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ON THE SPACE OF VECTOR-VALUED FUNCTIONS INTEGRABLE WITH RESPECT TO THE WHITE NOISE

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1. Introduction and notation. In this paper E denotes a Banach space with a norm $\|\cdot\|$, and E' stands for the dual space of E. Let $(\Omega, \mathscr{F}, P)$ be a probability space. A random vector $X\colon \Omega \to E$ denotes a strongly measurable function.

$L^p(\Omega, P; E)$, $0 \leqslant p \leqslant \infty$, denotes the Fréchet space (Banach space if $1 \leqslant p \leqslant \infty$) of random vectors $X\colon \Omega \to E$ for which

$$\|X\|_0 \overset{\text{df}}{=} \mathrm{E}\frac{\|X\|}{1+\|X\|} \quad \text{if } p = 0,$$

$$\|X\|_p \overset{\text{df}}{=} (\mathrm{E}\|X\|^p)^{1/r} < \infty \quad \text{if } 0 < p < \infty,\ r = \max\{1, p\},$$

and

$$\|X\|_\infty \overset{\text{df}}{=} \operatorname*{ess\,sup}_{\Omega}\|X\| < \infty \quad \text{if } p = \infty.$$

A random vector is called *symmetric* if $P(X \in A) = P(-X \in A)$ for every $A \in \mathscr{B}_E$, where $\mathscr{B}_E$ is the Borel σ-algebra on E.

A probability measure on $(E, \mathscr{B}_E)$ defined by

$$\mu(A) = P(X \in A) \quad \text{for every } A \in \mathscr{B}_E$$

is called the *distribution law* of X. The characteristic functional of a measure ν on $(E, \mathscr{B}_E)$ is defined by

$$\hat{\nu}(x') = \int_E \exp[i\langle x', x\rangle]\nu(dx) \quad \text{for every } x' \in E'.$$

A random vector X is *gaussian* if, for each $x' \in E'$, $\langle x', X\rangle$ is a gaussian random variable. X is *pregaussian* if there exists a gaussian measure γ on $(E, \mathscr{B}_E)$ such that

$$\hat{\gamma}(x') = \exp\left[-\frac{1}{2}E\langle x', X\rangle^2\right].$$

Let (S, Σ, μ) be a measure space with $\mu(S) = 1$. A mapping

$$W: \Sigma \to L^0(\Omega, \mathcal{F}, P)$$

is called a *gaussian random measure* on (S, Σ, μ) if

(a) for every sequence $A_1, A_2, \ldots$ of disjoint sets from Σ we have

$$W\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} W(A_n),$$

where the series converges with probability 1;

(b) for every sequence $A_1, \ldots, A_n$ of disjoint sets from Σ the random variables $W(A_1), \ldots, W(A_n)$ are independent;

(c) for every $A \in \Sigma$, $W(A)$ has a normal distribution with mean 0 and variance $\mu(A)$.

Let $f\colon S \to E$ be a simple function,

$$f = \sum_{i=1}^{n} x_i \mathbf{1}_{A_i},$$

where $A_i \in \Sigma$ are disjoint, $x_i \in E$, $i = 1, \ldots, n$. For each $B \in \Sigma$ we set

$$\int_B f\,dW = \sum_{i=1}^{n} x_i W(A_i \cap B).$$

$\int_S (\cdot)\,dW$ is a linear operator on the vector space of E-valued simple functions on S with values in the space of gaussian random vectors in $L^2(\Omega, P; E)$. If E is of type 2 (see Section 5 for the definition of Banach spaces of type 2), then, as Hoffmann-Jørgensen and Pisier [5] have shown, there exists a unique extension of this operator on $L^2(S, \mu; E)$. Using the idea of Urbanik and Woyczyński [13] we define a random integral of vector-valued functions with values in any Banach space.

The purpose of this paper is to study the class of random integrable functions with respect to a gaussian random measure. Section 2 of this paper contains the basic properties of random integrable functions. In Section 3 we give some counterexamples which show the difference between the random integral for Banach space valued functions and the random integral for Hilbert space valued functions. Section 4 contains a characterization of random integrable functions. In Section 5 we study properties of the random integral which depends on geometry of a Banach space. In Section 6 we investigate some properties of the space of functions which are integrable with respect to a gaussian random measure.

2. A gaussian random integral of vector-valued functions. Let (S, Σ, μ) be a measure space $\mu(S) = 1$ and let W be a gaussian random measure on (S, Σ, μ). Let E be a Banach space and let $f\colon S \to E$ be a simple function, i.e.

$$f = \sum_{i=1}^{n} x_i 1_{A_i},$$

where $A_i \in \Sigma$ are pairwise disjoint and $x_i \in E$. For every $B \in \Sigma$ we set

$$\int_B f dW \overset{\text{df}}{=} \sum_{i=1}^{n} x_i W(A_i \cap B).$$

Definition 2.1. A strongly measurable function $f\colon S \to E$ is said to be *integrable with respect to a gaussian random measure* W if there exists a sequence of simple functions $f_n\colon S \to E$ such that

(1) $f_n \to f$ in μ,

(2) $\int_B f_n dW$ converges in P for every $B \in \Sigma$.

Then for $B \in \Sigma$ we set

$$\int_B f dW \overset{\text{df}}{=} P\text{-lim} \int_B f_n dW.$$

This integral is uniquely determined. Definition 2.1 is the extension of the definition of Urbanik and Woyczyński (cf. [13]) of random integral in the case of Banach valued functions.

