Pub. Mat. UAB
Vol. 29 Nº 2-3 Nov. 1985

ON THE SPECTRAL MAPPING THEOREM FOR ESSENTIAL SPECTRA

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ABSTRACT.- B. Gramsch and D. Lay [2] have studied spectral mapping theorems for the essential spectra of an operator acting in a complex Banach space. Firstly they consider operators belonging to the Banach algebra of all bounded linear operators on the space, and later they derive the theorems for unbounded closed linear operators with non-empty resolvent from the above case; but bounded closed linear operators with domain a proper subspace are not included.

In this note we introduce a notion of extended essential spectra for any closed linear operator with non-empty resolvent, which covers the above cases. Then, in this more general context, we are able to prove the spectral mapping theorems by means of a more unified approach based on a factorization of the operators provided by the Dunford-Taylor calculus and well-known properties of products of operators present in Fredholm theory.

AMS Subject Classification (1980): Primary 47A60; secondary 47A53.

Let X be a complex Banach space, $C(X)$ the set of all closed linear operators in X, $L(X) := \{T \in C(X) \mid D(T) = X\}$, $T \in C(X)$; $D(T)$, $N(T)$, $R(T)$, $\alpha(T)$, $\beta(T)$, $i(T)$, $a(T)$ and $d(T)$ will denote the domain, kernel, range, nullity, defect, ascent and descent of T, respectively. We shall consider the following operator classes:

$\Phi_0 := \{T \in C(X) \mid \alpha(T) = \beta(T) = 0\}$

$\Phi_1 := \{T \in C(X) \mid \alpha(T),\ \beta(T) < \infty\}$

$\Phi_2 := \{T \in C(X) \mid \alpha(T) < \infty,\ R(T) \text{ complemented}\}$

$\Phi_3 := \{T \in C(X) \mid \beta(T) < \infty,\ N(T) \text{ complemented}\}$

$\Phi_4 := \{T \in C(X) \mid R(T) \text{ closed},\ \alpha(T) < \infty\}$

$\Phi_5 := \{T \in C(X) \mid \beta(T) < \infty\}$

$\Phi_6 := \Phi_4 \cup \Phi_5$

$\Phi_7 := \{T \in C(X) \mid \alpha(T) = \beta(T) < \infty\}$

$\Phi_8 := \{T \in \Phi_7 \mid a(T) = d(T) < \infty\}$

$\Phi_9 := \{T \in C(X) \mid a(T),\ d(T) < \infty\}$

$\Phi_{10} := \{T \in C(X) \mid R(T) \text{ closed}\}$.

The essential spectra $\sigma_i(T)$, $i = 0,1,..,10$, are defined in terms of the above classes:

$$\sigma_i(T) := \{\lambda \in \mathbb{C} \mid \lambda I - T \notin \Phi_i\},\quad \rho_i(T) := \mathbb{C} - \sigma_i(T)\ ,\ i = 0,1,..10;$$

note that $\rho_0(T) := \rho(T)$ and $\sigma_0(T) := \sigma(T)$, the usual resolvent and spectrum.

We now introduce the extended essential spectra.

Definition

Let $T \in C(X)$ with $\rho(T) \neq \emptyset$ and $\alpha \in \rho(T)$. For $i = 0,1,..,10$, we define

$$\sigma_{ie}(T) := \begin{cases} \sigma_i(T) & \text{if } (\alpha - T)^{-1} \in \Phi_i \\ \sigma_i(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

This definition does not depend of $\alpha \in \rho(T)$ because $\infty \in \sigma_{ie}(T)$ if and only if every $S \in L(X)$ such that $N(S) = \{0\}$ and $R(S) = D(T)$ does not belong to Φ_i . Notice that $\sigma_{oe}(T)$ is the usual extended spectrum $\sigma_e(T)$.

From now we consider $T \in C(X)$ with $\rho(T) \neq \emptyset$ and $\alpha \in \rho(T)$.

Let $\overline{C}$ be the extended complex plane and let $A(T)$ be the set of all functions $f : \overline{C} \longrightarrow \overline{C}$ with domain an open set $\Delta(f)$ such that $\sigma_e(T) \subset \Delta(f)$ and f holomorphic on $\Delta(f)$.

