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RESEARCH

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On the spectral norm of r-circulant matrices with the Pell and Pell-Lucas numbers

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Abstract

Let us define $A = C_r(a_0, a_1, \ldots, a_{n-1})$ to be a $n \times n$ r-circulant matrix. The entries in the first row of $A = C_r(a_0, a_1, \ldots, a_{n-1})$ are $a_i = P_i$, $a_i = Q_i$, $a_i = P_i^2$ or $a_i = Q_i^2$ $(i = 0, 1, 2, \ldots, n-1)$, where P_i and Q_i are the ith Pell and Pell-Lucas numbers, respectively. We find some bounds estimation of the spectral norm for r-Circulant matrices with Pell and Pell-Lucas numbers.

Keywords: Pell numbers; Pell-Lucas numbers; r-circulant matrix; spectral norm

1 Introduction

Special matrices is a widely studied subject in matrix analysis. Especially special matrices whose entries are well-known number sequences have become a very interesting research subject in recent years and many authors have obtained some good results in this area. For example, Bahşi and Solak have studied the norms of r-circulant matrices with the hyper-Fibonacci and Lucas numbers [1], Bozkurt and Tam have obtained some results belong to determinants and inverses of r-circulant matrices associated with a number sequence [2], Shen and Cen have made a similar study by using r-circulant matrices with the Fibonacci and Lucas numbers [3, 4] and He *et al.* have established on the spectral norm inequalities on r-circulant matrices with Fibonacci and Lucas numbers [5].

Lots of article have been written so far, which concern estimates for spectral norms of circulant and r-circulant matrices, which have connections with signal and image processing, time series analysis and many other problems.

In this paper, we derive expressions of spectral norms for r-circulant matrices. We explain some preliminaries and well-known results. We thicken the identities of estimations for spectral norms of r-circulant matrices with the Pell and Pell-Lucas numbers.

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 1, \qquad P_n = 2P_{n-1} + P_{n-2} \quad \text{for } n \geq 2$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2Q_{n-1} + Q_{n-2} \quad \text{for } n \geq 2.$$

If we start from $n = 0$, then the Pell and Pell-Lucas sequence are given by

n :	0	1	2	3	4	5	6	7	···
P_n :	0	1	2	5	12	29	70	169	···
Q_n :	2	2	6	14	34	82	198	478	···

The following sum formulas for the Pell and Pell-Lucas numbers are well known [6, 7]:

$$\sum_{k=1}^{n} P_k^2 = \frac{P_n P_{n+1}}{2}$$

and

$$\sum_{k=1}^{n} Q_k^2 = \frac{Q_{2n+1} + 2(-1)^n - 4}{2}.$$

A matrix $C = [c_{ij}] \in M_{n,n}(\mathbb{C})$ is called a r-circulant matrix if it is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i, \\ rc_{n+j-i}, & j < i. \end{cases}$$

Obviously, the r-circulant matrix C is determined by the parameter r and its first row elements $c_0, c_1, \ldots, c_{n-1}$, thus we denote $C = C_r(c_0, c_1, \ldots, c_{n-1})$. Especially, let $r = 1$, the matrix C is called a circulant matrix [3].

The Euclidean norm of the matrix A is defined as

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

The singular values of the matrix A are

$$\sigma_i = \sqrt{\lambda_i \left(A^* A \right)},$$

where λ_i is an eigenvalue of A^*A and A^* is conjugate transpose of matrix A. The square roots of the maximum eigenvalues of A^*A are called the spectral norm of A and are induced by $\|A\|_2$.

The following inequality holds:

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E.$$

Define the maximum column length norm c_1, and the maximum row length norm r_1 of any matrix A by

$$r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2}$$

and

$$c_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2},$$

respectively. Let A, B, and C be $m \times n$ matrices. If $A = B \circ C$ then

$$\|A\|_2 \leq r_1(B)c_1(C) \quad [8].$$

2 Result and discussion

Theorem 1 *Let $A = C_r(P_0, P_1, \ldots, P_{n-1})$ be a r-circulant matrix, where $r \in \mathbb{C}$. We have*

$$\text{(i)} \quad |r| \geq 1, \quad \sqrt{\frac{P_n P_{n-1}}{2}} \leq \|A\|_2 \leq |r|\sqrt{(n-1)\frac{P_n P_{n-1}}{2}},$$

$$\text{(ii)} \quad |r| < 1, \quad |r|\sqrt{\frac{P_n P_{n-1}}{2}} \leq \|A\|_2 \leq \sqrt{(n-1)\frac{P_n P_{n-1}}{2}}.$$

