



# On the spectral gap of the Kac walk and other binary collision processes

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**Abstract.** We give a new and elementary computation of the spectral gap of the Kac walk on  $S^N$ . The result is obtained as a by-product of a more general observation which allows to reduce the analysis of the spectral gap of an  $N$ -component system to that of the same system for  $N = 3$ . The method applies to a number of random “binary collision” processes with complete-graph structure, including non-homogeneous examples such as exclusion and colored exclusion processes with site disorder.

## 1. Introduction

The following model for energy preserving binary collisions was introduced by M. Kac in Kac (1956). Let  $\nu$  denote the uniform probability measure on the sphere

$$S^{N-1} = \left\{ \eta \in \mathbb{R}^N : \sum_{i=1}^N \eta_i^2 = 1 \right\},$$

and consider the  $\nu$ -reversible Markov process on  $S^{N-1}$  with infinitesimal generator given by

$$\mathcal{L}f(\eta) = \frac{1}{2N} \sum_{i,j=1}^N \frac{1}{2\pi} \int_0^{2\pi} \left[ f(R_\theta^{ij}\eta) - f(\eta) \right] d\theta, \quad (1.1)$$

where  $R_\theta^{ij}$ ,  $i \neq j$  is a clockwise rotation of angle  $\theta$  in the plane  $(\eta_i, \eta_j)$ . As a convention, we take  $R_\theta^{ii} = \text{Id}$ . In words, the associated Markov process goes as follows: we have independent Poisson clocks of rate  $1/2$  at each coordinate; when coordinate  $i$  rings we choose uniformly at random (with replacement) another coordinate  $j$ ; if  $j \neq i$  then we perform a rotation of angle  $\theta$  in the plane  $(\eta_i, \eta_j)$ , with  $\theta$  uniform over  $[0, 2\pi)$ , while if  $i = j$  we do nothing.

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Note that  $-\mathcal{L}$  is a non-negative, bounded self-adjoint operator on  $L^2(\nu)$ . Any constant is an eigenfunction with eigenvalue 0 and the spectral gap  $\lambda = \lambda(N)$  is defined as

$$\lambda(N) = \inf_{\substack{f \in L^2(\nu): \\ \nu(f) = 0}} \frac{\nu(f(-\mathcal{L})f)}{\nu(f^2)}, \quad (1.2)$$

where  $\nu(f)$  stands for the expectation  $\int f d\nu$ . M. Kac conjectured that  $\lambda(N)$  stays bounded away from 0 as  $N \rightarrow \infty$ . This conjecture was first proved by Janvresse (2001), where a powerful recursive approach due to H.T. Yau was used. After that, in the beautiful paper Carlen et al. (2003), Carlen, Carvalho and Loss introduced a new recursive approach which allows to actually compute the value of  $\lambda(N)$  for every  $N$ :

$$\lambda(N) = \frac{N+2}{4N}, \quad N \geq 2. \quad (1.3)$$

Around the same time, Maslen (2003) derived formulae for all eigenvalues of  $\mathcal{L}$  by means of harmonic analysis techniques. We refer to Carlen et al. (2003) for further background, motivation and references on Kac's conjecture. Our result below shows that the proof of (1.3) can be somewhat simplified. In particular, we do not need any recursive analysis: in one step we go from  $\lambda(N)$  to  $\lambda(3)$  and the conclusion follows by direct computations in the case  $N = 3$ .

**Theorem 1.1.** *For any  $N \geq 3$ :*

$$\lambda(N) = (3\lambda(3) - 1) \left(1 - \frac{2}{N}\right) + \frac{1}{N}. \quad (1.4)$$

*In particular, (1.3) follows from (1.4) with  $\lambda(3) = 5/12$ .*

The proof uses a well known equivalent characterization of the spectral gap as the largest constant  $\lambda$  such that the inequality

$$\nu((\mathcal{L}f)^2) \geq \lambda \nu(f(-\mathcal{L})f), \quad (1.5)$$

holds for all  $f \in L^2(\nu)$ . The equivalence of (1.2) and (1.5) follows from elementary spectral theory. A similar approach has been exploited recently in Boudou et al. (2006) to obtain spectral gap bounds for a class of interacting particle systems. A proof of Theorem 1.1 is given at the end of the introduction. In the following sections we shall show that variants of the same method can be used to obtain spectral gap estimates for several models sharing some of the features of the Kac walk. The argument turns out to be especially powerful in non-homogeneous cases where other known methods are harder to apply because of the lack of permutation symmetry. In particular, for exclusion processes with site disorder we obtain a remarkable simplification of a spectral gap estimate proved by the author in Caputo (2004). The latter estimate is at the heart of recent results on the hydrodynamic limit of disordered lattice gases, see Faggionato and Martinelli (2003); Quastel (2006). In the last section of this work we prove a new result which extends the spectral gap estimate to the case of colored particles.

On the other hand, we point out that for some of the physically more relevant generalizations of the Kac walk treated in Carlen et al. (2003) our argument will not necessarily yield sharp results. This is the case, for example, of the Kac model with momentum and energy conserving collisions discussed in the next section. The problem of the determination of the spectral gap for the latter model has been

recently solved in Carlen et al. (2007), where the authors develop an interesting extension of the recursive scheme introduced in Carlen et al. (2003).

1.1. *Proof of Theorem 1.1.* We start with some preliminaries. We write

$$E_b f(\eta) = \frac{1}{2\pi} \int_0^{2\pi} f(R_\theta^{ij} \eta) d\theta, \quad b = \{i, j\}, \quad (1.6)$$

for the binary average operator appearing in the definition of  $\mathcal{L}$ . Note that  $E_b$  is a projection which coincides with the  $\nu$ -conditional expectation given the  $\sigma$ -algebra  $\mathcal{F}_b$  generated by variables  $\{\eta_\ell, \ell \notin b\}$ . Thus, we rewrite the Markov generator as follows:

$$\mathcal{L}f(\eta) = \frac{1}{N} \sum_b D_b f(\eta), \quad (1.7)$$

where the sum runs over all  $\binom{N}{2}$  unordered pairs  $b$  and for each such pair

$$D_b = E_b - \text{Id}, \quad E_b f(\eta) = \nu(f | \mathcal{F}_b). \quad (1.8)$$

For each  $b$ ,  $D_b$  is a bounded self-adjoint operator in  $L^2(\nu)$  satisfying  $D_b^2 = -D_b$ . In particular,

$$\nu(f(-\mathcal{L})g) = \frac{1}{N} \sum_b \nu[D_b f D_b g]. \quad (1.9)$$

On the other hand we have

$$\nu((\mathcal{L}f)^2) = \frac{1}{N^2} \sum_{b, b'} \nu[D_b f D_{b'} f]. \quad (1.10)$$

We are going to use the expressions (1.9) and (1.10) in (1.5) to compute  $\lambda(N)$ . We start with the lower bound. We write  $b \sim b'$  when two unordered pairs have at least one common vertex (including the case  $b = b'$ ). Otherwise we write  $b \not\sim b'$ . We observe that if  $b \not\sim b'$ , then  $E_b$  and  $E_{b'}$  commute. Therefore, using  $D_b^2 = -D_b$  and self-adjointness

$$\nu[D_b f D_{b'} f] = -\nu[(D_{b'} D_b f)(D_{b'} f)] = \nu[(D_{b'} D_b f)^2] \geq 0, \quad b \not\sim b'. \quad (1.11)$$

It follows that

$$\nu((\mathcal{L}f)^2) \geq \frac{1}{N^2} \sum_{b, b': b \sim b'} \nu[D_b f D_{b'} f]. \quad (1.12)$$

Unordered triples  $\{i, j, k\}$  of distinct vertices are denoted by  $T$  (triangles). We say that  $b \in T$  if  $b = \{i, j\}$  and  $i, j \in T$ . Clearly, if  $b \sim b'$  and  $b \neq b'$  there is only one triangle  $T$  such that  $b, b' \in T$ . We may therefore write

$$\begin{aligned} \sum_{b, b': b \sim b'} \nu[D_b f D_{b'} f] &= \sum_{b, b': b \sim b', b \neq b'} \nu[D_b f D_{b'} f] + \sum_b \nu[(D_b f)^2] \\ &= \sum_T \sum_{b, b' \in T} \nu[D_b f D_{b'} f] - \sum_T \sum_{b \in T} \nu[(D_b f)^2] + \sum_b \nu[(D_b f)^2]. \end{aligned} \quad (1.13)$$

Since for every  $b$  there are exactly  $N - 2$  triangles  $T$  such that  $b \in T$  we see that

$$\begin{aligned} \sum_{b, b': b \sim b'} \nu[D_b f D_{b'} f] &= \sum_T \sum_{b, b' \in T} \nu[D_b f D_{b'} f] - (N - 3) \sum_b \nu[(D_b f)^2]. \end{aligned} \quad (1.14)$$