When $f \in A(T)$ we consider two open sets Δ, Δ' such that $\Delta(f) = \Delta \cup \Delta'$, $\Delta \cap \Delta' = \emptyset$, f is identically 0 on Δ' and is not identically 0 on each connected component of Δ; then $\sigma_e(T) = \sigma \cup \sigma'$ where $\sigma' := \sigma_e(T) \cap \Delta'$ and $\sigma := \sigma_e(T) \setminus \sigma'$. Moreover, E_σ will denote the projector associated with the function $e \in A(T)$ such that $e(\Delta) = \{1\}$ and $e(\Delta') = \{0\}$.

Lemma 1

" Let $i \in \{0,1,..,9\}$ and $\lambda \in C$. Then:

(a) $\lambda - T \in \Phi_i$ if and only if $(\lambda-T)(\alpha-T)^{-1} \in \Phi_i$.

(b) $E_\sigma \in \Phi_i$ if and only if $\sigma' \cap \sigma_{ie}(T)$ is empty ".

Proof. (a) As $R[(\alpha-T)^{-n}] = D(T^n)$ we have $R[(\lambda-T)^n] = R[((\lambda-T)(\alpha-T)^{-1})^n]$. On the other hand, since $(\lambda-T)^n(\alpha-T)^{-n}x = (\alpha-T)^{-n}(\lambda-T)^n x$ for $x \in D(T^n)$ and $(\alpha-T)^{-n}$ is injective, we have $N[(\lambda-T)^n] = N[((\lambda-T)(\alpha-T)^{-1})^n]$. Now the result is clear.

(b) We have $a(E_\sigma) = d(E_\sigma) \leq 1$, $\alpha(E_\sigma) = \beta(E_\sigma)$ and $R(E_\sigma)$ complemented because E_σ is a projector. Consequently, if $i = 0$ it is clear that $E_\sigma \in \Phi_0$ if and only if $\sigma' = \sigma' \cap \sigma_e(T)$ is empty; and for $i \neq 0$ we have $E_\sigma \in \Phi_i$ if and only if $\alpha(E_\sigma) < \infty$, that is, if and only if σ' is a finite set whose elements are poles of $(\lambda-T)^{-1}$ of finite rank, or equivalently $\sigma' \cap \sigma_{ie}$ is empty. #

As the zeros of $f \in A(T)$ are isolated points in Δ and σ is compact, there is only a finite number of them in σ, say $c_0 = \infty, c_1, \ldots, c_k$ with finite orders $m_0 \geq 0$, $m_i > 0$, $i = 1,\ldots,k$. Let $m := m_0 + m_1 + \ldots + m_k$ and $P(z) := \prod_{i=1}^{k} (c_i - z)^{m_i}$.

Lemma 2

" Let $f \in A(T)$. Then:

(a) If $T \in L(X)$, $f(T) = F(T)P(T)E_\sigma$ where $F(z)$ is locally holomorphic in $\Delta(f)$ with no zeros in $\sigma_e(T)$.

(b) If $T \notin L(X)$, $f(T) = F_\alpha(T)P(T)(\alpha-T)^{-m}E_\sigma$ where $F_\alpha(z)$ is locally holomorphic in $\Delta(f)$ with no zeros in $\sigma_e(T)$ ".

Proof. Define

$$F(z) := \begin{cases} 1 & \text{if } z \in \Delta' \\ f(z)P(z)^{-1} & \text{if } z \in \Delta \end{cases} \quad \text{and } F_\alpha(z) := \begin{cases} 1 & \text{if } z \in \Delta' \\ f(z)P(z)^{-1}(\alpha-z)^m & \text{if } z \in \Delta. \end{cases}$$

Now the result is evident from the properties of the Dunford-Taylor calculus, [5]. #

Remark

Since $F(z)$ and $F_\alpha(z)$ have no zeros in $\sigma_e(T)$, the operators $F(T), F_\alpha(T) \in L(X)$ are invertibles in $L(X)$. Moreover, if $T \in L(X)$, then $f(T)$ can be expressed as the product of the commuting operators $F(T)$, E_σ, $c_i - T \in L(X)$, $i = 1,\ldots,k$; when $T \notin L(X)$, then $f(T)$ is expressed as the product of the commuting operators $F_\alpha(T), E_\sigma, (\alpha-T)^{-1}, (c_i-T)(\alpha-T)^{-1} \in L(X)$, $i = 1,\ldots,k$.

