta_{s_\bullet}\colon [0,b] \to [0,1]$ is a smooth δ-cutoff function in the sense that*

$$\delta_{s_\bullet}(s) = \begin{cases} 1 & s \le s_\bullet \\ 0 & s \ge s_\bullet + \delta \end{cases}.$$

Proof. The proof is entirely analogous to that of [Che04, Proposition 2.3]. Note that because of eq. (4.15), there exists a $\delta > 0$ such that for all $s \in [s_3, s_4]$ and $t \in [0, \delta]$ we have

$$\kappa(s+t) < \frac{C_3}{2}\frac{\sin\theta(s)}{r(s)}.$$

Hence, we conclude that for all $t \in [0,\delta]$ and $s_\bullet \in [s_3, s_4]$

$$\tilde{\kappa}(s_\bullet + t) = \delta_{s_\bullet}(s_\bullet + t)\kappa(s_\bullet + t) \le \kappa(s_\bullet + t) < \frac{C_3}{2}\frac{\sin\theta(s_\bullet)}{r(s_\bullet)} \le \frac{C_3}{2}\frac{\sin\tilde{\theta}(s_\bullet + t)}{\tilde{r}(s_\bullet + t)}.$$

For $s \in [s_3, s_\bullet]$ both κ and $\tilde{\kappa}$ coincide, while for $t > \delta$, we have $\tilde{\kappa}(s_\bullet + t) = 0$. Thus, (4.11) is satified. □

Proof of Proposition 4.13. We will argue in two steps. First we will show that we can deform a curve γ that bends up to a straight line of small angle θ_0 within $[s_2, s_3]$ and bends down to meet the t-axis in a right angle to γ^0 maintaining C.

Using Proposition 4.7 Hoelzel concludes that for all $\nu \in \nu^1 N$ and $s \in [0, b)$

$$\|\tilde{R}_D(\nu, s) - \tilde{R}_M(\nu, r(s))\| \leq \sin^2\theta(s)(\sup\|R_M\| + \sup\|R_T\|)|_{\phi(r(s)\nu)} + \cos\theta(s)(1 - \cos\theta(s))C_1 + \frac{\theta'(s)\sin\theta(s)}{r(s)}C_2,$$

where the suprema are taken over the points in the tubular neighbourhood $D(\overline{r})$. As this is true for all $s \in [0, s_2]$ (where $\theta(s) \leq \theta_0$), it is easy to see that it remains satisfied, if we linearly decrease θ to 0.

Let $p\colon [0,1] \to I,\ t \mapsto s_4 - t(s_4 - s_3)$ be the linear path from s_4 to s_3. Choose $\delta > 0$ from Lemma 4.14 (whose assumptions are satisfied, which can be seen from (4.12)) and a δ-cutoff function $\delta_{s_4}\colon I \to [0,1]$. Note that we obtain a continuous family of δ-cutoff functions $\tilde{\delta}_{s_\bullet}(s) \coloneqq \delta_{s_4}(s + (s_4 - s_\bullet))$ depending on $s_\bullet \in [s_3, s_4]$. Now define for $t \in [0,1]$

$$\kappa_t(s) \coloneqq \begin{cases} \delta_{p(t)}(s)\kappa(s) & \text{if } 0 \leq s \leq p(t) + \delta \\ 0 & \text{if } p(t) + \delta \leq s \leq s_t \\ \delta_{s_t}(s_t + \delta - s)\varepsilon_t\kappa(s + (s_5 - s_t)) & \text{if } 0 \leq s_t \leq L_t \end{cases}$$

where s_t is the intersection of $\gamma(\tilde{\delta}_{p(t)}\kappa(s))$ with $\{r_4\} \times \mathbb{R}$ and ε_t, L_t are uniquely determined such that

$$\int_{[s_t, L_t]} \kappa_t(s)\,\mathrm{d}s = -\int_{[0, s_t]} \kappa_t(s)\,\mathrm{d}s \quad \text{and} \quad (\gamma(\kappa_t))(L_t) \in \{0\} \times \mathbb{R}\,.$$

The resulting isotopy $t \mapsto \gamma(\kappa_t)$ deforms $\gamma = \gamma(\kappa_0)$ into a curve of the form discussed in the beginnning. ☐

Corollary 4.16. *There exists a curve $\gamma \in \tilde{\Gamma}(\overline{r})$ and a continuous map*

$$\mathfrak{A}\colon S \times [0,1] \to \mathcal{R}_C(M),$$

with $\mathfrak{A}(\xi, 0) = g_\xi$ and $\mathfrak{A}(\xi, 1) = g_{\xi,\gamma}$.

Proof. By Proposition 4.13, we obtain an isotopy of curves and thus an isotopy of embeddings $M \hookrightarrow M \times \mathbb{R}$ whose corresponding metrics obtained via eq. (4.4) satisfy C. ☐

In the next step, we will show that we can deform metrics to become rotationally symmetric in a small normal tube.

Proposition 4.17. *There exists an $r_* > 0$ and a continuous map*

$$\mathfrak{B}\colon S \times [0,1] \to \mathcal{R}_C(M)$$

such that $\mathfrak{B}(\xi, 0) = g_\xi$ and $(\phi|_{\nu^{\leq r_*}N})^*\mathfrak{B}(\xi, 1)$ is the restriction of a connection metric on νN.

Moreover, if g_ξ is contained in $\mathcal{R}_C^{\mathrm{rot}}(M)$, then $t \mapsto \mathfrak{B}(\xi, t)$ is a path within $\mathcal{R}_C^{\mathrm{rot}}(M)$.

During the proof we will utilize the following technical result by Hoelzel.