Let $\mathscr{L}(S, W; E) \subset L^0(S, \mu; E)$ denote the set of all integrable functions with respect to the gaussian random measure W. The set $\mathscr{L}(S, W; E)$ is a vector space. Moreover, $\mathscr{L}(S, W; E)$ is a Fréchet space with F-norm

$$|||f|||_0 = \|f\|_0 + \Big\| \int_S f dW \Big\|_0,$$

and the set of simple functions is dense in $\mathscr{L}(S, W; E)$.

The following properties are immediate consequences of Definition 2.1 and we omit their proofs.

PROPOSITION 2.1. (1) *For every* $f, g \in \mathscr{L}(S, W; E)$ *and* $B \in \Sigma$ *we have*

$$\int_B (f+g) dW = \int_B f dW + \int_B g dW \quad P\text{-a.e.}$$

(2) *Let* E, F *be Banach spaces and let* $A\colon E \to F$ *be a continuous linear operator. If* $f \in \mathscr{L}(S, W; E)$, *then* $Af \in \mathscr{L}(S, W; F)$ *and*

$$A \int_B f dW = \int_B Af dW \quad P\text{-a.e.} \quad \text{for each } B \in \Sigma.$$

(2′) *In particular, if* $x' \in E'$, *then for every* $f \in \mathscr{L}(S, W; E)$ *we have* $\langle x', f \rangle \in \mathscr{L}(S, W; \mathbf{R})$ *and*

$$\Big\langle x', \int_B f dW \Big\rangle = \int_B \langle x', f \rangle dW \quad P\text{-a.e.}$$

(3) *If $f \in \mathcal{L}(S, W; E)$, then for every $B \in \Sigma$ we have $f1_B \in \mathcal{L}(S, W; E)$ and*

$$\int_B f dW = \int_S f 1_B dW.$$

PROPOSITION 2.2. *If $(f_n) \subset L^0(S, \mu; E)$ is a sequence of simple functions such that $f_n \to f$ in μ and $\int_S f_n dW$ converges in P, then $f \in \mathcal{L}(S, W; E)$.*

Proof. Let $B \in \Sigma$. The random vectors

$$\int_B (f_n - f_m) dW \quad \text{and} \quad \int_{S \setminus B} (f_n - f_m) dW$$

are independent and symmetric. By the inequality

$$P\left(\left\|\int_B (f_n - f_m) dW\right\| > \varepsilon\right) \leqslant 2P\left(\left\|\int_S (f_n - f_m) dW\right\| > \varepsilon\right)$$

for every $\varepsilon > 0$, we infer that $\int_B f_n dW$ converges in P.

In the sequel we shall use some other F-norms in the space $\mathcal{L}(S, W; E)$, equivalent to the original one. First we prove the following lemma:

LEMMA 2.1. *Let X_n be symmetric gaussian random vectors such that $X_n \to X$ in P. Then X is a symmetric gaussian random vector and, for every p, $0 < p < \infty$,*

$$\mathrm{E}\|X_n - X\|^p \to 0 \quad \textit{as } n \to \infty.$$

Proof. Let Y be a symmetric gaussian random vector. Combining the results of [3], [4] and [7], we infer that for every $p, q \in (0, \infty)$ there exists a constant $C_{p,q}$ (dependent only on E) such that

$$(1) \qquad (\mathrm{E}\|Y\|^p)^{1/p} \leqslant C_{p,q}(\mathrm{E}\|Y\|^q)^{1/q}.$$

Inequality (2) from I.5 in [8] applied to $\|Y\|^2$ gives

$$P(\|Y\|^2 > t\mathrm{E}\|Y\|^2) \geqslant (1-t)^2 \frac{(\mathrm{E}\|Y\|^2)^2}{\mathrm{E}\|Y\|^4} \geqslant (1-t)^2 C_{4,2}^{-4}$$

for every $t \in (0, 1)$. Putting $Y = X_n$ and $t = 1/2$ we get

$$(2) \qquad P\left(\|X_n\|^2 > \frac{1}{2}\mathrm{E}\|X_n\|^2\right) \geqslant 4^{-1} C_{4,2}^{-4} = \text{const} \quad \text{for every } n \in N.$$

By the assumptions of the lemma and inequality (2) we have

$$\sup_n \mathrm{E}\|X_n\|^2 < \infty,$$

and, by (1),

$$\sup_n \mathrm{E}\|X_n\|^p < \infty \quad \text{for each } p \geqslant 1.$$

By the Fatou lemma, $\mathrm{E}\|X\|^p < \infty$ for each $p > 0$. Let $p \in (0, \infty)$ be fixed. Using the Hölder inequality we infer that for each $\varepsilon > 0$

$$\mathrm{E}\|X_n - X\|^p \leqslant \int_{\{\|X_n - X\|^p > \varepsilon\}} \|X_n - X\|^p dP + \int_{\{\|X_n - X\|^p \leqslant \varepsilon\}} \|X_n - X\|^p dP$$
$$\leqslant C^{p/(p+1)} [P\{\|X_n - X\|^p > \varepsilon\}]^{p/(p+1)} + \varepsilon$$

where

$$C = \sup_n \mathrm{E}\|X_n\|^{p+1}.$$

Therefore, $\mathrm{E}\|X_n - X\|^p \to 0$ as $n \to \infty$. The fact that X is gaussian is trivial.

By Lemma 2.1 and Definition 2.1 we obtain immediately

COROLLARY 2.1. *Let for every $f \in \mathscr{L}(S, W; E)$*

$$|||f|||_p \overset{\text{df}}{=} \|f\|_0 + \Big\| \int_S f dW \Big\|_p, \quad 0 < p < \infty.$$

Then for every $p \in (0, \infty)$ the F-norm $|||\cdot|||_p$ is equivalent to $|||\cdot|||_0$.

Remark 2.1. In the definition of $|||\cdot|||_p$ the first component $\|\cdot\|_0$ cannot be omitted in general (see Example 3.3 in Section 3).