Let us now apply the inequality (1.5) to a fixed triangle  $T$ . Let  $\mathcal{F}_T$  denote the  $\sigma$ -algebra generated by  $\{\eta_\ell, \ell \notin T\}$ . The conditional probability  $\nu[\cdot | \mathcal{F}_T]$  coincides with the uniform probability measure on the sphere  $S^2(t)$  in  $\mathbb{R}^3$  with radius

$$t = \sqrt{1 - \sum_{\ell \notin T} \eta_\ell^2}.$$

Moreover, as noted in Carlen et al. (2003), it is not hard to show that the spectral gap of the Kac model does not depend on the radius of the sphere on which the walk is performed. Using (1.5) on each triangle  $T$ , we therefore have

$$\frac{1}{3} \sum_{b, b' \in T} \nu[D_b f D_{b'} f | \mathcal{F}_T] \geq \lambda(3) \sum_{b \in T} \nu[(D_b f)^2 | \mathcal{F}_T], \quad (1.15)$$

uniformly in  $\eta \in S^{N-1}$ . Taking  $\nu$ -expectation we can remove the conditioning on  $\mathcal{F}_T$  in (1.15). Using this in (1.14) gives

$$\begin{aligned} & \sum_{b, b' : b \sim b'} \nu[D_b f D_{b'} f] \\ & \geq 3 \lambda(3) (N-2) \sum_b \nu[(D_b f)^2] - (N-3) \sum_b \nu[(D_b f)^2] \\ & = ((3\lambda(3) - 1)(N-2) + 1) \sum_b \nu[(D_b f)^2]. \end{aligned}$$

From (1.5) we conclude that  $\lambda(N)$  is larger or equal than the right hand side of (1.4).

It remains to show that this bound is attained for a given  $f$ . To this end, take

$$f_N(\eta) = \sum_{i=1}^N \eta_i^4 + \text{const.} \quad (1.16)$$

Let us first check that  $\nu[D_b f_N D_{b'} f_N] = 0$  whenever  $b \not\sim b'$ , so that (1.12) is an equality for  $f = f_N$ . To this end, note that  $\nu[D_b f_N D_{b'} f_N] = \nu[f_N D_b D_{b'} f_N] = 0$ . Indeed, if  $b = \{i, j\}$  and  $b' \not\sim b$ , then  $D_{b'} f_N(\eta)$  depends on  $\eta_i, \eta_j$  only through  $\eta_i^2 + \eta_j^2 = 1 - \sum_{k \notin b} \eta_k^2$  and therefore  $D_b D_{b'} f_N = 0$ .

Next, we need that (1.15) is an equality as well. This requires checking that for  $N = 3$ , for any value of the conservation law  $\sum_{i=1}^3 \eta_i^2 = t > 0$ ,  $f_3$  is, up to additive constants (that may depend on  $t$ ), an eigenfunction of  $-\mathcal{L}$  with eigenvalue  $\lambda(3)$ . For the solution of this 3-dimensional problem, as well as for the calculation of  $\lambda(3) = 5/12$ , we refer to Carlen et al. (2003, Section 3). Once the estimates (1.12) and (1.15) can be turned into identities we see that all our bounds are saturated for the function (1.16). This completes the proof.  $\square$

## 2. Homogeneous models

The general setting can be described as follows. We consider a product space  $\Omega = X^N$ , where  $X$ , the single component space is a measurable space equipped with a probability measure  $\mu$ . On  $\Omega$  we consider the product measure  $\mu^N$ . Elements of  $\Omega$  will be denoted by  $\eta = (\eta_1, \dots, \eta_N)$ . Next, we take a measurable function

$\xi : X \rightarrow \mathbb{R}^d$ , for a given  $d \geq 1$ , and we define the probability measure  $\nu = \nu_{N,\omega}$  on  $\Omega$  as  $\mu^N$  conditioned on the event

$$\Omega_{N,\omega} := \left\{ \eta \in \Omega : \sum_{i=1}^N \xi(\eta_i) = \omega \right\}, \quad (2.1)$$

where  $\omega \in \mathbb{R}^d$  is a given parameter. We interpret the constraint on  $\Omega_{N,\omega}$  as a *conservation law*.

In all the examples considered below there are no difficulties in defining the conditional probability  $\nu$ , therefore we do not attempt here at a justification of this setting in full generality but rather refer to the examples for full rigor. The crucial property of  $\nu$  that will be repeatedly used below is that, for any set of indices  $A$ , conditioned on the  $\sigma$ -algebra  $\mathcal{F}_A$  generated by variables  $\eta_i$ ,  $i \notin A$ ,  $\nu$  becomes the  $\mu$ -product law over  $\eta_j$ ,  $j \in A$ , conditioned on the event

$$\sum_{j \in A} \xi(\eta_j) = \omega - \sum_{i \notin A} \xi(\eta_i).$$

We shall call this the *non-interference* property. In analogy with (1.7) we consider the binary collision Markov process described by the infinitesimal generator

$$\mathcal{L}f = \frac{1}{N} \sum_b [\nu(f | \mathcal{F}_b) - f], \quad (2.2)$$

where the sum runs over all  $\binom{N}{2}$  unordered pairs  $b = \{i, j\}$  and  $\nu[f | \mathcal{F}_b]$  is the  $\nu$ -conditional expectation of  $f$  given the variables  $\eta_\ell$ ,  $\ell \notin b$ . This defines a bounded self-adjoint operator on  $L^2(\nu)$ . Setting, as before,  $D_b = \nu[\cdot | \mathcal{F}_b] - \text{Id}$ , we see that  $D_b^2 = -D_b$  and the operator  $\mathcal{L}$  satisfies (1.9).

In principle, our arguments could be extended to more general processes. For instance, one can consider binary ‘‘collisions’’ which are not given by simple averages as in (2.2) but by other mechanisms which still preserve reversibility (see e.g. the non uniform models considered in Carlen et al. (2003)), or one could look at ‘‘collisions’’ which can involve more than two components at a time. However, we shall not investigate such extensions.

Returning to our model (2.2), the spectral gap is defined as in (1.2). Note that, by definition,  $\lambda(2) = \frac{1}{2}$  always, since for  $N = 2$  there is only one pair  $b$  and  $\nu((D_b f)^2) = \nu(f^2)$  for any  $f$  such that  $\nu(f) = 0$ . Here we shall assume that  $\lambda(3)$  is independent of the choice of  $\omega$ . This is a strong assumption which holds only for special choices of the model. However, it does hold in the examples considered below. More general models are treated in the next section.

**Theorem 2.1.** *Suppose  $\lambda(3)$  is independent of  $\omega$ . Then, for any  $N \geq 2$ :*

$$\lambda(N) \geq (3\lambda(3) - 1) \left(1 - \frac{2}{N}\right) + \frac{1}{N}. \quad (2.3)$$

If, in addition, there exists  $\varphi : X \rightarrow \mathbb{R}$  such that the function

$$f_3(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^3 \varphi(\eta_i), \quad (2.4)$$

satisfies, for  $N = 3$ ,  $\mathcal{L}f_3 = -\lambda(3)f_3 + \text{const.}$ , regardless of the value of  $\sum_{i=1}^3 \xi(\eta_i)$  (although the constant may depend on this value), then (2.3) can be turned into an identity for each  $N \geq 2$ .

*Proof:* We repeat the steps of the proof of Theorem 1.1. We start from (1.10) and arrive at (1.12) with the same commutation property used in (1.11). Indeed, this is a simple consequence of the non-interference property. The latter property also implies that the conditional probability  $\nu(\cdot | \mathcal{F}_T)$  is nothing but  $\nu_{3,\omega'}$  with  $\omega' = \omega - \sum_{\ell \notin T} \xi(\eta_\ell)$ . Since  $\lambda(3)$  is independent of the value  $\omega$  in the conservation law we may repeat the argument leading to (1.15). This proves the lower bound (2.3). As for the reverse direction, again the arguments given in the proof of Theorem 1.1 can be repeated line by line.  $\square$

Next, we examine some examples to which the theorem applies.

2.1. *Kac model.* The model discussed in the introduction can be seen as a special case of our general setting, so that Theorem 1.1 becomes a special case of Theorem 2.1. Here  $X = \mathbb{R}$ ,  $\mu$  is the centered Gaussian measure with variance  $\sigma^2 > 0$  and we take  $\xi(\eta_i) = \eta_i^2$  (with  $d = 1$ ). Then, for every  $\omega > 0$ ,  $\nu_{N,\omega}$  is the uniform probability measure on the sphere of radius  $\sqrt{\omega}$ . Clearly, the choice of  $\sigma^2 > 0$  is uninformative in the determination of  $\nu_{N,\omega}$ . As we have seen in the introduction, this model satisfies the two main assumptions in Theorem 2.1.