Theorem 4.18 ([Hoe16, Proposition 2.7])**.** *Let C be be a codimension c surgery stable curvature condition and let $N^k \subset M^n$ be a compact submanifold of the riemannian manifold (M, g_M) with codimension $n - k \geq c$. There exists an $r_* > 0$ such that for all $r \in (0, r_*)$ the riemannian manifold $(T(r) \times \mathbb{R}, g_{T(r)} + g_\mathbb{R})$ satisfies C. Moreover, there exists an $L > 0$ such that*

$$\tilde{R}_T(\nu, r) = B_{L/r}({}^1\!/\!_{r^2} R_{\mathbb{E}^{k+1} \times \mathbb{S}^{n-k-1}}) \subset C.$$

We note that, since all the metrics g_ξ are adjusted to ϕ on the $\overline{r}$-tube, we have $T(r) = \phi(\nu^r N)$ for every $r \leq \overline{r}$ when we apply this theorem to g_ξ.

Lemma 4.19. *Let $\gamma = (\delta \cos s, \delta \sin s)$ for a $s \in [0, \pi/2)$ be the curve in the (r,t)-plane following the arc of a circle. Let g_M be a riemannian metric on M and let $D_\gamma \subset \phi(\nu^{\leq \delta} N) \times \mathbb{R} \subset M \times \mathbb{R}$ be obtained from γ. Then there exists a $\delta > 0$, such that $\tilde{R}_D(\nu, s)$ satisfies C for all $\nu \in \nu^1 N$ and $s \in [0, \pi/2)$.*

Proof. The angular function θ is given by $\cos\theta(s) = \delta \sin(s)$. By Proposition 4.7, we have that

$$\begin{aligned}\tilde{R}_D(\nu, s) &= \cos^2\theta \tilde{R}_M(\nu, r(s)) + \sin^2\theta \tilde{R}_T(\nu, r(s)) + E(\nu, s)\\ &= \delta^2 \sin^2 s \tilde{R}_M(\nu, r(s)) + (1 - \delta^2 \sin^2 s)\tilde{R}_T(\nu, r(s)) + E(\nu, s),\end{aligned}$$

where $\|E(\nu, s)\| \leq \delta \sin s(1 - \delta \sin s) C_1 - \frac{\delta \cos s}{\sqrt{1-\delta^2 \sin^2 s}} C_2$. Moreover, we conclude from Theorem 4.18 that for δ small enough there exists an $L > 0$ such that

$$\begin{aligned}\tilde{R}_T(\nu, r(s)) &= \frac{1}{r(s)^2} R_{\mathbb{E}^{k+1} \times \mathbb{S}^{n-k-1}} + \tilde{E}(\nu, s)\\ &= \frac{1}{\delta^2 \cos^2 s} R_{\mathbb{E}^{k+1} \times \mathbb{S}^{n-k-1}} + \tilde{E}(\nu, s)\end{aligned}$$

where $\|\tilde{E}(\nu, s)\| \leq \frac{L}{r(s)} = \frac{L}{\delta \cos s}$. Combined, we have

$$\begin{aligned}\tilde{R}_D(\nu, s) = \delta^2 \sin^2 s \tilde{R}_M(\nu, r(s)) &+ \frac{1 - \delta^2 \sin^2 s}{\delta^2 \cos^2 s} R_{\mathbb{E}^{k+1} \times \mathbb{S}^{n-k-1}}\\ &+ (1 - \delta^2 \sin^2 s)\tilde{E}(\nu, s) + E(\nu, s).\end{aligned}$$

If we choose δ small enough we ensure that $\lambda(\delta) := \frac{1-\delta^2 \sin^2 s}{\delta^2 \cos^2 s}$ is large enough such that $\lambda(\delta) R_{\mathbb{E}^{k+1} \times \mathbb{S}^{n-k-1}}$ is contained in C. Because $\|(1 - \delta^2 \sin^2 s)\tilde{E}(\nu, s)\|$ grows slower than the cone opens, for a δ small enough $\tilde{R}_D(\nu, s)$ is contained in C. ☐

Proof of Proposition 4.17. Note that $\gamma \in \tilde{\Gamma}(\overline{r})$ as obtained by Proposition 4.8 comes with a partition $0 = s_0 \leq s_1 \leq \cdots \leq s_6 = b$. We denote $t_i = t(s_i)$ and define $I_1 := (-\infty, t_4]$, $I_2 = [t_4, t_5]$ and $I_3 = [t_5, \infty)$. As before, let $D(\overline{r}) := \phi(\nu^{\leq \overline{r}} N)$ and define $S_j := D(\overline{r}) \times I_J$. This partitions D_γ into $D_j := D_\gamma \cap S_j$ for $j = 1, 2, 3$ with the following descriptions (cf. fig. 4)

$$\begin{aligned}D_1 :=& \{(\phi(r(s)\nu), t(s)) \mid 0 \leq s \leq s_4, \nu \in \nu^1 N\},\\ D_2 :=& \{(\phi(r(s)\nu), t(s)) \mid s_4 \leq s \leq s_5, \nu \in \nu^1 N\},\\ D_3 :=& \{(\phi(r(s)\nu), t(s)) \mid s_5 \leq s < b, \nu \in \nu^1 N\} \cup \{(p, t(b)) \mid p \in N\}.\end{aligned}$$

Let h^{torp} denote a connection metric on νN obtained from a torpedo metric (as constructed in § 2.4).

Let $\xi \in S$ and let h be the pull-back of a connection metric on νN, which is adjusted to ϕ on the r-tube, along $(\phi|_{\nu^{\leq \overline{r}} N})^{-1}$ to $\phi(\nu^{\leq \overline{r}} N)$.