Lemma 3

" Let $T \in L(X)$, $f \in A(T)$ and $j = 1,..,k$. Then:

(a) $f(T) \in \Phi_i$ if and only if $c_j - T$, $E_\sigma \in \Phi_i$ for $i = 0,1,2,3,4,5,8$.

(b) If $f(T) \in \Phi_6$ we have $c_j - T$, $E_\sigma \in \Phi_6$.

(c) If $c_j - T$, $E_\sigma \in \Phi_i$ then $f(T) \in \Phi_i$, for $i = 7,9$ ".

Proof. Firstly we note that $F(T)$, $F_\alpha(T) \in \Phi_i$ for $i = 0,1,..,10$.

(a) The result follows from the following: Let $A,B \in L(X)$ with $AB = BA$; then $AB \in \Phi_i$ if and only if $A,B \in \Phi_i$. These are well-known results; see, for example, [3; S.3.1] for $i = 1,2,3$; [1;(1.3.3),(1.3.4),(1.3.5)] for $i = 4,5$; [1; (1.4.8)] and [4;Prop.9] for $i = 8$.

(b) It is consequence of (a) and the equality $\Phi_6 = \Phi_4 \cup \Phi_5$.

(c) For $i = 7$ it follows from the additivity of the index for the product of Fredholm operators.

On the other hand, we know that $A,B \in L(X) \cap \Phi_9$ and $AB = BA$ imply $AB \in \Phi_9$, [2;lemma 5]; now, the result follows for $i = 9$. #

Lemma 4

" Let $T \notin L(X)$, $f \in A(T)$ and $j = 1,..,k$. Then:

(a) For $i = 0,1,2,3,4,5,8$ we have $f(T) \in \Phi_i$ if and only if $(c_j - T)(\alpha - T)^{-1}, E_\sigma \in \Phi_i$, and $(\alpha - T)^{-1} \in \Phi_i$ if $m_0 \neq 0$.

(b) If $f(T) \in \Phi_6$ we have $(c_j - T)(\alpha - T)^{-1}, E_\sigma \in \Phi_6$, and $(\alpha - T)^{-1} \in \Phi_6$ if $m_0 \neq 0$.

(c) If $(c_j - T)(\alpha - T)^{-1}, E_\sigma \in \Phi_i$, and $(\alpha - T)^{-1} \in \Phi_i$ if $m_0 \neq 0$, then $f(T) \in \Phi_i$ for $i = 7,9$ ".

Proof. Imitate the proof of the lemma 3. #

We now show spectral mapping theorem for essential spectra.

Theorem

" Let $f \in A(T)$. The following statement hold:

(a) $\sigma_i[f(T)] = f[\sigma_{ie}(T)]$, $i = 0,1,2,3,4,5,8$,

(b) $\sigma_6(f(T)) \supset f[\sigma_{6e}(T)]$.

(c) $\sigma_i(f(T)) \subset f[\sigma_{ie}(T)]$, $i = 7,9$ ".

Proof. Given $\mu \in C$, let $g(z) := \mu - f(z)$. Then $\mu \notin \sigma_i(f(T))$ if and only if $g(T) \in \Phi_i$, $i = 0,1,..,9$.

(a) From lemmas 1, 3 and 4 we derive that $g(T) \in \Phi_i$ if and only if $g(z) \neq 0$ for every $z \in \sigma_{ie}(T)$, that is, if and only if $\mu \notin f[\sigma_{ie}(T)]$. Notice that lemma 2 is applied to the function $g \in A(T)$.

(b) and (c) are also obtained from lemmas 1, 3 and 4 in a similar way.

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Rebut el 16 de gener del 1985

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