2.2. *“Flat” Kac model.* Here  $X = \mathbb{R}^+$ ,  $\mu$  is the exponential law with parameter  $\gamma > 0$ . We take  $\xi(\eta_i) = \eta_i$  (with  $d = 1$ ). Then, for every  $\omega > 0$ , independently of the choice of  $\gamma > 0$ ,  $\nu_{N,\omega}$  is the uniform probability measure on the simplex  $\Omega_{N,\omega}$ . The binary collision process (2.2) associated to this setting does not appear explicitly in the literature, so we shall give more details on the computation of  $\lambda(N)$  in this case. Let us first check that  $\lambda(N)$  is independent of  $\omega$ , for any  $N$ . This is a consequence of the fact that  $\mathcal{L}$  commutes with the unitary change of scale from  $\Omega_{N,\omega}$  to  $\Omega_{N,\omega'}$ , for any  $\omega, \omega' > 0$ . Indeed,  $\nu_{N,\omega'}$  is the image of  $\nu_{N,\omega}$  under the map  $\mathcal{T} : \eta \rightarrow \omega' \eta / \omega$  and if  $f_{\mathcal{T}}(\eta) = f(\mathcal{T}\eta)$ , then

$$\nu_{N,\omega}(f_{\mathcal{T}} | \mathcal{F}_b)(\eta) = \nu_{N,\omega'}(f | \mathcal{F}_b)(\mathcal{T}\eta), \quad (2.5)$$

for all  $\eta \in \Omega_{N,\omega}$  and for all pairs  $b$ .

Next, we shall prove that  $\lambda(3) = \frac{4}{9}$  and that the eigenfunction for  $N = 3$  is given by

$$f_3(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^3 \eta_i^2 + \text{const.} \quad (2.6)$$

for any value of the constraint  $\sum_{i=1}^3 \eta_i = \omega$  (of course, the constant will be given by  $-3\nu_{3,\omega}(\eta_1^2)$ , since we must have  $\nu_{3,\omega}(f) = 0$ ). From these facts and Theorem 2.1 we therefore obtain, for all  $N \geq 2$ :

$$\lambda(N) = \frac{N+1}{3N}. \quad (2.7)$$

To solve the 3-dimensional problem, we observe that when  $N = 3$ , then  $\mathcal{L} + 1$  coincides with the *average operator*  $P$  introduced in Carlen et al. (2003). Therefore we can apply the general analysis of Section 2 in Carlen et al. (2003) or equivalently that of Theorem 4.1 in Caputo (2003). The outcome is that

$$\lambda(3) \geq \frac{1}{3} \min\{2 + \mu_1, 2 - 2\mu_2\}, \quad (2.8)$$

where the parameters  $\mu_1, \mu_2$  are given by

$$\mu_1 = \inf_{\varphi} \nu(\varphi(\eta_1)\varphi(\eta_2)), \quad \mu_2 = \sup_{\varphi} \nu(\varphi(\eta_1)\varphi(\eta_2)) \quad (2.9)$$

with  $\varphi$  chosen among all functions  $\varphi : X \rightarrow \mathbb{R}$  satisfying  $\nu(\varphi(\eta_1)^2) = 1$  and  $\nu(\varphi(\eta_1)) = 0$ . Here  $\nu$  stands for  $\nu_{3,\omega}$ , but we have removed the subscripts for simplicity. As in (2.5) one checks that the parameters  $\mu_1, \mu_2$  do not depend on  $\omega$ . Write  $\mathcal{K}\varphi(\alpha) = \nu[\varphi(\eta_2) | \eta_1 = \alpha]$ ,  $\alpha \geq 0$ . This defines a self-adjoint Markov operator on  $L^2(\nu_1)$ , where  $\nu_1$  is the marginal on  $\eta_1$  of  $\nu$ . In particular, the spectrum  $\text{Sp}(\mathcal{K})$  of  $\mathcal{K}$  contains 1 (with eigenspace given by the constants). Then  $\mu_1, \mu_2$  are, respectively, the smallest and the largest value in  $\text{Sp}(\mathcal{K}) \setminus \{1\}$ , as we see by writing  $\nu(\varphi(\eta_1)\varphi(\eta_2)) = \nu[\varphi(\eta_1)\mathcal{K}\varphi(\eta_1)]$ . This is now a one-dimensional problem and  $\mu_1, \mu_2$  can be computed as follows. To fix ideas we use the value  $\omega = 1$  for the conservation law  $\eta_1 + \eta_2 + \eta_3$ . In this case  $\nu_1$  is the law on  $[0, 1]$  with density  $2(1 - \eta_1)$ . Moreover,

$$\mathcal{K}\varphi(\eta_1) = \frac{1}{1 - \eta_1} \int_0^{1-\eta_1} \varphi(\eta_2) d\eta_2, \quad \eta_1 \in [0, 1).$$

( $\mathcal{K}\varphi(1) = \varphi(0)$ ). In particular,  $\varphi_1(\alpha) = \alpha - \frac{1}{3}$  is an eigenfunction of  $\mathcal{K}$  with eigenvalue  $-1/2$ . Moreover,  $\mathcal{K}$  preserves the degree of polynomials so that if  $Q_n$  denotes the space of all polynomials of degree  $d \leq n$  we have  $\mathcal{K}Q_n \subset Q_n$ . By induction we see that for each  $n \geq 1$  the polynomial  $\alpha^n + q_{n-1}(\alpha)$ , for a suitable  $q_{n-1} \in Q_{n-1}$ , is an eigenfunction with eigenvalue  $\mu_n = \frac{(-1)^n}{n+1}$ , and it is orthogonal to  $Q_{n-1}$  in  $L^2(\nu_1)$ . Since the union of  $Q_n$ ,  $n \geq 1$ , is dense in  $L^2(\nu_1)$  this shows that there is a complete orthonormal set of eigenfunctions  $\varphi_n$ , where  $\varphi_n$  is a polynomial of degree  $n$  with eigenvalue  $\mu_n$  and  $\text{Sp}(\mathcal{K}) = \{\mu_n, n = 0, 1, \dots\}$ . Therefore we can take  $\mu_1 = -\frac{1}{2}$  and  $\mu_2 = \frac{1}{3}$  in the formula (2.8). We conclude that  $\lambda(3) \geq \frac{4}{9}$ .

To end the proof we take  $f = \eta_1^2 + \eta_2^2 + \eta_3^2$  and, using  $\nu[\eta_1^2 | \eta_2] = \frac{1}{3}(\eta_2^2 - 2\eta_2 + 1)$  we compute

$$\mathcal{L}f(\eta) = -\frac{4}{9}f(\eta) + \text{const.}$$

Thus,  $\lambda(3) = \frac{4}{9}$  and the eigenfunction is given by (2.6). Clearly, the unitary change of scale  $\mathcal{T}$  introduced above does not alter the form of the eigenfunction so that all the hypothesis of Theorem 2.1 apply and (2.7) follows.

**2.3. Momentum and energy conserving collision model.** Here  $X = \mathbb{R}^3$ , and  $\mu$  is a centered 3-dimensional Gaussian law  $N(0, C)$  with covariance matrix  $C$  given by a multiple of the identity. Each coordinate  $\eta_i$  is a 3-dimensional velocity vector  $\eta_i^\alpha$ ,  $\alpha = 1, 2, 3$ . We have  $d = 4$  conservation laws, with  $\xi^\alpha(\eta_i) = \eta_i^\alpha$ ,  $\alpha = 1, 2, 3$  (momentum conservation) and  $\xi^4(\eta_i) = |\eta_i|^2 = \sum_{\alpha=1}^3 (\eta_i^\alpha)^2$  (energy conservation). For any  $\omega = (\omega^1, \dots, \omega^4) \in \mathbb{R}^4$  with  $\omega^4 > 0$ ,  $\nu_{N,\omega}$  is the uniform probability measure on the manifold  $\Omega_{N,\omega}$  (whenever  $\Omega_{N,\omega} \neq \emptyset$ ). We refer to Carlen et al. (2003) for an explicit description of the probability measure and its main properties. It is still the case that  $\lambda(3)$  is independent of the conservation law, see Carlen et al. (2003). In particular, the lower bound (2.3) holds in this case. However, a computation of  $\lambda(3)$  for this model shows that  $\lambda(3) = \frac{1}{3}$  (see (2.11) below) and therefore the estimate becomes  $\lambda(N) \geq \frac{1}{N}$  which is rather poor. Indeed, it was shown in Carlen

et al. (2003) that the the spectral gap is bounded away from zero uniformly in  $N$ :

$$\inf_{N \geq 2} \lambda(N) > 0. \quad (2.10)$$

Recently, by a very deep analysis of the Jacobi polynomials naturally associated to this model Carlen, Jeronimo and Loss succeeded in computing  $\lambda(N)$  exactly for every  $N$  (see Carlen et al. (2007)):

$$\lambda(N) = \frac{1}{3}, \quad N \geq 3. \quad (2.11)$$

This shows our approach is too rough here. As we know from Theorem 2.1 the loss must come from the lack of the second property required in that theorem. It was shown in Carlen et al. (2007) that the eigenspace of  $\lambda(N)$ , for the choice  $\omega = (0, 0, 0, 1)$ , is spanned by the functions

$$f_{N,\alpha}(\eta) = \sum_{i=1}^N |\eta_i|^2 \eta_i^\alpha, \quad \alpha = 1, 2, 3.$$

One cannot expect that the change of scale from  $\omega$  to  $\omega'$  transforms a linear combination of  $f_{N,\alpha}$ 's into itself (up to multiplicative and additive constants), and the second property in Theorem 2.1 must fail here.