Fix a small $\varepsilon > 0$ and define for $l \in [0, 1]$ a riemannian metric

$$G_{\xi, l} := \begin{cases} g_\xi + \mathrm{d}t^2 & \text{on } D(\overline{r}) \times (I_1 \cup [t_4, t_4 + \varepsilon)) \\ ((1-l)g_\xi + lh) + \mathrm{d}t^2 & \text{on } D(\overline{r}) \times ((t_5 - \varepsilon, t_5] \cup I_3). \end{cases}$$

For every $l \in [0, 1]$ continue this to a smooth metric on $D(\overline{r}) \times \mathbb{R}$ by choosing a family of smooth paths of metrics $P_l \colon [t_4 + \varepsilon, t_5 - \varepsilon] \to \mathcal{R}(M)$, which are adjusted to ϕ on some tube. By an application of Proposition 2.33, we can assume that the paths are through metrics adjusted to ϕ on the r_0-tube for some $r_0 < \overline{r}$. Moreover, we choose a path through connection metrics, in case g_ξ is a connection metric to begin with.

In total, we obtain a family of metrics $\{G_{\xi, l}\}_{(\xi, l) \in S \times [0,1]}$.

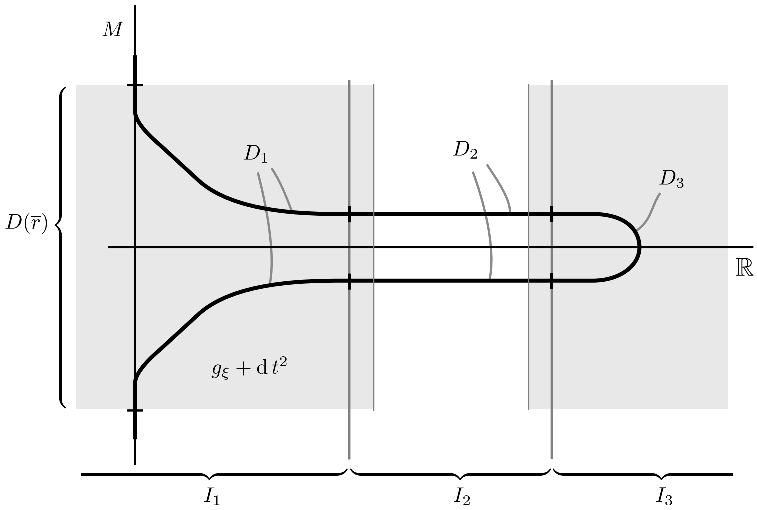


FIGURE 4. Partition of D_γ into D_1, D_2 and D_3.

Since the metrics $G_{\xi,l}$ restricted to $M \times \{t\}$ are adjusted to ϕ on the r_0-tube, D_2 is the distance tube around $S_2 \cap (N \times \{0\})$ for every $(\xi, l) \in S \times [0,1]$. Thus, by Theorem 4.18 the metric restricted to D_2 satisfies C, if $r(s_4) = r(s_5) =: \delta$ is small enough. This can be accomplished according to the construction of γ (cf. Proposition 4.8).

In S_3, the submanifold D_3 is determined by γ following the arc of a circle. Hence, we can apply Lemma 4.19 to see that for δ small enough C is satified for all $(\xi, l) \in S \times [0,1]$. Because h is a connection metric and modified by γ following an arc only radially, the metric induced on D_3 by $G_{\xi,1}$ is a connection metric, as well.

Therefore, the metric induced on D_γ from $g_{\xi,l}$ satisfies C and if we let $r_* := \overline{r}(1 - {}^{t_4}/_{b})$, then

$$\mathfrak{B}\colon S \times [0,1] \to \mathcal{R}_C(M), \quad (\xi, l) \mapsto \begin{cases} \mathfrak{A}(\xi, 2l) & \text{for } l \in [0, {}^1/_2], \\ \phi_\gamma^*(G_{\xi,2l-1}|_{D_\gamma}) & \text{for } l \in [{}^1/_2, 1] \end{cases}$$

has the properties claimed.

In particular, if $g_\xi \in \mathcal{R}_C^{\mathrm{rot}}(M)$, then by Remark 4.5 the metrics $\mathfrak{A}(\xi, \cdot\,)$ are contained in $\mathcal{R}_C^{\mathrm{rot}}(M)$. During the second part of the deformation the restriction of $G_{\xi,l}$ to $M \times \{t\}$ is a connection metric and thus $\mathfrak{B}(\xi, \cdot\,)$ is contained in $\mathcal{R}_C^{\mathrm{rot}}(M)$. □

The final step amounts to an adjustment of the metric we produced to the tubular map ϕ using a suitable radial diffeomorphism.

Proof of Proposition 3.2. Consider the deformation map $\mathfrak{B}$ as constructed in Proposition 4.17. We can choose a family of radial diffeomorphisms $\Phi\colon [0,1] \to \mathrm{Diff}(M)$, $t \mapsto \Phi_t$, such that

(1) $\Phi_0 \equiv \mathrm{id}_M$,
(2) Φ_t is the identity outside of $\phi(\nu^{\leq {}^3/_2 \overline{r}} N)$,
(3) $\Phi_1(\phi(\nu^{\leq \overline{r}} N)) \subset \phi(\nu^{\leq r_*} N)$.

Now define the final deformation map

$$\Pi\colon [0,1]\times S \to \mathcal{R}_C(M), \quad (l,\xi)\mapsto \begin{cases} \mathfrak{B}(2l,\xi) & \text{for } l\in[0,1/2] \\ \Phi^*_{2l-1}\mathfrak{B}(1,\xi) & \text{for } l\in[1/2,1] \end{cases}.$$

This map has the desired properties and in particular $\Pi(1,\xi) = \Phi_1^*\mathfrak{B}(1,\xi)$ is contained in $\mathcal{R}_C^{\mathrm{rot}}(M)$. □

§5. Rotationally symmetric metrics

The main goal of this section is the proof of Proposition 3.3, which will complete our proof of the main theorem. It will follow from an analysis of rotationally symmetric metrics on the disc. As before, we are generalizing the method laid out in Chernysh's [Che04] to the case of certain curvature conditions.