3. Some counterexamples. The examples given in the sequel show that the basic properties of the space $\mathscr{L}(S, W; E)$ and of the random integral, which are evidently fulfilled in Hilbert spaces, are not usually fulfilled in arbitrary Banach spaces.

PROPOSITION 3.1. *Let*

$$f = \sum_{n=1}^{\infty} x_n 1_{A_n},$$

where $x_n \in E$, $A_n \in \Sigma$ are disjoint, $n = 1, 2, \ldots$ and $\bigcup_{n=1}^{\infty} A_n = S$. Then $f \in \mathscr{L}(S, W; E)$ if and only if $\sum_{n=1}^{\infty} x_n W(A_n)$ converges a.s. Moreover,

$$\int_S f dW = \sum_{n=1}^{\infty} x_n W(A_n).$$

The proposition is an immediate consequence of Proposition 2.2 and the theorem of Ito and Nisio [6].

Let $S = [0, 1]$, $\Sigma = \mathscr{B}_{[0,1]}$, $\mu(dt) = dt$ and let W be the random measure generated by the Brownian motion w_t on $[0, 1]$, i.e. $W((s, t]) = w_t - w_s$ for $s, t \in [0, 1]$.

Example 3.1. *There exist a Banach space E and f: $[0, 1] \to E$ strongly measurable such that*

$$\sup_{t \in [0,1]} \|f(t)\| = 1 \quad \textit{and} \quad f \notin \mathscr{L}([0, 1], W; E).$$

Let $E = l^p$, $1 \leqslant p < 2$, and let, for $1 < r < 2p^{-1}$,

$$c = \sum_{n=1}^{\infty} n^{-r}.$$

Let $e_1, e_2, \ldots$ be the standard Schauder basis in l^p. Let

$$t_0 = 0, \quad t_n = c^{-1} \sum_{i=1}^{n} i^{-r} \text{ for } n \geqslant 1.$$

We set

$$f(t) = e_n \text{ for } t \in (t_{n-1}, t_n], \quad f(0) = e_1.$$

Then $\|f(t)\| = 1$ for each $t \in [0, 1]$.

We assume that $f \in \mathscr{L}(S, W; l^p)$. Then, by Proposition 3.1,

$$\int_0^1 f dW = \sum_{n=1}^{\infty} e_n (w_{t_n} - w_{t_{n-1}}) \text{ a.s.}$$

Moreover,

$$\mathrm{E} \Big\| \int_0^1 f dW \Big\|^p = \sum_{n=1}^{\infty} \mathrm{E} |w_{t_n} - w_{t_{n-1}}|^p = \sum_{n=1}^{\infty} \mathrm{E} \left| \frac{X}{(cn^r)^{1/2}} \right|^p$$
$$= \mathrm{E} |X|^p c^{-p/2} \sum_{n=1}^{\infty} \frac{1}{n^{rp/2}} = \infty,$$

where X is a gaussian random variable, $X \sim \mathscr{N}(0, 1)$. This gives a contradiction.

Example 3.2. *There exist a Banach space E and a sequence $(f_n) \subset \mathscr{L}([0, 1], W; E)$ such that*

$$\sup_{t \in [0,1]} \|f_n(t)\| \to 0 \quad \text{as } n \to \infty$$

and $\int_0^1 f_n dW$ *diverges in* P.

Let $E = l^p$, $1 \leqslant p < 2$, and let $e_1, e_2, \ldots$ be the standard Schauder basis in l^p. We set

$$f_n(t) = n^{-r/p} \sum_{k=1}^{n} e_k \mathbf{1}_{((k-1)/n, k/n]}(t) \quad \text{for } t \in (0, 1],$$
$$f_n(0) = n^{-r/p} e_1,$$

where $0 < r < 1 - p/2$.

We obtain

$$\Big\| \int_0^1 f_n dW \Big\|^p = \mathrm{E} \Big(\sum_{k=1}^{n} n^{-r} |w_{k/n} - w_{(k-1)/n}|^p \Big) = \mathrm{E} |X|^p n^{1-p/2-r} \to \infty \quad \text{as } n \to \infty$$

($X \sim \mathcal{N}(0,1)$) and

$$\sup_t \|f_n(t)\| = n^{-r/p} \to 0.$$

Example 3.3. *There exist a Banach space E and a sequence $(f_n) \subset \mathscr{L}([0,1], W; E)$ such that*

$$\int_0^1 f_n dW \to 0 \quad \text{in } P$$

and (f_n) diverges in μ.

Let $E = l^p$, $p > 2$, and let $e_1, e_2, \ldots$ be the standard Schauder basis in l^p. We set

$$f_n(t) = n^{r/p} \sum_{k=1}^{n} e_k \mathbf{1}_{((k-1)/n, k/n]}(t) \quad \text{for } t \in (0,1],$$

$$f_n(0) = n^{r/p} e_1,$$

where $0 < r < p/2-1$.