Let us show that our approach can nevertheless be used to prove the weaker result (2.10) without any recursive analysis. Namely, we prove that if  $\lambda(4) > 1/4$  then (2.10) holds. The choice of triangles in the proof of Theorem 1.1 and Theorem 2.1 can be replaced by the choice of larger cliques (i.e. complete subgraphs) of the original complete graph. Namely, if in (1.13) we sum over cliques with 4 vertices instead of triangles we shall obtain the bound, for  $N \geq 4$ :

$$\lambda(N) \geq (4\lambda(4) - 1) \left( \frac{1}{2} - \frac{1}{N} \right) + \frac{1}{N}, \quad (2.12)$$

instead of (2.3). To see this, set for simplicity  $a_{b,b'} := \nu[D_b f D_{b'} f]$ , for a given  $f \in L^2(\nu)$ , and recall that  $N^2 \nu[(\mathcal{L}f)^2] = \sum_{b,b'} a_{b,b'}$ . Denote by  $Q$  the cliques of 4 vertices and note that: for every  $b$  there are  $\frac{1}{2}(N-2)(N-3)$   $Q$ 's such that  $b \in Q$ ; for any  $b \neq b'$  with  $b \sim b'$  there are  $(N-3)$   $Q$ 's such that  $b, b' \in Q$ ; for any  $b, b'$  with  $b \not\sim b'$  there is only one  $Q$  such that  $b, b' \in Q$ . Then

$$\begin{aligned} \sum_{b,b': b \neq b', b \sim b'} a_{b,b'} &= \frac{1}{N-3} \sum_Q \sum_{b,b' \in Q: b \neq b', b \sim b'} a_{b,b'} \\ &= \frac{1}{N-3} \sum_Q \left\{ \sum_{b,b' \in Q} a_{b,b'} - \sum_{b,b' \in Q: b \not\sim b'} a_{b,b'} - \sum_{b \in Q} a_{b,b} \right\} \\ &\geq \frac{1}{N-3} \sum_Q (4\lambda(4) - 1) \sum_{b \in Q} a_{b,b} - \frac{1}{N-3} \sum_Q \sum_{b,b' \in Q: b \not\sim b'} a_{b,b'} \\ &= \frac{1}{2}(N-2)(4\lambda(4) - 1) \sum_b a_{b,b} - \frac{1}{N-3} \sum_{b,b': b \not\sim b'} a_{b,b'}. \end{aligned}$$



Therefore

$$\begin{aligned} \sum_{b,b'} a_{b,b'} &= \sum_{b \neq b' : b \sim b'} a_{b,b'} + \sum_{b,b' : b \not\sim b'} a_{b,b'} + \sum_b a_{b,b} \\ &\geq \frac{1}{2} [(N-2)(4\lambda(4) - 1) + 2] \sum_b a_{b,b} + \left(1 - \frac{1}{N-3}\right) \sum_{b,b' : b \not\sim b'} a_{b,b'} \\ &\geq \frac{1}{2} [(N-2)(4\lambda(4) - 1) + 2] \sum_b a_{b,b}, \end{aligned}$$

where, in the last line, we have used (1.11). Since  $-N\nu(f\mathcal{L}f) = \sum_b a_{b,b}$ , this proves the claim (2.12). Note that this argument applies in the more general setting of Theorem 2.1, and similar computations can in principle be carried out for any choice of cliques of  $k < N$  vertices. Therefore, to prove (2.10) it suffices to prove  $\lambda(4) > \frac{1}{4}$ . With the value  $\lambda(4) = \frac{1}{3}$  from (2.11) this gives  $\lambda(N) \geq \frac{1}{6} + \frac{2}{3N}$ , for all  $N \geq 4$ .

### 3. Non-homogeneous models

We generalize the setting introduced above as follows. As before we take  $\Omega = X^N$  but now each copy of  $X$  will be equipped with possibly distinct probability measures  $\mu_i$ ,  $i = 1, \dots, N$ . Again we consider conservation laws as in (2.1), associated to a given function  $\xi$  on  $X$  and a given parameter  $\omega$ , and the probability  $\nu$  given by the product  $\mu_1 \times \dots \times \mu_N$  conditioned on  $\eta \in \Omega_{N,\omega}$ . Note that the non-interference property still holds for this setting. The binary collision process is defined as in (2.2). Again,  $-\mathcal{L}$  is a non-negative self-adjoint operator on  $L^2(\nu)$  and we may define its spectral gap just as in (1.2). This time, however, to keep track of the conservation law we shall write  $\lambda(N, \omega)$  instead of just  $\lambda(N)$ . Note that  $\lambda(2, \omega) = \frac{1}{2}$  always. We define

$$\bar{\lambda}(N) = \inf_{\omega} \lambda(N, \omega), \quad (3.1)$$

where the infimum ranges over the set of admissible values of the parameter  $\omega$ . This set depends on the choice of the model and, as usual, we refer to the examples for fully rigorous formulations of the results. As a convention we may set  $\lambda(N, \omega) = +\infty$  if  $\omega$  is such that the measure  $\nu$  becomes a Dirac delta. Thanks to the non-interference property of  $\nu$  there is no difficulty in repeating the previous arguments to prove the following estimate.

**Theorem 3.1.** *For any  $N \geq 2$  and any  $\omega$ :*

$$\bar{\lambda}(N) \geq (3\bar{\lambda}(3) - 1) \left(1 - \frac{2}{N}\right) + \frac{1}{N}. \quad (3.2)$$

Let us investigate some specific models.

**3.1. Non-homogeneous Kac models.** Consider the following non-homogeneous version of the “flat” Kac model introduced in Section 2. Take  $X = \mathbb{R}_+$  and  $\mu_i$  the probability on  $\mathbb{R}_+$  with density

$$\frac{1}{z_i} \exp(-\eta_i + b_i(\eta_i)),$$

where  $b_i$  are bounded measurable functions and  $z_i = \int_0^\infty dx \exp(-x + b_i(x))$  is the normalizing constant. We set  $B := \sup_i |b_i|_\infty$ . For this model, any value  $\omega > 0$  of

the conservation law is allowed in the definition (3.1) of  $\bar{\lambda}(N)$ . We claim that for every  $\varepsilon < \frac{1}{3}$  there is  $\delta > 0$  such that

$$\liminf_{N \rightarrow \infty} \bar{\lambda}(N) \geq \varepsilon, \tag{3.3}$$

provided  $B < \delta$ . Thanks to Theorem 3.1, to prove (3.3) it suffices to show that  $\lambda(3, \omega) \geq \frac{4}{9}(1 - \varepsilon(B))$  with some  $\varepsilon(B) \rightarrow 0$  as  $B \rightarrow 0$ , uniformly in  $\omega > 0$ , the value of the conservation law  $\sum_{i=1}^3 \eta_i = \omega$ . To this end we shall use a standard perturbation argument. Let  $\nu$  denote the measure  $\mu_1 \times \mu_2 \times \mu_3(\cdot | \sum_{i=1}^3 \eta_i = \omega)$  and call  $\nu_0$  the same measure in the case  $b_1 = b_2 = b_3 = 0$ . Thus, for any bounded measurable function  $g$  we have

$$\nu(g) = \frac{1}{Z_\omega} \int_0^\omega d\eta_1 \int_0^{\omega - \eta_1} d\eta_2 g(\eta_1, \eta_2, \omega - \eta_1 - \eta_2) u(\eta_1, \eta_2), \tag{3.4}$$

where  $u(x, y) = e^{-b_1(x) - b_2(y) - b_3(\omega - x - y)}$  and  $Z_\omega = \int_0^\omega dx \int_0^{\omega - x} dy u(x, y)$ . Using

$$e^{-3B} \leq u(\eta_1, \eta_2) \leq e^{3B},$$

it is easily seen that, for any bounded  $f$  we have the bound between variances

$$\text{Var}_\nu(f) \leq e^{6B} \text{Var}_{\nu_0}(f). \tag{3.5}$$

(Use  $g = (f - \nu_0(f))^2$  in (3.4) and the fact that  $\nu(g) \geq \text{Var}_\nu(f)$ ). The same reasoning shows that

$$\nu(f(-\mathcal{L})f) \geq e^{-10B} \nu_0(f(-\mathcal{L}_0)f), \tag{3.6}$$

where  $\mathcal{L}_0$  is the generator corresponding to the choice  $b_i \equiv 0$ . Indeed,

$$\begin{aligned} \nu(f(-\mathcal{L})f) &= \frac{1}{3} \sum_{i=1}^3 \nu [(\nu[f | \eta_i] - f)^2] \\ &= \frac{1}{3} \sum_{i=1}^3 \nu (\text{Var}_\nu(f | \eta_i)), \end{aligned}$$

(where  $\text{Var}_\nu(f | \eta_i)$  denotes the variance of  $f$  w.r.t.  $\nu(\cdot | \eta_i)$ ). For each  $i = 1, 2, 3$  we have (as above)

$$\text{Var}_\nu(f | \eta_i) \geq e^{-4B} \text{Var}_{\nu_0}(f | \eta_i), \tag{3.7}$$

uniformly in  $\eta$ . A further comparison gives  $\nu(\text{Var}_\nu(f | \eta_i)) \geq e^{-10B} \nu_0(\text{Var}_{\nu_0}(f | \eta_i))$  which implies (3.6).