§5.1 Metrics on the disc. In the following fix a deformable, codimension c surgery stable curvature condition $C\subset \mathcal{C}_{\mathrm{B}}(\mathbb{E}^n)$, $\delta>0$ and $q\geq c$.

Definition 5.1. Define

$$\mathcal{R}^{\mathrm{rot}} := \{g\in\mathcal{R}(\mathbb{R}^q) \mid g=\alpha^2(t)\,\mathrm{d}t^2+\beta^2(t)g_{\mathbb{S}^{q-1}} \text{ and } \alpha(t)\neq 0\ \forall t, \beta\geq 0\}$$

in spherical coordinates $(0,\infty]\times S^{q-1}\cong S^q\setminus\{0\}$, where $\alpha,\beta\colon (0,\delta]\to\mathbb{R}$. Further, we denote by

$$\mathcal{R}_C^{\mathrm{rot}} := \{g\in\mathcal{R}^{\mathrm{rot}} \mid g_{\mathbb{E}^{n-q}}+g\in\mathcal{R}_C(\mathbb{R}^{n-q}\times\mathbb{R}^q)\}$$

the space of rotationally symmetric metrics satisfying C.

Definition 5.2. On $\mathcal{R}^{\mathrm{rot}}$ the *radius* defines a continuous map

$$\mathrm{rad}\colon \mathcal{R}^{\mathrm{rot}}\to\mathbb{R}_{>0}, \quad g\mapsto \int_0^\delta g(\gamma',\gamma')^{\frac12}\,\mathrm{d}t,$$

where γ is a radial path from the centre to a boundary point of $D^q(\delta)$. In the spherical coordinates $(0,\mathrm{rad}(g)]\times S^{n-1}\cong D^n(\delta)\setminus\{0\}$ the metric g is of the form $\mathrm{d}t^2+\beta g_{\mathbb{S}^{n-1}}$ for $\beta\colon (0,\mathrm{rad}\,g]\to\mathbb{R}$, which is the restriction of a smooth odd function $\beta\colon\mathbb{R}\to\mathbb{R}$ with $\beta'(0)=1$ and $\beta^{(\mathrm{even})}(0)=0$. The group of diffeomorphisms $\mathrm{Diff}_0(\mathbb{R})=\{\psi\in\mathrm{Diff}(\mathbb{R}) \mid \psi^{(k)}(0)=0\ \forall k\geq 0, \psi'(0)=1\}$ acts continuously on rotationally symmetric metrics as follows:

$$\mathrm{Diff}_0(\mathbb{R})\times\mathcal{R}^{\mathrm{rot}}\to\mathcal{R}^{\mathrm{rot}},$$
$$(\psi, g=\mathrm{d}t^2+\beta^2 g_{\mathbb{S}^{q-1}})\mapsto \psi\star g := \mathrm{d}t^2+(\beta\circ\psi)^2 g_{\mathbb{S}^{q-1}},$$

where the expression for $\psi\star g$ is understood in spherical co-ordinates $(0,\psi^{-1}(\mathrm{rad}(g))]\times S^{q-1}\cong D^n(\delta)\setminus\{0\}$.

Remark 5.3. (1) For the radius we have $\mathrm{rad}(\psi\star g)=\psi^{-1}(\mathrm{rad}\,g)$.
(2) If R^β is the curvature operator corresponding to $g=\mathrm{d}t^2+\beta^2 g_{\mathbb{S}^{q-q}}$ (cf. (2.18)), then $R^{\beta\circ\psi}$ is the curvature operator corresponding to $\psi\star g$.

Proposition 5.4. *In the situation of the above definition there exists a continuous function* $\sigma\colon\mathcal{R}_C^{\mathrm{rot}}\to(0,\frac{\delta}{2}]$ *and a continuous deformation of the identity* $\Psi_1\colon\mathcal{R}_C^{\mathrm{rot}}\times[0,1]\to\mathcal{R}_C^{\mathrm{rot}}$ *with*

(1) $\Psi_1(\,\cdot\,,0)\equiv\mathrm{id}$,
(2) $\Psi_1(\,\cdot\,,s)\equiv\mathrm{id}$ *near* $\partial D^q(\delta)$ *for all* $s\in[0,1]$,
(3) $\Psi_1(g,1)=\mathrm{d}t^2+\beta(t)g_{\mathbb{S}^{q-1}}$ *with*
  (i) $\beta^{(l)}|_{\sigma(g)}=0$ *for all* $l\geq 1$,
  (ii) $0\leq\beta'\leq 1$ *and* $\beta''\leq 0$ *on* $[0,\sigma(g)]$.

The proof will be based on the observation that the warping function of a metric $g \in \mathcal{R}_C^{\mathrm{rot}}$ behaves in a certain way near 0 combined with explicit deformations of the metric corresponding to this region. We will separate these steps in the following lemmata.

