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ON THE SPECTRAL FUNCTION OF A NORMAL OPERATOR

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Let T be a normal operator in Hilbert space and let $E(.)$ be its spectral measure; it is desirable to have formulas which describe the projections $E(.)$ explicitly in terms of the action of the operator on the vectors of the underlying space. Such formulas have been given in [1] and [2]. In the present note we present a simple characterization which does not involve taking powers of the operators and has the further advantage of being purely algebraic. The method is related to that of [2].

Notation. Let H be a Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. For each $W \in B(H)$, we denote by $\mathscr{R}(W)$ its range. If $T \in B(H)$ is normal, we denote by $E(.)$ its spectral measure. For each Borel set M in the complex plane let $H_T(M)$ be the range of the projection $E(M)$. We intend to prove the following

Theorem. *For each closed set F in the complex plane*

$$H_T(F) = \bigcap_{\lambda \notin F} \mathscr{R}(\lambda - T).$$

Proof. Denote by G the complement of F and let us show first that, for each $x \in \bigcap_{\lambda \in G} \mathscr{R}(\lambda - T)$, we have $E(G)\, x = 0$. Since every open set in the complex plane may be represented as a union of a sequence of closed squares with sides parallel to the axes, it suffices to prove that, for each such square M, we have

$$E(M)\,\big(\bigcap_{\lambda \in M} \mathscr{R}(\lambda - T)\big) = 0.$$

Now let x be a fixed element of the intersection $\bigcap_{\lambda \in M} \mathscr{R}(\lambda - T)$. We shall now construct, by induction, a sequence of sets M_n with the following properties

1° $M = M_0$

2° the closure of each set M_n is a closed square with sides parallel to the axes

3° each M_n^- is one of the four closed squares obtained by halving the sides of M_{n-1}^-

4° $|E(M)\,x|^2 \leqq 4^n|E(M_n)\,x|^2$

Suppose we have already constructed the set M_n. Let C_1, C_2, C_3, C_4 be four disjoint Borel subsets of M_n the closures of which are the four squares obtained by halving the sides of M_n^-. Since $|E(M_n)\,x|^2 = \sum_{i=1}^{4} |E(C_i)\,x|^2$, we have $\frac{1}{4}|E(M_n)\,x|^2 \leqq |E(C_i)\,x|^2$ for at least one index i. Let M_{n+1} be this C_i; it follows that $|E(M_n)\,x|^2 \leqq 4|E(M_{n+1})\,x|^2$. The sequence $M = M_0^- \supset M_1^- \supset M_2^- \supset \ldots$ has exactly one point λ_0 in its intersection. Since $\lambda_0 \in M$, there exists a $y \in H$ such that $x = (\lambda_0 - T)\,y$. Now we can write

$$|E(M)\,x|^2 \leqq 4^n|E(M_n)\,x|^2 = 4^n|E(M_n)(\lambda_0 - T)\,y|^2 =$$

$$= 4^n(E(M_n)(\lambda_0 - T)\,y, (\lambda_0 - T)\,y) = 4^n(E(M_n)(\lambda_0 - T)^*(\lambda_0 - T)\,y, y) =$$

$$= 4^n \int_{M_n} |\lambda_0 - \lambda|^2 \, \mathrm{d}(E(.)\,y, y) = 4^n \int_{M_n \setminus \{\lambda_0\}} |\lambda_0 - \lambda|^2 \, \mathrm{d}(E(.)\,y, y) \leqq 4^n \, \mathrm{d}_n^2 v(M_n \setminus \{\lambda_0\})$$

where d_n is the diameter of the set M_n and $v(S)$ is the variation of the measure $(E(.)\,y, y)$ on the set S. Since the sets $M_n \setminus \lambda_0$ form a decreasing sequence with empty intersection, the sequence $v(M_n \setminus \lambda_0)$ tends to zero. It follows that $E(M)\,x = 0$ and completes the proof of the inclusion $\bigcap_{\lambda \in G} \mathscr{R}(\lambda - T) \subset H_T(F)$.

On the other hand, suppose that $x \in H_T(F)$ and let $\lambda_0 \in G$ be given. Let g be the function defined for all complex λ by setting $g(\lambda) = (\lambda_0 - \lambda)^{-1}\, c_F(\lambda)$ where c_F is the characteristic function of the set F. Then $x = (\lambda_0 - T)\,y$ for $y = g(T)\,x$. This proves the other inclusion and completes the proof.

References

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