Recall that the spectral gap for  $\mathcal{L}_0$  is equal to  $4/9$  regardless of the value of  $\omega > 0$ . The previous estimates therefore imply that

$$\lambda(3, \omega) \geq \frac{4}{9} e^{-16B}.$$

This proves our claim with  $\varepsilon(B) = 1 - e^{-16B}$ , from which (3.3) follows.

The same argument can be used to produce uniform lower bounds on the gap of non-homogeneous versions of the Kac walk on  $S^{N-1}$  and of the momentum and energy conserving collision model, under the assumption of small perturbations. In the latter model we need the argument in (2.12) to obtain a uniform estimate  $\inf_N \bar{\lambda}(N) > 0$ .

3.2. *Non-uniform random permutations.* We take  $X = \{1, \dots, N\}$  and, for each  $i = 1, \dots, N$  we consider a probability  $\mu_i$  on  $X$  given by  $\mu_i(\eta_i = j) = \frac{e^{-b_i(j)}}{Z_i}$ , where  $b_i : X \rightarrow \mathbb{R}$  are bounded functions, and  $Z_i = \sum_{j=1}^N e^{-b_i(j)}$ . To model permutations we use  $N$  conservation laws that will force all components of  $\eta$  to have distinct values: set  $d = N$  and define  $\xi = (\xi^j)_{j=1}^N$  with  $\xi^j(\eta_i) = 1_{\{\eta_i=j\}}$ . Fixing  $\omega = (1, 1, \dots, 1)$  we see that the set  $\Omega_{N,\omega}$  coincides with the set of  $N!$  permutations of  $N$  letters. We define the probability  $\nu$  as usual by  $\mu_1 \times \dots \times \mu_N(\cdot | \eta \in \Omega_{N,\omega})$ . Note that if the bias functions  $b_i$  are all 0 then  $\nu$  is simply the uniform probability measure over permutations.

The binary collision is now a *random transposition* process. Note that only the value  $\omega = (1, 1, \dots, 1)$  is considered for the conservation law so that  $\bar{\lambda}(N) = \lambda(N, \omega)$  for this model (there is no real infimum in (3.1) here). In the uniform case ( $b_i \equiv 0$ ) a simple variant of the argument of Theorem 2.1 proves that  $\lambda(N) = \frac{1}{2}$  for every  $N \geq 2$ .

While relaxation to equilibrium for the uniform case is well known, the non-uniform case is certainly less understood. Here we can show that if  $B = \sup_i |b_i|_\infty$  is sufficiently small we have  $\inf_N \bar{\lambda}(N) > 0$ . More precisely, for every  $\varepsilon < \frac{1}{2}$  there exists  $\delta > 0$  such that  $B < \delta$  implies

$$\liminf_{N \rightarrow \infty} \bar{\lambda}(N) \geq \varepsilon,$$

for all  $N \geq 2$ . To prove this we use exactly the same argument we have used to prove (3.3). In particular, it suffices to show that  $\bar{\lambda}(3) \rightarrow \frac{1}{2}$  as  $B \rightarrow 0$ . The same perturbation argument will yield the desired estimate. To avoid repetitions we leave the details to the reader.

3.3. *Exclusion processes with site disorder.* Here we consider a non-homogeneous version of what is sometimes called the Bernoulli–Laplace process. The inhomogeneous distribution models site impurities or site disorder. We take  $X = \{0, 1\}$ ,  $\mu_i$  is a Bernoulli law with parameter  $p_i \in (0, 1)$ , i.e.  $\mu_i(\eta_i = 1) = p_i$  and  $\mu_i(\eta_i = 0) = 1 - p_i$ . The value of  $\eta_i$  is interpreted as the presence ( $\eta_i = 1$ ) or absence ( $\eta_i = 0$ ) of a particle at the vertex  $i$ . The function  $\xi$  is given by  $\xi(\eta_i) = \eta_i$  so that for any integer  $\omega \in \{0, 1, \dots, N\}$ , the set  $\Omega_{N,\omega}$  denotes the configurations of  $\omega$  particles over  $N$  vertices. The binary collision process (2.2) becomes nothing but the well known exclusion process on the complete graph  $\{1, \dots, N\}$ . This can be seen as follows. Given a pair  $b = \{i, j\}$  and a configuration  $\eta \in \Omega_{N,\omega}$ , write  $\omega_{ij} = \omega - \sum_{\ell \notin b} \eta_\ell$ . Clearly,  $\omega_{ij} \in \{0, 1, 2\}$ . Observe that, if  $\omega_{ij} = 1$  then  $\nu(f | \mathcal{F}_b)(\eta)$  is given by

$$\frac{p_i(1-p_j)}{p_i(1-p_j) + p_j(1-p_i)} f(\eta; 1, 0) + \frac{p_j(1-p_i)}{p_i(1-p_j) + p_j(1-p_i)} f(\eta; 0, 1)$$

where, for simplicity we write explicitly the  $i$ -th and  $j$ -th entries in  $f(\eta) = f(\eta; \eta_i, \eta_j)$ . On the other hand in the case  $\omega_{ij} \in \{0, 2\}$  we have  $\nu(f | \mathcal{F}_b)(\eta) = f(\eta)$ . Setting

$$c_b(\eta) = \frac{p_i(1-p_j)\eta_j(1-\eta_i)}{p_i(1-p_j) + p_j(1-p_i)} + \frac{p_j(1-p_i)\eta_i(1-\eta_j)}{p_i(1-p_j) + p_j(1-p_i)}, \quad (3.8)$$

we therefore obtain, for any  $\eta \in \Omega_{N,\omega}$ ,

$$\nu(f | \mathcal{F}_b)(\eta) - f(\eta) = c_b(\eta) (f(\eta^b) - f(\eta)), \quad (3.9)$$

where  $\eta^b$  denotes the configuration in which  $\eta_i$  and  $\eta_j$  have been exchanged. From (3.9) we see that  $\mathcal{L}$  has the familiar form of the exclusion process.

If we proceed by perturbative arguments (as in the previous two subsections) we would be able to prove a result of the form: if  $p_i$  are (uniformly) sufficiently close to  $\frac{1}{2}$  then we have a uniform bound from below on the spectral gap. However, we shall prove here a much stronger result. We assume there exists  $\varepsilon > 0$  such that the parameters  $p_i$  satisfy

$$\varepsilon \leq p_i \leq 1 - \varepsilon, \quad i = 1, \dots, N. \quad (3.10)$$

The minimal spectral gap  $\bar{\lambda}(N)$  is defined as usual by (3.1), where the infimum ranges over all  $\omega \in \{0, 1, \dots, N\}$ , with the convention that  $\lambda(N, 0) = \lambda(N, N) = \infty$ . Under the same assumption the following theorem was proved in Caputo (2004) by means of rather technical local limit theorem estimates. It is surprising that the simple argument of Theorem 3.1 allows a straightforward proof. The uniform spectral gap bound below is an important step in the recent works Faggionato and Martinelli (2003); Quastel (2006) establishing hydrodynamic limits for exclusion processes with disorder.