Lemma 5.5. *For every $g = \mathrm{d}t^2 + \beta^2 g_{\mathbb{S}^{q-1}} \in \mathcal{R}_C^{\mathrm{rot}}$ there exists a t^* such that $0 < \beta' < 1$ and $\beta'' < 0$ on $(0, t^*]$.*

Proof. Assume this would not be the case, i.e. there exists a point t_0 with $\beta' \geq 1$ or $\beta'' \geq 0$ on $[0, t_0]$. Since we have $\beta'(0) = 1$ and $\beta''(0) = 0$, both inequalities are actually equivalent to convexity of β on the interval $[0, t_0]$. The curvature operator corresponding to β is

$$R^\beta = \underbrace{\frac{1 - \beta'^2}{\beta^2}}_{=:\lambda(\beta)} R_{\mathbb{E}\times\mathbb{S}^{q-1}} \underbrace{-\frac{\beta''}{\beta}}_{=:\mu(\beta)} L_q \tag{5.6}$$

and for $t \in [0, t_0]$ we thus have $\lambda(\beta)(t) \leq 0$ and $\mu(\beta)(t) \leq 0$.
Since we know that C satisfies an inner ray condition with respect to $s(R_{\mathbb{E}\times\mathbb{S}^{q-1}} + \mu L)$ for $\mu \geq 0$ and $s > 0$, we conclude that with $\lambda \leq 0$, $\mu \leq 0$ the curvature condition C would contain the flat curvature operator 0, which is a contradiction to our assumption that C is deformable. □

Lemma 5.7 ([Che04, Lemma 3.5])**.** *For $0 < C_1 \leq 1$, $0 < t^*$, $0 < t_* < 1/2\ t^*$, $s \in [0, 1]$ let $0 < a = a(t_*, t^*) < b = b(t_*, t^*) < t_*$ be continuous functions. There exist continuous functions $0 < C_2 = C_2(C_1, t^*) \leq 1$, $e = e(s)$ and an isotopy through diffeomorphisms $\phi_s\colon \mathbb{R} \to \mathbb{R}$ such that for $b < c = \phi_s^{-1}(8/10\ t^*) < d = \phi_s^{-1}(9/10\ t^*) < e$*

(i) $\phi_0 \equiv \mathrm{id}$,
(ii) $\phi_s(0) = 0$ *and* $\phi_s(e(s)) = t^*$,
(iii) $\phi_s'|_{[0,a]\cup[d,\infty)} \equiv 1$ *and* $\phi_s'|_{[b,c]} \equiv 1 - sC_2$, $\phi_s^{(n)}|_{[b,c]} \equiv 0$ *for all* $n \geq 2$,
(iv) $0 \leq 1 - C_2 \leq \phi_s' \leq 1$,
(v) $\phi_s'' \leq C_1$,
(vi) $\phi_s''|_{[a,b]} \leq 0$ *and* $\phi_s''|_{[c,d]} \geq 0$.

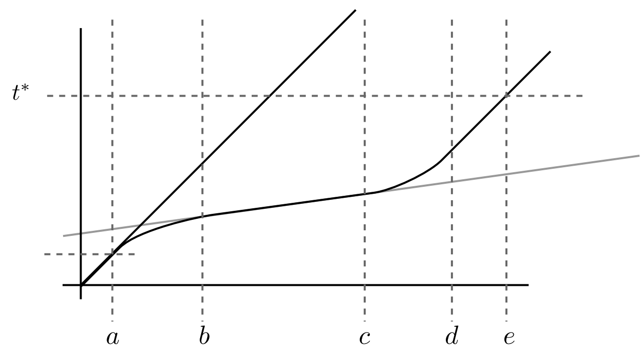


FIGURE 5. First deformation family as described in Lemma 5.7

Lemma 5.8 ([Che04, Lemma 3.7])**.** *Let $0 < D_1 \leq 1$, $0 < t^{**}$, $s \in [0, 1]$. There exist continuous functions $0 < \bar{b} = \bar{b}(D_1, t^{**}) < \frac{t^{**}}{2}$, $\bar{e} = \bar{e}(s)$ and an isotopy through diffeomorphisms $\psi_s\colon \mathbb{R} \to \mathbb{R}$ such that for $0 < \bar{a} = 1/10\ \bar{b} < \bar{c} = \psi_s^{-1}(9/10\ t^{**}) < \bar{e}$*

(i) $\psi_0 \equiv \mathrm{id}$,

(ii) $\psi_s(0) = 0$, $\psi_s(\overline{d}(s)) = t^{**}$,
(iii) $0 \leq \psi_s' \leq 1$,
(iv) $\psi_s'|_{[0,\overline{a}]\cup[\overline{c},\infty)} \equiv 1$,
(v) $\psi_s''|_{(0,\infty)}(t) \leq {}^{D_1}/_t$,
(vi) $\psi_1^{(n)}(\overline{b}(1)) = 0$ *for all* $n \geq 1$.

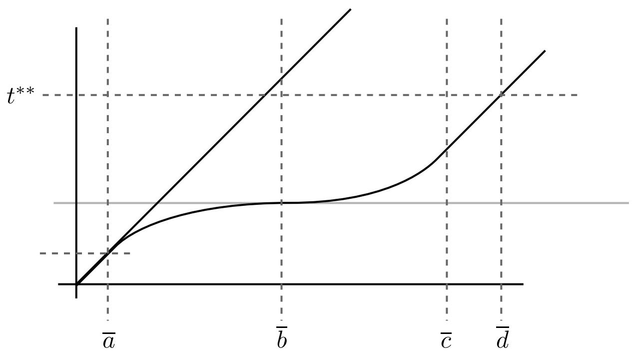


FIGURE 6. Second deformation family as described in Lemma 5.8

The proof of these two lemmata is a straight forward explicit construction by means of ordinary differential equations.

Proof of Proposition 5.4. We consider $g = \mathrm{d}t^2 + \beta^2 g_{\mathbb{S}^{q-1}} \in \mathcal{R}_C^{\mathrm{rot}}$. By Lemma 5.5, there exists a t^* such that $0 < \beta' < 1$ and $\beta'' < 0$ on $(0, t^*]$. Without loss of generality, we can assume that $t^* \leq {}^{\delta}/_2$.
Now we invoke Lemma 5.7 with the parameters C_1, t^* and the Lemma 5.8 with the parameters D_1, t^{**} to define

$$\Psi_1 \colon (g, s) \mapsto \begin{cases} \phi_{2s} \star \beta & \text{for } s \in [0, {}^1/_2] \\ (\phi_1 \circ \psi_{2s-1}) \star \beta & \text{for } s \in [{}^1/_2, 1] \end{cases}.$$

We need to make sure that during both parts of the deformation the corresponding metrics stay within the space $\mathcal{R}_C^{\mathrm{rot}}$.