We have

$$\mathbf{E}\left\|\int_0^1 f_n dW\right\|^p = n^{1-p/2+r}\mathbf{E}|X|^p \to 0 \quad \text{as } n \to \infty$$

($X \sim \mathcal{N}(0,1)$) and

$$\|f_n(t)\| = n^{r/p} \to \infty \quad \text{for each } t \in [0,1].$$

Example 3.4. *There exist a Banach space and E a function f in $\mathscr{L}([0,1], W; E)$ such that for each $r > 0$*

$$\int_0^1 \|f(t)\|^r dt = \infty.$$

Let $E = l^p$, $p > 2$, and let $e_1, e_2, \ldots$ be the standard Schauder basis in l^p. Let

$$c_n = (\log^{1/2} n)\log\log n \quad \text{if } n \geqslant 3 \text{ and } c_1 = c_2 = 1.$$

We have

$$c = \sum_{n=1}^{\infty} (nc_n^2)^{-1} < \infty.$$

Let

$$t_0 = 0, \quad t_n = \frac{1}{c}\sum_{k=1}^{n} (kc_k^2)^{-1} \text{ for } n \geqslant 1.$$

We set

$$f(t) = \sum_{n=1}^{\infty} c_n e_n \mathbf{1}_{(t_{n-1}, t_n]}(t).$$

By Proposition 3.1 we have

$$\mathrm{E}\left\|\int_0^1 f dW\right\|^p = \sum_{n=1}^{\infty} c_n^p \mathrm{E}\,|w_{t_n} - w_{t_{n-1}}|^p = \mathrm{E}\,|X|^p \sum_{n=1}^{\infty} c_n^p (t_n - t_{n-1})^{p/2} < \infty,$$

where $X \sim \mathscr{N}(0, 1)$, but

$$\int_0^1 \|f(t)\|^r dt = \sum_{n=1}^{\infty} c_n^r (t_n - t_{n-1}) = \sum_{n=1}^{\infty} (n c_n^{2-r})^{-1} = \infty$$

for each $r > 0$.

4. Characterization of elements in $\mathscr{L}(S, W; E)$. In this section the characterization of elements in $\mathscr{L}(S, W; E)$ is given. As some applications of this characterization we study the definition of random integral in the sense of Pettis and we give a description of $\mathscr{L}(S, W; l^p)$ for $1 \leqslant p < \infty$.

Suppose that $f \in \mathscr{L}(S, W; E)$; then for each $x' \in E'$

$$\langle x', f \rangle \in \mathscr{L}(S, W; \boldsymbol{R}) = L^2(S, \mu; \boldsymbol{R})$$

and, by Corollary 5.31 in [12], f is integrable in the sense of Pettis.

LEMMA 4.1. *If $f\colon S \to E$ is strongly measurable and integrable in the sense of Pettis, then there exists a sequence of finite σ-algebras $\Sigma_1 \subset \Sigma_2 \subset \ldots \ldots \subset \Sigma$ such that*

$$\mathrm{E}_\mu(f\,|\,\Sigma_n) \to f \textit{ strongly } \mu\textit{-a.s.},$$

where $\mathrm{E}_\mu(f\,|\,\Sigma_n)$ denotes the weak conditional expectation.

Proof. Notice first that if Σ' is a finite sub-σ-algebra of Σ, then it is generated by atoms $A_1, \ldots, A_n$ and

$$\mathrm{E}_\mu(f\,|\,\Sigma') = \sum_{i=1}^{n} [\mu(A_i)]^{-1} \int_{A_i} f d\mu\, 1_{A_i},$$

where $\int_{A_i} f d\mu$ denotes the Pettis integral and we take $[\mu(A_i)]^{-1} \doteq 0$ if $\mu(A_i) = 0$.

Now, since f is strongly measurable, for each $n \in N$ there exists a disjoint covering of S by sets $A_1^n, \ldots, A_{k_n}^n, A_{k_n+1}^n \in \Sigma$ such that

$$\mu\Big(\bigcup_{i=1}^{k_n} A_i^n\Big) > 1 - 2^{-n}$$

and

$$\operatorname{diam}\{f(A_i^n)\} = \sup\{\|f(t) - f(s)\|\colon t, s \in A_i^n\} \leqslant 2^{-n}$$

for $i = 1, \ldots, k_n$.

Write

$$\Sigma_1 = \sigma(A_1^1, \ldots, A_{k_1}^1, A_{k_1+1}^1)$$

and

$$\Sigma_n = \sigma(A_1^n, \ldots, A_{k_n}^n, A: A \in \Sigma_{n-1}) \quad \text{for } n \geqslant 2.$$

Then $\Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma$ and Σ_n is finite for each n.

Let $n \in N$ be fixed and let $B_1, \ldots, B_{r_n} \in \Sigma_n$ be generators of Σ_n, i.e. B_i are pairwise disjoint, $\bigcup_{i=1}^{r_n} B_i = S$ and each set in Σ_n is the union of some number of sets B_i. Let

$$I = \{1 \leqslant i \leqslant r_n : B_i \subset \bigcup_{j=1}^{k_n} A_j^n\}.$$

We have

$$[\mu(B_i)]^{-1} \int_{B_i} f d\mu \in \overline{\operatorname{conv}(f(B_i))}$$

(see, e.g., [11], Theorem 3.1) and for each $i \in I$ there exists a j, $1 \leqslant j \leqslant k_n$, such that $B_i \subset A_j^n$, which gives

$$\operatorname{diam}\{\overline{\operatorname{conv}(f(B_i))}\} \leqslant \operatorname{diam}\{f(A_j^n)\} \leqslant 2^{-n}.$$

Thus for each $i \in I$ and $s \in B_i$ we have

$$\Big\|f(s) - [\mu(B_i)]^{-1} \int_{B_i} f d\mu\Big\| \leqslant 2^{-n}$$

and

$$\mu\big(\bigcup_{i \in I} B_i\big) \geqslant 1 - 2^{-n}.$$

We obtain

$$\mu\{\|f - \mathrm{E}_\mu(f|\Sigma_n)\| \leqslant 2^{-n}\} \geqslant 1 - 2^{-n},$$

and this completes the proof.

THEOREM 4.1. *A strongly measurable function $f: S \to E$ is integrable with respect to a gaussian random measure W associated with μ if and only if*

(i) $\int_S \langle x', f\rangle^2 d\mu < \infty$ *for each* $x' \in E'$,

(ii) $\varphi(x') = \exp\big[-\frac{1}{2}\int_S \langle x', f\rangle^2 d\mu\big]$ *is the characteristic functional of some measure on* $(E, \mathscr{B}_E)$.