Proof The matrix A is of the form

$$A = \begin{bmatrix} P_0 & P_1 & \ldots & P_{n-2} & P_{n-1} \\ rP_{n-1} & P_0 & \ldots & P_{n-3} & P_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rP_2 & rP_3 & \ldots & P_0 & P_1 \\ rP_1 & rP_2 & \ldots & rP_{n-1} & P_0 \end{bmatrix}.$$

Then we have

$$\|A\|_E^2 = \sum_{i=0}^{n-1}(n-i)P_i^2 + \sum_{i=1}^{n-1} i|r|^2 P_i^2;$$

hence, when $|r| \geq 1$ we obtain

$$\|A\|_E^2 \geq \sum_{i=0}^{n-1}(n-i)P_i^2 + \sum_{i=1}^{n-1} iP_i^2 = n\sum_{i=0}^{n-1} P_i^2 = n\frac{P_n P_{n-1}}{2},$$

that is,

$$\frac{1}{\sqrt{n}}\|A\|_E \geq \sqrt{\frac{P_n P_{n-1}}{2}} \quad \Rightarrow \quad \|A\|_2 \geq \sqrt{\frac{P_n P_{n-1}}{2}}.$$

On the other hand, let the matrices B and C be

$$B = \begin{bmatrix} P_0 & 1 & \ldots & 1 & 1 \\ r & P_0 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \ldots & P_0 & 1 \\ r & r & \ldots & r & P_0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} P_0 & P_1 & \ldots & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & \ldots & P_{n-3} & P_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & \ldots & P_0 & P_1 \\ P_1 & P_2 & \ldots & P_{n-1} & P_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{nj}|^2} = \sqrt{|r|^2(n-1)} = |r|\sqrt{(n-1)} \quad \text{and}$$

$$c_1(C) = \max_j \sqrt{\sum_i |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} P_i^2} = \sqrt{\frac{P_n P_{n-1}}{2}}.$$

We have

$$\|A\|_2 \leq |r|\sqrt{(n-1)\frac{P_n P_{n-1}}{2}}.$$

When $|r| < 1$ we also obtain

$$\|A\|_E^2 \geq \sum_{i=0}^{n-1}(n-i)|r|^2 P_i^2 + \sum_{i=1}^{n-1} i|r|^2 P_i^2 = n|r|^2 \frac{P_n P_{n-1}}{2},$$

that is,

$$\frac{1}{\sqrt{n}}\|A\|_E \geq |r|\sqrt{\frac{P_n P_{n-1}}{2}} \quad \Rightarrow \quad \|A\|_2 \geq |r|\sqrt{\frac{P_n P_{n-1}}{2}}.$$

On the other hand, let the matrices B and C be

$$B = \begin{bmatrix} P_0 & 1 & \dots & 1 & 1 \\ r & P_0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \dots & P_0 & 1 \\ r & r & \dots & r & P_0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} P_0 & P_1 & \dots & P_{n-2} & P_{n-1} \\ P_{n-1} & P_0 & \dots & P_{n-3} & P_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_2 & P_3 & \dots & P_0 & P_1 \\ P_1 & P_2 & \dots & P_{n-1} & P_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n-1} \quad \text{and}$$

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} P_i^2} = \sqrt{\frac{P_n P_{n-1}}{2}}.$$

We have

$$\|A\|_2 \leq \sqrt{(n-1)\frac{P_n P_{n-1}}{2}}.$$

Thus, the proof is completed. □

Corollary 2 *Let $A = C_r(P_0^2, P_1^2, \dots, P_{n-1}^2)$ be a r-circulant matrix, where $r \in \mathbb{C}$, $|r| \geq 1$; we have*

$$\|A\|_2 \leq (n-1)|r|\frac{P_n P_{n-1}}{2},$$

where $\|\cdot\|_2$ *is the spectral norm and* P_n *denotes the nth Pell number.*

Proof Since $A = C_r(P_0^2, P_1^2, \ldots, P_{n-1}^2)$ is a r-circulant matrix, if the matrices $B = C_r(P_0, P_1, \ldots, P_{n-1})$ and $C = C(P_0^2, P_1^2, \ldots, P_{n-1}^2)$ we get $A = B \circ C$; thus, we obtain

$$\|A\|_2 \leq (n-1)|r|\frac{P_n P_{n-1}}{2}. \qquad \square$$

Theorem 3 *Let* $A = C_r(Q_0, Q_1, \ldots, Q_{n-1})$ *be a r-circulant matrix, where* $r \in \mathbb{C}$.