(1) We need to make sure that the rotationally symmetric metric given by the warping function $\beta \circ \phi_{2s}$, which has the curvature operator
$$R^{\beta\circ\phi_{2s}} = \frac{1 - \beta'(\phi_{2s})^2 \phi_{2s}'^2}{\beta(\phi_{2s})^2} R_{\mathbb{E}\times\mathbb{S}^{q-1}} - \frac{\beta''(\phi_{2s})\phi_{2s}'^2 + \beta'(\phi_{2s})\phi_{2s}''}{\beta(\phi_{2s})} L_q, \tag{5.9}$$
satisfies C, as well. On $[0, a]$, we have $\phi_{2s} \equiv \mathrm{id}$ and there is nothing to check. On $[d, \phi_{2s}^{-1}(\mathrm{rad}\, g)]$ we have $\phi_{2s}' \equiv 1$ and $\phi_{2s}'' \equiv 0$, thus, $R^{\beta\circ\phi_{2s}}$ at t equals R^β at $\phi_{2s}^{-1}(t)$, which of course satisfies C.
On $[b, c]$ we have
$$\begin{aligned} R^{\beta\circ\phi_{2s}} &= \frac{1 - \beta'(\phi_{2s})^2(1 - sC_2)^2}{\beta(\phi_{2s})^2} R_{\mathbb{E}\times\mathbb{S}^{q-1}} - \frac{\beta''(\phi_{2s})(1 - sC_2)^2}{\beta(\phi_{2s})} L_q \\ &= (1 - sC_2)^2 R^\beta|_{\psi_{2s}} \in C. \end{aligned}$$
In the remaining part $[a, b] \cup [c, d]$ we will use the deformability (cf. Definition 2.19) of the curvature condition C. We can conclude that the first summand in (5.9) is contained in C because $\frac{1-\beta'^2(\phi_{2s})\phi_{2s}'^2}{\beta^2(\phi_{2s})} > 0$. The whole sum is clearly contained in C, if $-\frac{\beta''(\phi_{2s})\phi_{2s}'^2}{\beta(\phi_{2s})} - \frac{\beta'(\phi_{2s})\phi_{2s}''}{\beta(\phi_{2s})} \geq 0$, i.e. if $-\frac{\beta''(\phi_{2s})\phi_{2s}'^2}{\beta'(\phi_{2s})} \geq$

ϕ''_{2s}. Since $\beta'' < 0$, $1 - C_2(C_1, t^*) \leq \phi'_{2s} \leq 1$ and $0 < \beta' < 1$, we can choose C_1 in Lemma 5.7 such that this is satisfied.

(2) Now we need to conclude the same for $\beta \circ \phi_1 \circ \psi_{2s-1}$, where this time we denote $\tilde{\beta} := \beta \circ \phi_1$. We let $b < t^{**} < c$ from Lemma 5.7. The curvature operator here is

$$R^{\beta\circ\phi_1\circ\psi_{2s-1}} = \frac{1 - (\tilde{\beta}')^2(\psi_{2s-1})(\psi'_{2s-1})^2}{\tilde{\beta}^2(\psi_{2s-1})} R_{\mathbb{E}\times\mathbb{S}^{q-1}} - \frac{\tilde{\beta}''(\psi_{2s-1})\psi'^2_{2s-1} + \tilde{\beta}'(\psi_{2s-1})\psi''_{2s-1}}{\tilde{\beta}(\psi_{2s-1})} L_q \tag{5.10}$$

again, by the same reasoning, the first summand is contained in C. And again on $[0, \overline{a}] \cup [\overline{c}, (\phi_1 \circ \psi_{2s-1})^{-1}(\operatorname{rad} g)]$ there is nothing to check. Within $[\overline{a}, \overline{c}]$ by deformability, the sum in (5.10) would be contained in C, if $-\frac{\tilde{\beta}''\psi'^2_{2s-1}}{\tilde{\beta}} \geq \psi''_{2s-1}$. Unfortunately, the left-hand side of this inequality is not necessarily strictly positive and hence can't be used to saturate D_1 in Lemma 5.8. Using the fact that C satisfies an inner cone condition with respect to $R_{\mathbb{E}\times\mathbb{S}^{q-1}}$, we conclude that actually the factor in front of L_q has to be larger than a negative constant depending on the cone opening.

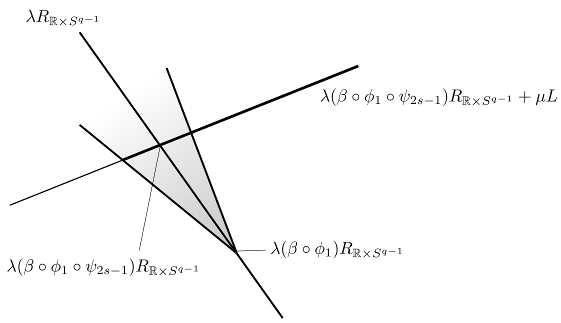


FIGURE 7. Intersection of the cone condition with the $\lambda R_{\mathbb{E}\times\mathbb{S}^{q-1}} + \mu L_q$ plane. Here $\lambda(\beta) := \frac{1-\beta'^2}{\beta^2}$.