Proof. It follows non-trivially from (i) and (ii) that f is integrable with respect to W, i.e. there exist simple functions f_n such that $f_n \to f$ in μ and $\int_S f_n dW$ converges in P (Proposition 2.2). By (i), f is Pettis integrable (see [12], Corollary 5.31) and, by Lemma 4.1, there exists a sequence of finite σ-algebras $\Sigma_1 \subset \Sigma_2 \subset \ldots \subset \Sigma$ such that

$$\mathrm{E}_\mu(f|\Sigma_n) \to f \text{ strongly } \mu\text{-a.s.}$$

Put $f_n = \mathrm{E}_\mu(f|\Sigma_n)$. The functions $\{f_n\}$ are simple and we have to prove that $\int_S f_n dW$ converges in P. First we prove that $\int_S f_n dW$ are partial sums of some series of independent gaussian random vectors.

Let

$$X_1 = \int_S f_1 dW, \quad X_n = \int_S f_n dW - \int_S f_{n-1} dW \text{ for } n \geqslant 2.$$

Since W is gaussian by the linearity of random integral, it is sufficient to prove that

$$\mathrm{E}\langle x', X_n\rangle\langle y', X_m\rangle = 0 \quad \text{for each } x', y' \in E' \text{ and } n \neq m.$$

We have

$$\mathrm{E}\langle x', X_n\rangle\langle y', X_m\rangle = \mathrm{E}\int_S \langle x', f_n - f_{n-1}\rangle dW \int_S \langle y', f_m - f_{m-1}\rangle dW$$
$$= \int_S \langle x', f_n - f_{n-1}\rangle\langle y', f_m - f_{m-1}\rangle d\mu = 0,$$

since $\{f_n, \Sigma_n\}$ is a martingale in E.

We infer that

$$\int_S f_n dW = \sum_{i=1}^{n} X_i$$

and $\{X_i\}_{i\geqslant 1}$ are independent symmetric gaussian random vectors.

Let $x' \in E'$. We have

$$\mathrm{E}\exp\Big[i\Big\langle x', \sum_{i=1}^{n} X_i\Big\rangle\Big] = \mathrm{E}\exp\Big[i\Big\langle x', \int_S f_n dW\Big\rangle\Big]$$
$$= \exp\Big[-\frac{1}{2}\int_S \langle x', f_n\rangle^2 d\mu\Big] \to \exp\Big[-\frac{1}{2}\int_S \langle x', f\rangle^2 d\mu\Big] \quad \text{as } n \to \infty.$$

By (ii) and by the theorem of Ito and Nisio [6], the sums

$$\sum_{i=1}^{n} X_i = \int_S f_n dW$$

converge a.s. This completes the proof.

COROLLARY 4.1. *Let the function $f\colon S \to E$ be strongly measurable. Then $f \in \mathscr{L}(S, W; E)$ if and only if f is pregaussian (as a random element on the probability space (S, Σ, μ)).*

The random integral with vector-valued functions may be defined in the sense of Pettis (see also [15]). Namely, a strongly measurable function $f\colon S \to E$ is *weakly integrable with respect to a gaussian random measure* W if for each $x' \in E'$ the integral $\int \langle x', f\rangle dW$ exists and for each $B \in \Sigma$ there exists a random vector X_B such that for each $x' \in E'$

$$\langle x', X_B\rangle = \int_B \langle x', f\rangle dW \text{ a.s.}$$

In view of Theorem 4.1 we obtain

COROLLARY 4.2. *A function $f\colon S \to E$ is weakly integrable with respect to a gaussian random measure W if and only if $f \in \mathscr{L}(S, W; E)$.*

Proof. Indeed, let f be weakly integrable. Then

$$\int_S \langle x', f\rangle^2 d\mu < \infty$$

and

$$\varphi(x') = \exp\left[-\frac{1}{2}\int_S \langle x', f\rangle^2 d\mu\right]$$

is the characteristic functional of the random vector X_S. Conditions (i) and (ii) of Theorem 4.1 are fulfilled.

COROLLARY 4.3. *Let $f\colon S \to l^p$, where $f = (f_n)_{n\geq 1}$, $1 \leqslant p < \infty$, be measurable. Then f is integrable with respect to a gaussian random measure W if and only if*

$$\sum_{n=1}^{\infty}\Big(\int_S f_n^2 d\mu\Big)^{p/2} < \infty.$$

The corollary is a consequence of Theorem 4.1 and Vakhania's characterization of covariance operators of gaussian measures in l^p (see [14]).

5. Random integral in the spaces of type and cotype 2. Let E be a Banach space. We say that E is of *type p, $p \in (1, 2]$* (*cotype q, $q \in [2, \infty)$*) if for a Rademacher sequence (r_n) and for every $(x_n) \subset E$ the following implication holds (see [4] and [10]):

if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$, then $\sum_{n=1}^{\infty} r_n x_n$ converges a.e.

(if $\sum_{n=1}^{\infty} r_n x_n$ converges a.e., then $\sum_{n=1}^{\infty} \|x_n\|^q < \infty$).

For example, the spaces L^p and l^p are of type 2 and cotype p if $2 \leqslant p < \infty$ and of type p and cotype 2 if $1 \leqslant p < 2$.