**Theorem 3.2.** *Assume (3.10) for some  $\varepsilon > 0$ . Then, there exists  $c_\varepsilon > 0$  such that for all  $N \geq 2$*

$$\bar{\lambda}(N) \geq c_\varepsilon. \quad (3.11)$$

*Proof:* Thanks to Theorem 3.1, all we have to do is prove that

$$\bar{\lambda}(3) \geq \frac{1}{3} + c_\varepsilon, \quad (3.12)$$

for some  $c_\varepsilon > 0$ . We fix three vertices  $i = 1, 2, 3$ , with their occupation probabilities  $p_i$  satisfying (3.10) and with  $\sum_{i=1}^3 \eta_i = \omega$ . We may assume that  $\omega = 1$ , i.e. there is one particle. Indeed if there are two we may look at occupied vertices as empty and vice-versa, if there is none (or three) the measure is a Dirac delta and by our convention (see discussion after (3.1)) the estimate  $\lambda(3, \omega) \geq \frac{1}{3} + c_\varepsilon$  becomes obvious. Since there is one particle we shall call  $x$ , respectively  $y$ , the probability that the particle is at  $i = 1$ , respectively  $i = 2$ . We set  $z = 1 - x - y$  for the probability that the particle is at  $i = 3$ . Note that, thanks to (3.10),  $x, y, z$  are all bounded away from 0 and 1. For instance,

$$x = \frac{p_1(1-p_2)(1-p_3)}{p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) + (1-p_1)(1-p_2)p_3}.$$

It is easily checked that the process generated by  $\mathcal{L}$  on our three sites becomes a 3-state Markov chain with the  $3 \times 3$  transition matrix  $P = \mathcal{L} + \text{Id}$  given by

$$P = \frac{1}{3} \begin{pmatrix} 1 + \frac{x}{x+y} + \frac{x}{x+z} & \frac{y}{x+y} & \frac{z}{x+z} \\ \frac{x}{x+y} & 1 + \frac{y}{x+y} + \frac{y}{y+z} & \frac{z}{y+z} \\ \frac{x}{x+z} & \frac{y}{y+z} & 1 + \frac{z}{x+z} + \frac{z}{y+z} \end{pmatrix}. \quad (3.13)$$

We need to estimate the eigenvalues of  $P$ . Clearly, one eigenvalue is 1. From (3.13) we see that  $\text{Tr}(P) = 2$ . Therefore the other two eigenvalues must satisfy  $\lambda_1 = \text{Tr}(P) - 1 - \lambda_2 = 1 - \lambda_2$ . Note that

$$\lambda(3, \omega) = \min\{1 - \lambda_1, 1 - \lambda_2\}.$$

To estimate  $\lambda_i, i = 1, 2$  we compute the determinant of  $P$ . The next lemma shows that  $\det(P) > \frac{2}{9}$ . Therefore, for both  $i = 1, 2$  we have  $\lambda_i(1 - \lambda_i) > \frac{2}{9}$ . This implies  $|\lambda_i - \frac{1}{2}| < \frac{1}{6}$ . In particular, it shows that  $\lambda_i < \frac{2}{3}$ . In conclusion:  $\bar{\lambda}(3) > \frac{1}{3}$  as claimed. (Note that  $\lambda_i = \frac{1}{2}$  would be the value in the homogeneous case  $p_i \equiv \frac{1}{2}$ .)

**Lemma 3.3.**

$$\det(P) = \frac{2}{9} \left( 1 + \frac{xyz}{(1-x)(1-y)(1-z)} \right) \quad (3.14)$$

*Proof:* Set  $\Gamma = 3P$ . Also, set

$$\delta = \frac{x}{x+z}, \quad \beta = \frac{x}{x+y}, \quad \gamma = \frac{y}{y+z}.$$

From (3.13) we compute

$$\begin{aligned} \det(\Gamma) &= (1 + \beta + \delta) (6 - 3\beta - 2\delta + \beta\delta - \gamma\delta + \gamma\beta) \\ &\quad + (\beta - 1) (3\beta - \beta\delta - \gamma\beta - \delta + \gamma\delta) \\ &\quad + (1 - \delta) (\beta\delta - \gamma\delta + \gamma\beta - 2\delta). \end{aligned}$$

Simplifying we arrive at

$$\det(\Gamma) = 6 + 3\delta - 3\beta\delta - 3\gamma\delta + 3\beta\gamma.$$

Rewriting this in terms of the probabilities  $x, y, z$  we obtain

$$\det(\Gamma) = 6 + \frac{6xyz}{(1-x)(1-y)(1-z)}. \quad (3.15)$$

This implies (3.14).  $\square$

**3.4. Colored exclusion processes with site disorder.** A natural generalization of the previous model is a system where particles can be of several different kinds - or colors. Namely, suppose there are  $m$  colors and let each particle be painted with one of the available colors. Configurations of colored particles are denoted by  $\eta \in \Omega := \{0, 1, \dots, m\}^N$  with the interpretation that  $\eta_i = 0$  means that  $i$  is empty, while  $\eta_i = k$ ,  $k \in \{1, \dots, m\}$  means that  $i$  is occupied by a particle with color  $k$ . Thus, the single state space  $X$  is  $\{0, \dots, m\}$ . The conservation laws are given by  $\xi^k(\eta_i) = 1_{\{\eta_i=k\}}$ ,  $k = 1, \dots, m$  and the vector  $\omega = (\omega_1, \dots, \omega_m)$ , where  $\omega_i$  are non-negative integers such that  $\sum_{k=1}^m \omega_k \leq N$ . Thus, the set  $\Omega_{N,\omega}$  denotes the configurations of particles over  $N$  vertices, with  $\omega_k$  particles with color  $k$ . We shall use the notation  $\psi_i = 1_{\{\eta_i \geq 1\}}$  so that the variables  $\psi \in \{0, 1\}^N$  denote the configuration of occupied sites. Let  $p_i$ ,  $i = 1, \dots, N$  be given parameters satisfying (3.10). Let  $\mu_i$  denote the probability on  $X$  such that

$$\mu_i(\eta_i = k) = \frac{1}{Z_i} (p_i 1_{\{k \geq 1\}} + (1 - p_i) 1_{\{k=0\}}),$$

where  $Z_i = (m-1)p_i + 1$  is the normalization. In particular, w.r.t.  $\mu_i$  the occupation variable  $\psi_i$  is a Bernoulli random variable with parameter  $mp_i/Z_i$ . We call, as usual,  $\nu$  the product  $\mu_1 \times \dots \times \mu_N$  conditioned on  $\Omega_{N,\omega}$ . To avoid degenerate cases we take  $1 \leq \sum_{k=1}^m \omega_k \leq N-1$ . We are interested in the following exclusion-type dynamics. For any configuration  $\eta$  and any edge  $b = \{i, j\}$  we write  $\eta^b$  for the configuration where the variables  $\eta_i, \eta_j$  have been exchanged. For every  $\gamma \geq 0$  we define the Markov generator by

$$\mathcal{L}_\gamma f(\eta) = \frac{1}{N} \sum_b c_b^\gamma(\eta) (f(\eta^b) - f(\eta)), \quad (3.16)$$

where the rates are given, for  $b = (i, j)$ , by

$$c_b^\gamma(\eta) = c_b(\psi) + \frac{\gamma}{2} 1_{\{\psi_i = \psi_j\}} \quad (3.17)$$

where  $c_b$  are the functions defined in (3.8), but now evaluated at the occupation variables  $\psi = \psi(\eta)$  defined above. It is easily checked that the rates (3.17) are reversible w.r.t.  $\nu$ : for any  $\eta$  and any  $b$

$$\nu(\eta)c_b^\gamma(\eta) = \nu(\eta^b)c_b^\gamma(\eta^b). \quad (3.18)$$

The latter statement is equivalent to self-adjointness of  $\mathcal{L}_\gamma$  in  $L^2(\nu)$ . Moreover,

$$-\nu(f\mathcal{L}_\gamma f) = \frac{1}{2} \sum_b \nu [c_b^\gamma (f^b - f)^2], \quad (3.19)$$

where  $f^b(\eta) := f(\eta^b)$ . If  $1 \leq \sum_{k=1}^m \omega_k \leq N - 1$  the Markov chain generated by  $\mathcal{L}_\gamma$  is irreducible. We shall consider the cases  $\gamma = 0$  and  $\gamma = 1$ . The difference between  $\mathcal{L}_0$  and  $\mathcal{L}_1$  is that in  $\mathcal{L}_1$  we have added the possibility of “stirring” between particles, i.e. exchange of positions of particles of different colors. The case  $\gamma = 1$  coincides then with the usual binary collision dynamics given by local averages (2.2). On the other hand, the case  $\gamma = 0$  is a true *exclusion* process, with particles jumping only to empty sites. The addition of stirring can result in a faster relaxation to the equilibrium distribution  $\nu$ , or equivalently in a larger spectral gap. Note, however, that for the case  $m = 1$  there is no difference: in this case  $\mathcal{L}_\gamma = \mathcal{L}_0$  for all  $\gamma$ . The case  $m = 1$ , of course, is the one analyzed in Theorem 3.2. From now on we take  $m > 1$ .