On $[\overline{a}, \overline{c}]$ we have $\tilde{\beta}'' = \beta''(\phi_1)\phi_1'^2 + \beta'(\phi_1)\phi_1'' < 0$, because $\phi_1'' \leq 0$, $\beta'' < 0$, $0 < \phi_1' \leq 1$ and $0 < \beta' < 1$, as well as $0 < \tilde{\beta}' = \beta'(\phi_1)\phi_1' < 1$. Now eq. (5.10) is contained in C, if

$$-\frac{\tilde{\beta}''(\psi_{2s-1})\psi'^2_{2s-1} + \tilde{\beta}'(\psi_{2s-1})\psi''_{2s-1}}{\tilde{\beta}} \geq -C^{**},$$

where $C^{**} > 0$ depends on the cone opening $\rho(\lambda(\beta\circ\phi_1)R_{\mathbb{E}\times\mathbb{S}^{q-1}}) > 0$ (cf. Definition 2.6 and fig. 7) and the difference $\lambda(\beta\circ\phi_1\circ\psi_{2s-1}) - \lambda(\beta\circ\phi_1) > 0$. This is equivalent to

$$\frac{\tilde{\beta}(\psi_{2s-1})C^{**} - \tilde{\beta}''(\psi_{2s-1})\psi'^2_{2s-1}}{\tilde{\beta}'} \geq \psi''_{2s-1},$$

where the left-hand side is strictly positive, which enables us to choose D_1 such that $R^{\beta\circ\phi_1\circ\psi_{2s-1}}$ is contained in C.

Finally, let $\sigma(g) := b$ from Lemma 5.8. □

Definition 5.11. Let $g_{\mathrm{torp}} = \mathrm{d}t^2 + \beta_{\mathrm{torp}}(t)^2 g_{\mathbb{S}^{n-k-1}}$ be the torpedo metric on $D^{n-k}(\delta)$. We define

$$\mathcal{R}_C^{\mathrm{loc}} := \{g \in \mathcal{R}_C^{\mathrm{rot}} \mid g = \left(\tfrac{\delta}{\delta^*}\right)^2 \mathrm{d}t^2 + \beta_{\mathrm{torp}}\left(\tfrac{\delta}{\delta^*}t\right)^2 g_{\mathbb{S}^{q-1}} \text{ with } \delta^* \in (0,\delta]\}$$

and refer to it as the *space of locally torpedo metrics.* The radius δ^* on which a metric $g \in \mathcal{R}_C^{\mathrm{loc}}$ is torpedo yields a continuous map $\delta^*\colon \mathcal{R}_C^{\mathrm{loc}} \to (0,\delta]$.

Proposition 5.12. *In the situation of Proposition 5.4 there exists a continuous map*

$$\Psi_2\colon \mathcal{R}_C^{\mathrm{rot}} \times [0,1] \to \mathcal{R}_C^{\mathrm{rot}}$$

such that

(1) $\Psi_2(\,\cdot\,,0) \equiv \mathrm{id}$,
(2) $\Psi_2(\,\cdot\,,s) \equiv \mathrm{id}$ *near* $\partial D^q(\delta)$ *for all* $s \in [0,1]$,
(3) $\Psi_2(g,1) = \left(\frac{\delta}{\sigma}\right)^2 \mathrm{d}t^2 + \beta_{\mathrm{torp}}\left(\frac{\delta}{\sigma}t\right)^2 g_{\mathbb{S}^{q-1}}$ *on* $D^q(\sigma)$,

Proof. For the first part of the deformation we use Ψ_1, i.e.

$$\Psi_2\colon (g,s) \mapsto \Psi_1(g,2s) \text{ for } s \in [0,1/2].$$

Now let $\Psi_1(g,1) = \mathrm{d}t^2 + \beta^2 g_{\mathbb{S}^{q-1}}$. We will describe a deformation on $D^q(\sigma(g))$. For $u \in [0,1]$ let $\beta_u := (1-u)\beta + u\beta_{\mathrm{torp}}$ we observe the following. If $\beta(t) \le \beta_u(t) \le \beta_{\mathrm{torp}}(t)$, we have

$$\frac{1-\beta_{\mathrm{torp}}'^2(t)}{\beta_{\mathrm{torp}}(t)} \le \frac{1-\beta_u'^2(t)}{\beta_u(t)} \le \frac{1-\beta'^2(t)}{\beta(t)},$$

since $0 \le \beta' \le 1$ and $0 \le \beta'_{\mathrm{torp}} \le 1$ (we obtain the opposite inequalities if $\beta(t) \ge \beta_u(t) \ge \beta_{\mathrm{torp}}(t)$). Moreover

$$\frac{\beta_u''}{\beta_u} = \frac{(1-u)\beta + u\beta_{\mathrm{torp}}}{(1-u)\beta + u\beta_{\mathrm{torp}}} \le 0,$$

because $\beta'' \le 0$ and $\beta_{\mathrm{torp}} \le 0$. Hence, by deformability β_u is contained in C and since we have created a collar at $\sigma(g)$ (i.e. $\beta^{(l)}(\sigma(g)) = 0$ for all $l \ge 1$), we can deform β on a small interval to connect $\beta_u(\sigma(g))$ with $\beta(\sigma(g))$. Denote the result by $\tilde\beta_u$ and define $\Psi_2\colon (g,s) \mapsto \mathrm{d}t^2 + \tilde\beta_{2s-1}^2 g_{\mathbb{S}^{q-1}}$ for $s \in [1/2,1]$. ☐

§5.2 Rotationally symmetric metrics around a submanifold. Finally, we return to the case of an embedded submanifold with the assumptions as listed in Theorem 3.1.

Recall that we can use $g_N, h^{\nu N}, \omega$ and a rotationally symmetric metric g_{rot} on $\mathbb{R}^{n-k}$ to define a connection metric $h^\nabla(g_{\mathrm{rot}}) := g_N \oplus_\omega g_{\mathrm{rot}}$ on νN.