PROPOSITION 5.1 (cf. [4] and [10]). *The following statements are equivalent:*

(a) *E is of type p (cotype q).*

(b) *There exists a constant C_1 (C_2) depending only on E such that*

$$\mathbf{E}\Big\|\sum_{i=1}^{n} \eta_i x_i\Big\|^p \leqslant C_1 \sum_{i=1}^{n} \|x_i\|^p$$

$$\Big(\sum_{i=1}^{n} \|x_i\|^q \leqslant C_2 \mathbf{E}\Big\|\sum_{i=1}^{n} \eta_i x_i\Big\|^q\Big)$$

for every $n \geqslant 1$ and $x_1, \ldots, x_n \in E$, where (η_n) is a sequence of independent normally distributed random variables with mean 0 *and variance* 1.

(c) *If $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$, then $\sum_{n=1}^{\infty} \eta_n x_n$ converges a.s.*

(if $\sum_{n=1}^{\infty} \eta_n x_n$ converges a.s., then $\sum_{n=1}^{\infty} \|x_n\|^q < \infty$),

where η_n are as in (b).

The gaussian random integral, as a linear operator on $L^2(S, \mu; E)$ (E of type 2), was constructed by Hoffmann-Jørgensen and Pisier [5]. The next proposition follows immediately from Proposition 5.1 (see also [5]).

PROPOSITION 5.2. *Let E be a Banach space of type* 2. *Then*

$$L^2(S, \mu; E) \subset \mathscr{L}(S, W; E).$$

Moreover, the identity embedding

$$I\colon L^2(S, \mu; E) \to \mathscr{L}(S, W; E)$$

is continuous and there exists a constant C such that for each $f \in L^2(S, \mu; E)$

$$\mathrm{E}\Big\|\int_S f\,dW\Big\|^2 \leqslant C \int_S \|f\|^2 d\mu.$$

PROPOSITION 5.3. *Let E be a Banach space of cotype* 2. *Then*

$$\mathscr{L}(S, W; E) \subset L^2(S, \mu; E).$$

Moreover, the identity embedding

$$I\colon \mathscr{L}(S, W; E) \to L^2(S, \mu; E)$$

is continuous and there exists a constant C such that for each $f \in \mathscr{L}(S, W; E)$

$$\int_S \|f\|^2 d\mu \leqslant C\mathrm{E}\Big\|\int_S f\,dW\Big\|^2. \tag{3}$$

Proof. Let E be of cotype 2. By Proposition 5.1 we obtain (3) for each simple function f. If $f \in \mathscr{L}(S, W; E)$, then there exists a sequence (f_n) of simple functions such that $f_n \to f$ in μ and $\int_S f_n\,dW$ converges in P. Therefore, by Lemma 2.1,

$$\int_S (f_n - f_m)\,dW \to 0 \quad \text{in } L^2(\Omega, P; E) \text{ as } n, m \to \infty,$$

and so $f_n \to f$ in $L^2(S, \mu; E)$.

This shows that inequality (3) holds for each $f \in \mathscr{L}(S, W; E)$ and that the identity embedding of $\mathscr{L}(S, W; E)$ in $L^2(S, \mu; E)$ is continuous.

COROLLARY 5.1. *If E is of cotype 2, then $\mathscr{L}(S, W; E)$ is the Banach space with the norm*

$$|||f|||_p = \Big(\mathrm{E}\Big\|\int_S f\,dW\Big\|^p\Big)^{1/p}, \qquad p \geq 1$$

($|||\cdot|||_p$ are equivalent as norms on $\mathscr{L}(S, W; E)$ for each $p \geq 1$).

For example, we consider the space $\mathscr{L}(S, W; l^p)$ for $1 \leq p \leq 2$. Let F be a Banach space and $p \geq 1$. By $l^p(F)$ we denote the Banach space of all sequences $(x_n) \subset F$ for which

$$\|(x_n)\| \overset{\mathrm{df}}{=} \Big(\sum_{n=1}^{\infty} \|x_n\|^p\Big)^{1/p} < \infty.$$

COROLLARY 5.2. *Let $1 \leq p \leq 2$. The space $\mathscr{L}(S, W; l^p)$ is isomorphic and isometric with $l^p(L^2(S, \mu; \boldsymbol{R}))$.*

Proof. If $f = (f_n) \in \mathscr{L}(S, W; l^p)$, then, in view of Corollary 4.3, $f \in l^p(L^2(S, \mu; \boldsymbol{R}))$.

Conversely, if $f \in l^p(L^2(S, \mu; \boldsymbol{R}))$, then by the inequality

$$\int_S |f_n|^p d\mu \leq \Big(\int_S f_n^2 d\mu\Big)^{p/2} \qquad \text{for } 1 \leq p \leq 2$$

we have $f \in L^0(S, \mu; l^p)$ and, by Corollary 4.3, $f \in \mathscr{L}(S, W; l^p)$. Since l^p is of cotype 2 $(1 \leq p \leq 2)$, by Corollary 5.1 the space $\mathscr{L}(S, W; l^p)$ is a Banach space with the norm

$$|||f|||_p = \Big(\mathrm{E}\Big\|\int_S f\,dW\Big\|^p\Big)^{1/p}.$$

We have

$$|||f|||_p^p = \mathrm{E}\Big\|\int_S f\,dW\Big\|_{l^p}^p = \sum_n \mathrm{E}\Big|\int_S f_n\,dW\Big|^p = c_p \sum_n \Big[\mathrm{E}\Big(\int_S f_n\,dW\Big)^2\Big]^{p/2}$$
$$= c_p \sum_n \Big(\int_S f_n^2 d\mu\Big)^{p/2} = c_p \|f\|_{l^p(L^2(S,\mu;\boldsymbol{R}))}^p,$$

where

$$c_p = (2\pi)^{-1/2}\Big(\int_{-\infty}^{\infty} |x|^p \exp\Big[-\frac{x^2}{2}\Big]dx\Big).$$

We infer that the operator $I(f) = c_p^{1/p} f$ forms an isometry between $\mathscr{L}(S, W; l^p)$ and $l^p(L^2(S, \mu; \boldsymbol{R}))$.