We denote by  $\lambda_\gamma(N, \omega)$  the spectral gap of the generator  $\mathcal{L}_\gamma$ . The following theorem shows that in the case  $\gamma = 1$  we have a uniform lower bound  $\lambda_1(N, \omega) \geq c_\varepsilon$  as in Theorem 3.2 and, in the case  $\gamma = 0$  we have  $\lambda_0(N, \omega) \geq c_\varepsilon(1 - \rho)$ , where  $\rho$  is the global density:

$$\rho = \frac{1}{N} \sum_{k=1}^m \omega_k. \quad (3.20)$$

The slow-down in the limit  $\rho \rightarrow 1$  is natural in view of the absence of stirring. Moreover, we shall show that, up to a constant the reverse inequality  $\lambda_0(N, \omega) \leq C(1 - \rho)$  holds as well for some choices of  $\omega$ , see the remark after the end of the proof. Similar results had been obtained in Quastel (1992) in the homogeneous case  $p_i \equiv \frac{1}{2}$ .

**Theorem 3.4.** *Assume (3.10) for some  $\varepsilon > 0$ . For  $m > 1$ ,  $\gamma = 1$ , there exists  $c_\varepsilon > 0$  such that for any  $N \geq 2$  and any  $\omega$  such that  $0 < \rho < 1$ :*

$$\lambda_1(N, \omega) \geq c_\varepsilon. \quad (3.21)$$

*For  $m > 1$ ,  $\gamma = 0$ , there exists  $c_\varepsilon > 0$  such that for any  $N \geq 2$  and any  $\omega$  with  $0 < \rho < 1$ :*

$$\lambda_0(N, \omega) \geq c_\varepsilon(1 - \rho). \quad (3.22)$$

*Proof:* We start with some preliminary facts. Consider functions  $f$  of the occupation variables  $\psi = \{\psi_i\}$  only. Let  $\mathcal{H}_0$  denote the space of all such functions and observe that  $\mathcal{L}_\gamma \mathcal{H}_0 \subset \mathcal{H}_0$ , i.e.  $\mathcal{H}_0$  is invariant, for both  $\gamma = 0, 1$ . This follows from the fact that the only dependence of the rates on the configuration  $\eta$  is through the variables  $\{\psi_i\}$ , see (3.17). In particular, under the generator  $\mathcal{L}_\gamma$ , the variables  $\{\psi_i\}$  evolve as the Markov chain of the case  $m = 1$ . Therefore, the spectral gap estimate of Theorem 3.2 applies to functions in  $\mathcal{H}_0$ , for both  $\gamma = 0, 1$ .

For any  $f \in L^2(\nu)$  we may write  $f = f_0 + f_0^\perp$ , where  $f_0 = \nu(f | \mathcal{F}_\psi) \in \mathcal{H}_0$  and  $f_0^\perp = f - f_0 \in \mathcal{H}_0^\perp$ . Here  $\mathcal{F}_\psi$  denotes the  $\sigma$ -algebra generated by the functions  $\psi_i$  and  $\mathcal{H}_0^\perp$  is the orthogonal complement of  $\mathcal{H}_0$ , i.e. the space of  $f$  such that

$\nu(fg) = 0$  for all  $g \in \mathcal{H}_0$ . Since  $\mathcal{L}_\gamma$  leaves  $\mathcal{H}_0$  invariant, by self-adjointness it follows that  $\mathcal{L}_\gamma \mathcal{H}_0^\perp \subset \mathcal{H}_0^\perp$ . In conclusion

$$\nu(f\mathcal{L}_\gamma f) = \nu(f_0\mathcal{L}_\gamma f_0) + \nu(f_0^\perp\mathcal{L}_\gamma f_0^\perp).$$

Moreover,  $\text{Var}_\nu(f) = \text{Var}_\nu(f_0) + \text{Var}_\nu(f_0^\perp)$ . From these simple observations it is clear that the constant  $\lambda_\gamma(N, \omega)$  satisfies

$$\lambda_\gamma(N, \omega) \geq \min\{\lambda_\gamma^0(N, \omega), \lambda_\gamma^\perp(N, \omega)\}, \quad (3.23)$$

where  $\lambda_\gamma^0(N, \omega), \lambda_\gamma^\perp(N, \omega)$  are the constants obtained in the variational principle (1.2) – applied to  $\mathcal{L}_\gamma$  – by restricting to  $f \in \mathcal{H}_0$  and  $f \in \mathcal{H}_0^\perp$  respectively. From Theorem 3.2 we know that

$$\lambda_\gamma^0(N, \omega) \geq c_\varepsilon, \quad (3.24)$$

for some  $c_\varepsilon$ , for both  $\gamma = 0, 1$ . To prove Theorem 3.4 we thus have to estimate from below the constant  $\lambda_\gamma^\perp(N, \omega)$ .

3.4.1. *The case  $\gamma = 1$ .* If  $f \in \mathcal{H}_0^\perp$ , then we must have  $\nu(f | \mathcal{F}_\psi) = 0$ . Therefore

$$\text{Var}_\nu(f) = \nu[\text{Var}(f | \mathcal{F}_\psi)] + \text{Var}_\nu(\nu(f | \mathcal{F}_\psi)) = \nu[\text{Var}(f | \mathcal{F}_\psi)]. \quad (3.25)$$

Here  $\text{Var}(\cdot | \mathcal{F}_\psi)$  denotes variance w.r.t.  $\nu(\cdot | \mathcal{F}_\psi)$ , the conditional probability given the color-blind configuration  $\psi$ , and we have used the standard decomposition of variance under conditioning. Let  $S = S(\eta) := \{i \in \{1, \dots, N\} : \psi_i = 1\}$  denote the set of occupied sites. Clearly  $|S| = \rho N$ , the total number of particles. Observe that once  $\psi$  is known then the distribution of the variables  $\eta$  is given by  $\eta_i = 0$  for  $i \notin S$  (deterministically) and  $\eta_i \in \{1, \dots, m\}$  on  $S$  uniformly with the constraint that  $\sum_{i \in S} 1_{\{\eta_i = k\}} = \omega_k$ ,  $k \in \{1, \dots, m\}$ . In particular there is no inhomogeneity once the set  $S$  (or, equivalently the configuration  $\psi$ ) is given. Therefore  $\text{Var}(f | \mathcal{F}_\psi)$  can be estimated for every  $\eta$  with the well known bound for random transpositions without disorder (see e.g. Caputo (2004)):

$$\text{Var}(f | \mathcal{F}_\psi) \leq \frac{1}{4|S|} \sum_{i,j \in S} \nu [(\nabla_{ij} f)^2 | \mathcal{F}_\psi]. \quad (3.26)$$

Here we use the notation  $\nabla_{ij} f(\eta) = f(\eta^b) - f(\eta)$ ,  $b = \{i, j\}$ , for the exchange gradient. Averaging with  $\nu$  and using (3.25) we obtain (since  $|S| = \rho N$ , deterministically)

$$\text{Var}_\nu(f) \leq \frac{1}{4\rho N} \nu \left( \sum_{i,j \in S} (\nabla_{ij} f)^2 \right). \quad (3.27)$$

Suppose first  $\rho \geq \frac{1}{2}$ . Then (3.27), (3.19) and a uniform lower bound on the rates  $c_b$  imply

$$\text{Var}_\nu(f) \leq \frac{1}{2N} \sum_{i,j} \nu [(\nabla_{ij} f)^2] \leq C_\varepsilon \nu(f(-\mathcal{L}_1)f),$$

for some constant  $C_\varepsilon < \infty$ , with the sum now extending to all pairs  $i, j$ . This shows that, with  $c_\varepsilon = 1/C_\varepsilon$ :

$$\lambda_1^\perp(N, \omega) \geq c_\varepsilon, \quad \rho \geq \frac{1}{2}. \quad (3.28)$$

Note that here we have used  $\gamma = 1$  (if  $\gamma = 0$  there is no uniform lower bound on the rates  $c_b$ ).