We note that every rotationally symmetric metric around N in the sense of Definition 2.34 is given by a rotationally symmetric on the disc, that is contained in $\mathcal{R}_C^{\mathrm{rot}}$, in particular. More precisely, for $\delta = R$ in Definition 5.1 we have for $g \in R_C^{\mathrm{rot}}(M)$ that $\phi|_{\nu^{\le R}N}^* g = h^\nabla(g_{\mathrm{rot}})|_{\nu^{\le R}N}$, where $g_{\mathrm{rot}}|_{D^{n-k}(R)} \in \mathcal{R}_C^{\mathrm{rot}}$. As g is adjusted to ϕ on the R-tube, we have that $g_{\mathrm{rot}} = \mathrm{d}r^2 + \beta(r)^2 g_{\mathbb{S}^{n-k-1}}$.

We note that a continuous alteration of g_{rot}, which does not alter the metric near $S^{n-k}(R) \subset \mathcal{R}^{n-k}$ results in a continuous alteration of the metric g near N.

Definition 5.13. Analogous to Definition 5.11 we define

$$\mathcal{R}_C^{\mathrm{loc}}(M) := \{g \in \mathcal{R}_C(M) \mid \phi|_{\nu^{\le R}N}^* g = h^\nabla(g_{\mathrm{rot}})|_{\nu^{\le R}N} \text{ with } g_{\mathrm{rot}}|_{D^{n-k}(R)} \in \mathcal{R}_C^{\mathrm{loc}}\}.$$

Lemma 5.14. *There exists a continuous deformation map* $\Phi\colon \mathcal{R}_C^{\mathrm{loc}}(M) \times [0,1] \to \mathcal{R}_C^{\mathrm{loc}}(M)$ *such that*

(i) $\Phi(\,\cdot\,,0) \equiv \mathrm{id}$,
(ii) $\Phi(\,\cdot\,,1) \in \mathcal{R}_C^{\mathrm{torp}}(M)$.

Proof. For a small $\varepsilon > 0$ choose a family of radial diffeomorphisms on the disc $\phi\colon D^{n-k}(R+\varepsilon) \times (0,R] \times [0,1] \to D^{n-k}(R+\varepsilon)$ such that

(i) $\phi(\,\cdot\,,r,0) \equiv \mathrm{id}$,
(ii) $\phi(x,r,1) = \frac{r}{R}x$ on $x \in D^{n-k}(R)$.
(iii) Denote by $\tilde{\delta} = \tilde{\delta}(r)$ the value where $\phi(\,\cdot\,,r,1)^{-1}(S^{n-k}(R)) = S^{n-k}(\tilde{\delta})$. If $\tilde{\delta} \in [\delta, \delta+\varepsilon]$, then $\phi(\,\cdot\,,r,1)$ is a radial euclidean isometry in a tubular neighbourhood of $S^{n-k}(\tilde{\delta})$.

These give rise to a family of diffeomorphisms on the normal bundle $\phi\colon \nu^{\leq R+\varepsilon}N \times (0,R] \times [0,1] \to \nu^{\leq R+\varepsilon}N$. Further, by Definition 5.11 we obtain a continuous function $\delta^*\colon \mathcal{R}_C^{\mathrm{loc}}(M) \to (0,\delta]$ such that we can define

$$\Phi(g,s) \coloneqq \phi(\,\cdot\,,\delta^*(g),s)^* g,$$

which is the deformation map we were seeking to construct. □

Now we are set up to give a proof of Proposition 3.3, i.e. to show that $\mathcal{R}_C^{\mathrm{torp}}(M)$ is a weak deformation retract of $\mathcal{R}_C^{\mathrm{rot}}(M)$.

Proof of Proposition 3.3. The idea is to construct a retraction map

$$r\colon \mathcal{R}_C^{\mathrm{rot}}(M) \xrightarrow{\Psi} \mathcal{R}_C^{\mathrm{loc}}(M) \xrightarrow{\Phi} \mathcal{R}_C^{\mathrm{torp}}(M),$$

where Ψ is a replacement of the rotationally symmetric metric by local torpedo metric, while Φ is induced by a family of radial diffeomorphisms on the disc.

The map Ψ_2 constructed in Proposition 5.12 gives rise to a continuous deformation map $\Psi\colon \mathcal{R}_C^{\mathrm{rot}}(M) \times [0,1] \to \mathcal{R}_C^{\mathrm{rot}}(M)$ with $\Psi(\,\cdot\,,1) \subset \mathcal{R}_C^{\mathrm{loc}}(M)$ by replacing the rotationally symmetric metric in the connection metric with the metrics obtained during the deformation Ψ_2. Now use this Ψ and the map Φ constructed in Lemma 5.14 to define $r \coloneqq \Phi(\,\cdot\,,1) \circ \Psi_2(\,\cdot\,,1)$. The homotopy $i \circ r \simeq \mathrm{id}_{\mathcal{R}_C^{\mathrm{rot}}(M)}$, where i is the inclusion, is given by

$$(g,s) \mapsto \begin{cases} \Psi(g,2s) & \text{for } s \in [0, 1/2] \\ \Phi(\Psi(g,1), 2s-1) & \text{for } s \in [1/2, 1] \end{cases}.$$

For the other homotopy $r \circ i \simeq \mathrm{id}_{\mathcal{R}_C^{\mathrm{torp}}(M)}$, we note that $\Psi_2(\,\cdot\,,1)$ does not alter the torpedo metric within the annulus $D^{n-k}(R/2, R)$, since $t^* \leq R/2$ in the proof of Proposition 5.4. Let $g \in \mathcal{R}_C^{\mathrm{torp}}(M)$. On $D^{n-k}(R)$ the metric $r(g)$ is torpedo, i.e. agrees with $\tau^* g$, while on the annulus $D^{n-k}(R, R+\varepsilon)$ the metric might not agree with the original one. This can be resolved by straightening this to a collar region and shrinking it down afterwards, which is possible because $\tilde{\delta} \in [R, R+\varepsilon]$ in the proof of Lemma 5.14. □

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