6. Some questions concerning the space $\mathscr{L}(S, W; E)$. Examples 3.1 and 3.4 show that bounded functions are not always integrable with respect to a gaussian random measure and that $f \in \mathscr{L}(S, W; E)$ does not always imply that $f \in L^r(S, \mu; E)$ for some $r > 0$. Corollary 5.1 shows that if E is of cotype 2, then $\mathscr{L}(S, W; E)$ is the Banach space.

In this section we answer the following questions:

For which Banach spaces E are the following conditions satisfied:

(A) $L^\infty(S, \mu; E) \subset \mathscr{L}(S, W; E)$,

(B) $\mathscr{L}(S, W; E) \subset \bigcup_{r>0} L^r(S, \mu; E)$,

(C) $\mathscr{L}(S, W; E)$ admits a Banach norm equivalent to $|||\cdot|||_0$?

In this section we assume that μ is atomless.

The following proposition answers the question (A).

PROPOSITION 6.1. *The following conditions are equivalent*:

(a) $L^\infty(S, \mu; E) \subset \mathscr{L}(S, W; E)$;

(b) *E is of type* 2.

Proof. Let $L^\infty(S, \mu; E) \subset \mathscr{L}(S, W; E)$. We have to show that E is of type 2. Suppose that this is not true. Then there exists a sequence $(x_n) \subset E$,

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty,$$

such that the series $\sum_{n=1}^{\infty} x_n \eta_n$ diverges a.e., where η_n are independent random variables, $\eta_n \sim N(0, 1)$, $n = 1, 2, \ldots$ (Proposition 5.1).

Let

$$c = \sum_n \|x_n\|^2.$$

Since μ is atomless, there exists a partition $A_1, A_2, \ldots$ of S such that A_n are disjoint, $A_n \in \Sigma$ and $\mu(A_n) = c^{-1}\|x_n\|^2$, $n = 1, 2, \ldots$

We set

$$f = \sum_{n=1}^{\infty} x_n [\mu(A_n)]^{-1/2} 1_{A_n}.$$

We have $\|f(t)\| = c^{1/2}$ for each $t \in S$, and so $f \in L^\infty(S, \mu; E)$.

On the other hand, the series

$$\sum_n x_n [\mu(A_n)]^{-1/2} W(A_n)$$

diverges a.e. and, consequently, $f \notin \mathscr{L}(S, W; E)$, which gives a contradiction (Proposition 3.1).

The inverse implication of this proposition follows from Proposition 5.2.

The following proposition answers the question (B).

PROPOSITION 6.2. *The following conditions are equivalent*:

(a) $\mathscr{L}(S, W; E) \subset \bigcup_{p>0} L^p(S, \mu; E)$;

(b) *E is of cotype* 2.

In order to prove this proposition we need the following lemma:

Lemma 6.1. *If (a_n) is a sequence of real numbers such that*

$$0 \leqslant a_n \leqslant M \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n = \infty,$$

then there exists a sequence $b_n \searrow 0$ such that

$$\sum_{n=1}^{\infty} a_n b_n < \infty$$

and, for each $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n b_n^{1-\varepsilon} = \infty.$$

Proof. Let

$$n_1 = \max\Big\{n\colon M \leqslant \sum_{i=1}^{n} a_i < 2M\Big\}, \quad n_k = \max\Big\{n\colon M \leqslant \sum_{i=n_{k-1}+1}^{n} a_i < 2M\Big\}.$$

We have $0 = n_0 < n_1 < n_2 < \ldots$

Let $b_j = (k\log^2 k)^{-1}$ for $n_{k-1} < j \leqslant n_k$, $k = 1, 2, \ldots$ Then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty} (k\log^2 k)^{-1} \sum_{j=n_{k-1}+1}^{n_k} a_j \leqslant 2M \sum_{k=1}^{\infty} (k\log^2 k)^{-1} < \infty$$

and

$$\sum_{n=1}^{\infty} a_n b_n^{1-\varepsilon} = \sum_{k=1}^{\infty} (k\log^2 k)^{-1+\varepsilon} \sum_{j=n_{k-1}+1}^{n_k} a_j$$

$$\geqslant M \sum_{k=1}^{\infty} (k\log^2 k)^{-1+\varepsilon} = \infty \quad \text{for } \varepsilon > 0.$$

Proof of Proposition 6.2. Suppose that E is not of cotype 2. Then there exists a sequence $(x_n) \subset E$ such that $\sum_n x_n \eta_n$ converges a.e., where η_n are independent, $\eta_n \sim N(0,1)$, and $\sum_n \|x_n\|^2 = \infty$ (Proposition 5.1).

The sequence $(\|x_n\|)$ is bounded. Indeed, the convergence of the series $\sum_n x_n \eta_n$ implies that, for each $x' \in E'$,

$$\mathbf{E}\Big\langle x', \sum_n x_n \eta_n\Big\rangle^2 = \sum_n \langle x', x_n\rangle^2 < \infty,$$

so (x_n) is weakly bounded and, therefore, by the Banach-Steinhaus theorem (x_n) is strongly bounded.

By Lemma 6.1 there exists a sequence $b_n \searrow 0$ such that

$$\sum_n b_n \|x_n\|^2 < \infty$$

and, for each $\varepsilon > 0$,

$$\sum_n b_n^{1-\varepsilon} \|x_n\|^2 = \infty.$$

Let

$$b = \sum_n b_n \|x_n\|^2$$

and let $(A_n) \subset \Sigma$ be a partition of S such that $\mu(A_n) = b^{-1} b_n \|x_n\|^2$.