We turn to the proof in the case  $\rho \leq \frac{1}{2}$ . We rewrite (3.27) as

$$\begin{aligned} \text{Var}_\nu(f) &\leq \frac{1}{4\rho N} \sum_{i,j} \nu [(\nabla_{ij}f)^2 1_{\{i \in S\}} 1_{\{j \in S\}}] \\ &= \frac{1}{4\rho(1-\rho)N^2} \sum_{i,j,\ell} \nu [(\nabla_{ij}f)^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}}], \end{aligned} \quad (3.29)$$

where we use the identity  $(1-\rho)N = N - \sum_{k=1}^m \omega_k = \sum_\ell 1_{\{\ell \notin S\}}$  for the number of empty sites. Let  $\eta \in \Omega_{N,\omega}$  be fixed. Pick  $i, j \in S(\eta)$  and write  $\eta^{i,j}$  for the exchanged configuration. Observe that for any  $\ell \notin S(\eta)$  we can write

$$\eta^{i,j} = \left( (\eta^{i,\ell})^{i,j} \right)^{j,\ell}.$$

Therefore

$$\begin{aligned} \nabla_{ij}f(\eta) &= [f(\eta^{i,j}) - f(\eta)] \\ &= \nabla_{j\ell}f \left( (\eta^{i,\ell})^{i,j} \right) + \nabla_{ij}f(\eta^{i,\ell}) + \nabla_{i\ell}f(\eta). \end{aligned}$$

Each term in the sum appearing in (3.29) can then be estimated by

$$\begin{aligned} &\nu [(\nabla_{ij}f)^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}}] \\ &\leq 3\nu \left[ \left\{ \left( \nabla_{j\ell}f \left( (\eta^{i,\ell})^{i,j} \right) \right)^2 + \left( \nabla_{ij}f(\eta^{i,\ell}) \right)^2 + \left( \nabla_{i\ell}f \right)^2 \right\} 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}} \right]. \end{aligned} \quad (3.30)$$

Next, we claim that there exists  $C_1 = C_1(\varepsilon) < \infty$  such that

$$\begin{aligned} &\nu \left[ \left( \nabla_{j\ell}f \left( (\eta^{i,\ell})^{i,j} \right) \right)^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}} \right] \\ &\leq C_1 \nu \left[ \left( \nabla_{j\ell}f(\eta) \right)^2 1_{\{i \in S\}} 1_{\{j \notin S\}} 1_{\{\ell \in S\}} \right]. \end{aligned} \quad (3.31)$$

Note the change of the indicator functions in (3.31). To prove (3.31) we write, with the change of variables  $\varphi := \eta^{i,\ell}$  and  $\beta := \varphi^{i,j}$

$$\begin{aligned} &\nu \left[ \left( \nabla_{j\ell}f \left( (\eta^{i,\ell})^{i,j} \right) \right)^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}} \right] \\ &= \sum_{\eta} \nu(\eta) \left( \nabla_{j\ell}f \left( (\eta^{i,\ell})^{i,j} \right) \right)^2 1_{\{i \in S(\eta)\}} 1_{\{j \in S(\eta)\}} 1_{\{\ell \notin S(\eta)\}} \\ &= \sum_{\varphi} \nu(\varphi^{i,\ell}) \left( \nabla_{j\ell}f(\varphi^{i,j}) \right)^2 1_{\{i \notin S(\varphi)\}} 1_{\{j \in S(\varphi)\}} 1_{\{\ell \in S(\varphi)\}} \\ &= \sum_{\beta} \nu((\beta^{i,j})^{i,\ell}) \left( \nabla_{j\ell}f(\beta) \right)^2 1_{\{i \in S(\beta)\}} 1_{\{j \notin S(\beta)\}} 1_{\{\ell \in S(\beta)\}} \\ &= \nu \left[ \chi_{i,j,\ell} \left( \nabla_{j\ell}f \right)^2 1_{\{i \in S\}} 1_{\{j \notin S\}} 1_{\{\ell \in S\}} \right], \end{aligned}$$

where

$$\chi_{i,j,\ell}(\eta) := \frac{\nu((\eta^{i,j})^{i,\ell})}{\nu(\eta)},$$

is the change of measure. Since the variables  $p_i$  defining our measure satisfy (3.10) it is not hard to check that  $\chi_{i,j,\ell}(\eta) \leq C_\varepsilon$  uniformly for some constant  $C_\varepsilon$ . This proves (3.31).



Moreover, in a similar way one proves that there is a constant  $C_2 = C_2(\varepsilon) < \infty$  such that

$$\begin{aligned} & \nu \left[ (\nabla_{ij} f(\eta^{i,\ell}))^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}} \right] \\ & \leq C_2 \nu \left[ (\nabla_{ij} f(\eta))^2 1_{\{i \notin S\}} 1_{\{j \in S\}} 1_{\{\ell \in S\}} \right]. \end{aligned} \quad (3.32)$$

From (3.29) and (3.30), using (3.31) and (3.32) we obtain for a suitable constant  $C_3$ :

$$\begin{aligned} \text{Var}_\nu(f) & \leq \frac{C_3}{\rho(1-\rho)N^2} \sum_{i,j,\ell} \left\{ \nu \left[ (\nabla_{j\ell} f)^2 1_{\{i \in S\}} 1_{\{j \notin S\}} 1_{\{\ell \in S\}} \right] \right. \\ & \left. + \nu \left[ (\nabla_{ij} f)^2 1_{\{i \notin S\}} 1_{\{j \in S\}} 1_{\{\ell \in S\}} \right] + \nu \left[ (\nabla_{i\ell} f)^2 1_{\{i \in S\}} 1_{\{j \in S\}} 1_{\{\ell \notin S\}} \right] \right\}. \end{aligned} \quad (3.33)$$

Since  $\sum_i 1_{\{i \in S\}} = \rho N$ , setting  $C_4 = 3C_3$  we see that (3.33) can be written as

$$\text{Var}_\nu(f) \leq \frac{C_4}{(1-\rho)N} \sum_{i,j} \nu \left[ (\nabla_{ij} f)^2 1_{\{i \in S\}} 1_{\{j \notin S\}} \right]. \quad (3.34)$$

Using  $\rho \leq \frac{1}{2}$  and (3.19) we see that

$$\text{Var}_\nu(f) \leq C \nu(f(-\mathcal{L}_1)f),$$

for some constant  $C = C(\varepsilon) < \infty$  and for all  $f \in \mathcal{H}_0^\perp$ . Therefore we have proved that  $\lambda_1^\perp(N, \omega) \geq c_\varepsilon$ . Together with (3.23), (3.24) and (3.28) we see that  $\lambda_1(N, \rho)$  is uniformly bounded from below and the claim (3.21) holds. This proves Theorem 3.4 in the case  $\gamma = 1$ .

3.5. *The case  $\gamma = 0$ .* Here we cannot use the argument leading to (3.28) above. However, we can use the argument giving (3.34) without modification (it never used the fact that  $\gamma = 1$ ). In particular, (3.34) holds for any  $0 < \rho < 1$  and any  $f \in \mathcal{H}_0^\perp$ . Now, observe that for any edge  $b = \{i, j\}$  such that  $i \in S$  and  $j \notin S$  the rate  $c_b^0$  defined in (3.17) is uniformly bounded away from zero with constants only depending on the  $\varepsilon$  from (3.10) (this follows from the fact that for such cases either  $\psi_i(1 - \psi_j) = 1$  or  $\psi_j(1 - \psi_i) = 1$ ). Therefore, there exists  $C = C(\varepsilon) < \infty$  such that for any  $f \in \mathcal{H}_0^\perp$ :

$$\text{Var}_\nu(f) \leq \frac{C}{(1-\rho)} \nu(f(-\mathcal{L}_0)f). \quad (3.35)$$

This proves that  $\lambda_0^\perp(N, \omega) \geq c(1-\rho)$ . From (3.23) and (3.24) we see that  $\lambda_0(N, \omega)$  satisfies the claim (3.22). This proves Theorem 3.4 in the case  $\gamma = 0$ .  $\square$

Let us briefly address the issue of comparable upper bounds on spectral gaps. For example, to prove that  $\lambda_0(N, \omega) = O(1-\rho)$ , as  $\rho \rightarrow 1$  one can consider the case of  $m = 2$  colors, with  $\omega_1 = \omega_2$  such that  $\omega_1 + \omega_2 = \rho N$  with  $\rho \rightarrow 1$ . Then choose  $f$  in the variational principle (1.2) as the indicator function of the event that a given vertex  $i$  is occupied by a particle of color 1. The variance of  $f$  is of order 1. On the other hand, since the number of empty sites that can be used to change the value of  $\eta_i$  is  $(1-\rho)N$ , it is not hard to show that  $\nu(f(-\mathcal{L}_0)f)$  is of order  $(1-\rho)$ . It follows that  $\lambda_0(N, \omega) = O(1-\rho)$ . Of course, if  $\gamma = 1$  then  $\nu(f(-\mathcal{L}_1)f)$  is of order 1 and the gap does not vanish as  $\rho \rightarrow 1$  in that case.

Finally, we point out an interesting problem concerning local versions of the exclusion dynamics described by (3.19). The local dynamics is obtained by summing

over pairs  $b$  which are given by the edges of a small subgraph of the complete graph, such as e.g. the box of side  $L \sim N^{1/d}$  in a  $d$ -dimensional grid:  $\Lambda_L = [1, L]^d \cap \mathbb{Z}^d$ . In the latter cases one expects the gap to scale as  $L^{-2}$ . In the case  $m = 1$  there is a nice argument in Quastel (2006, Lemma 5.1 and Lemma 5.2) which allows to derive such an estimate from the complete-graph bound (3.11). On the other hand, the case  $m > 1$  with  $\gamma = 0$  seems to be much more complicated and we are not aware of any result in this direction, except for the homogeneous case considered in Quastel (1992).

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