Setting

$$f(t) = \sum_{n=1} x_n [\mu(A_n)]^{-1/2} 1_{A_n}(t),$$

we infer that

$$\int_S f dW = \sum_n x_n [\mu(A_n)]^{-1/2} W(A_n) \text{ converges a.e.}$$

Thus $f \in \mathscr{L}(S, W; E)$.

Let $r > 0$. Then

$$\int_S \|f\|^r d\mu = \sum_n \|x_n\|^r [\mu(A_n)]^{-r/2} \mu(A_n) = b^{r/2-1} \sum_n b_n^{1-r/2} \|x_n\|^2 = \infty.$$

The inverse implication of this proposition follows from Proposition 5.3.

The following proposition answers the question (C).

PROPOSITION 6.3. *The following conditions are equivalent*:

(a) $\mathscr{L}(S, W; E)$ *admits a Banach norm equivalent to* $|||\cdot|||_0$;

(b) E *is of cotype* 2.

In the proof of this proposition we use the following lemma:

LEMMA 6.2. *Suppose that E is not of cotype 2. Then there exists a sequence of simple functions (f_n) such that $\int_S f_n dW$ are bounded in P and*

$$\inf_{s \in S} \|f_n(s)\| \to \infty \quad \textit{as } n \to \infty.$$

Proof. Since E is not of cotype 2, then there exists a sequence $(x_n) \subset E$ such that $\sum_n \eta_n x_n$ converges a.s. and

$$\sum_n \|x_n\|^2 = \infty$$

($\eta_n \sim \mathscr{N}(0, 1)$ independent). Put

$$a_n = \sum_{i=1}^n \|x_i\|^2.$$

Then $a_n \to \infty$.

Let $n \in N$ be fixed and let $A_1, \ldots, A_n$ be a partition of S such that $\mu(A_i) = a_n^{-1} \|x_i\|^2$, $i = 1, \ldots, n$.

We write

$$f_n = a_n^{1/2} \sum_{i=1}^{n} \|x_i\|^{-1} x_i 1_{A_i}.$$

We have

$$\int_S f_n dW = a_n^{1/2} \sum_{i=1}^{n} \|x_i\|^{-1} x_i W(A_i) = \sum_{i=1}^{n} x_i \eta_i \text{ in law},$$

and for each $s \in S$

$$\|f_n(s)\| = a_n^{1/2} \to \infty \quad \text{as } n \to \infty.$$

Proof of Proposition 6.3. Suppose that there exists a Banach norm $|\cdot|$ on $\mathscr{L}(S, W; E)$ equivalent to $|||\cdot|||_1$ (which is equivalent to $|||\cdot|||_0$). Thus there exist r_1 and r_2, $0 < r_1 < \frac{1}{2}$, $r_2 > 0$, such that $S_0 \supset S_1 \supset S_2$, where

$$S_0 = \{f\colon |f| < 1\},$$
$$S_1 = \{f\colon |||f|||_1 < r_1\},$$
$$S_2 = \{f\colon |f| < r_2\}.$$

Suppose, to the contrary, that E is not of cotype 2. Then by Lemma 6.2 there exist simple functions f_n such that

$$b_n^2 = \inf_{s\in S} \|f_n(s)\| \to \infty$$

and $\int_S f_n dW$ are bounded in P.

Put $g_n = b_n^{-1} f_n$, $n = 1, 2, \ldots$

We have

$$\inf_{s\in S} \|g_n(s)\| = b_n \to \infty \quad \text{and} \quad \int_S g_n dW \to 0 \text{ in } P.$$

Let $A_1, \ldots, A_k$ be a partition of S such that $\mu(A_i) < \frac{1}{2} r_1$.

We write $h_i^n = g_n 1_{A_i}$, $i = 1, \ldots, k$. Then

$$\|h_i^n\|_0 < \mu(A_i) < \frac{1}{2} r_1.$$

Let N be a positive integer such that for each $n \geqslant N$

$$\mathrm{E}\left\|\int_S g_n dW\right\| < \frac{1}{2} r_1.$$

Take fixed $n \geqslant N$. We have

$$\mathrm{E}\left\|\int_S h_i^n dW\right\| = \mathrm{E}\left\|\int_{A_i} g_n dW\right\| \leqslant \mathrm{E}\left\|\int_S g_n dW\right\| < \frac{1}{2} r_1,$$

whence

$$|||h_i^n|||_1 = \|h_i^n\|_0 + \mathrm{E}\left\|\int_S h_i^n dW\right\| < r_1,$$

so $h_i^n \in S_1$ for each $i = 1, \ldots, k$ and $n \geqslant N$. Consequently,

$$k^{-1}g_n = k^{-1}\sum_{i=1}^{k} h_i^n \in S_0 \quad \text{and} \quad k^{-1}r_2 g_n \in S_2,$$

whence $k^{-1}r_2g_n \in S_1$. We obtain a contradiction, since

$$\inf_{s\in S} \|k^{-1}r_2 g_n(s)\| \to \infty \quad \text{as } n \to \infty.$$

COROLLARY 6.1. *$\mathscr{L}(S, W; E) = L^2(S, \mu; E)$ if and only if E is isomorphic to a Hilbert space.*

This follows from the result of Kwapień [9], which states that E is isomorphic to a Hilbert space if and only if E is of type 2 and of cotype 2, and from Propositions 6.1 and 6.2.

Remark 6.1. Chobanian and Tarieladze (Theorem 4.1 in [1]) have shown that if there exists a $p > 0$ such that each pregaussian measure on a Banach space E has the p-th strong order, then E is of cotype 2. From Theorem 4.1 and Proposition 6.2 we obtain:

If each pregaussian measure μ on E has some p-th strong order, then E is of cotype 2.

This strengthens the above-mentioned result of Chobanian and Tarieladze [1].

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