$$\text{(i)} \quad |r| \geq 1, \quad \begin{cases} \sqrt{\frac{Q_{2n-1}+6}{2}} \leq \|A\|_2 \leq |r|\sqrt{n\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{\frac{Q_{2n-1}+2}{2}} \leq \|A\|_2 \leq |r|\sqrt{n\frac{Q_{2n-1}+2}{2}}, & n \text{ even}, \end{cases}$$

$$\text{(ii)} \quad |r| < 1, \quad \begin{cases} |r|\sqrt{\frac{Q_{2n-1}+6}{2}} \leq \|A\|_2 \leq \sqrt{n\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ |r|\sqrt{\frac{Q_{2n-1}+2}{2}} \leq \|A\|_2 \leq \sqrt{n\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

Proof The matrix A is of the form

$$A = \begin{bmatrix} Q_0 & Q_1 & \ldots & Q_{n-2} & Q_{n-1} \\ rQ_{n-1} & Q_0 & \ldots & Q_{n-3} & Q_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rQ_2 & rQ_3 & \ldots & Q_0 & Q_1 \\ rQ_1 & rQ_2 & \ldots & rQ_{n-1} & Q_0 \end{bmatrix}.$$

Then we have

$$\|A\|_E^2 = \sum_{i=0}^{n-1}(n-i)Q_i^2 + \sum_{i=1}^{n-1} i|r|^2 Q_i^2;$$

hence, when $|r| \geq 1$ we obtain

$$\|A\|_E^2 \geq \sum_{i=0}^{n-1}(n-i)Q_i^2 + \sum_{i=1}^{n-1} iQ_i^2 = n\sum_{i=0}^{n-1} Q_i^2 = \begin{cases} \sqrt{n\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{n\frac{Q_{2n-1}+2}{2}}, & n \text{ even}, \end{cases}$$

that is,

$$\frac{1}{\sqrt{n}}\|A\|_E \geq \|A\|_2 \geq \begin{cases} \sqrt{\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

On the other hand, let the matrices B and C be

$$B = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ r & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \ldots & 1 & 1 \\ r & r & \ldots & r & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} Q_0 & Q_1 & \ldots & Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_0 & \ldots & Q_{n-3} & Q_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & \ldots & Q_0 & Q_1 \\ Q_1 & Q_2 & \ldots & Q_{n-1} & Q_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{|r|^2(n-1)+1} \quad \text{and}$$

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} Q_i^2} = \begin{cases} \sqrt{\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

We have

$$\|A\|_2 \leq \begin{cases} \sqrt{(|r|^2(n-1)+1)(\frac{Q_{2n-1}+6}{2})}, & n \text{ odd}, \\ \sqrt{(|r|^2(n-1)+1)(\frac{Q_{2n-1}+2}{2})}, & n \text{ even}. \end{cases}$$

When $|r| < 1$ we also obtain

$$\|A\|_E^2 \geq \sum_{i=0}^{n-1} (n-i)|r|^2 Q_i^2 + \sum_{i=1}^{n-1} i|r|^2 Q_i^2 = \begin{cases} |r|\sqrt{n(\frac{Q_{2n-1}+6}{2})}, & n \text{ odd}, \\ |r|\sqrt{n(\frac{Q_{2n-1}+2}{2})}, & n \text{ even}, \end{cases}$$

that is,

$$\frac{1}{\sqrt{n}}\|A\|_E \geq \|A\|_2 \geq \begin{cases} |r|\sqrt{\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ |r|\sqrt{\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

On the other hand, let the matrices B and C be

$$B = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ r & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r & r & \dots & 1 & 1 \\ r & r & \dots & r & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} Q_0 & Q_1 & \dots & Q_{n-2} & Q_{n-1} \\ Q_{n-1} & Q_0 & \dots & Q_{n-3} & Q_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_2 & Q_3 & \dots & Q_0 & Q_1 \\ Q_1 & Q_2 & \dots & Q_{n-1} & Q_0 \end{bmatrix}$$

such that $A = B \circ C$. Then

$$r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2} = \sqrt{\sum_{j=0}^{n-1} |b_{nj}|^2} = \sqrt{n} \quad \text{and}$$

$$c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2} = \sqrt{\sum_{i=0}^{n-1} |c_{in}|^2} = \sqrt{\sum_{i=0}^{n-1} Q_i^2} = \begin{cases} \sqrt{\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

We have

$$\|A\|_2 \leq \begin{cases} \sqrt{n\frac{Q_{2n-1}+6}{2}}, & n \text{ odd}, \\ \sqrt{n\frac{Q_{2n-1}+2}{2}}, & n \text{ even}. \end{cases}$$

Thus, the proof is completed. □

Corollary 4 *Let $A = C_r(Q_0^2, Q_1^2, \ldots, Q_{n-1}^2)$ be a r-circulant matrix, where $r \in \mathbb{C}$, $|r| \geq 1$,*

$$\|A\|_2 \leq \begin{cases} n|r|\frac{Q_{2n-1}+6}{2}, & n \text{ odd}, \\ n|r|\frac{Q_{2n-1}+2}{2}, & n \text{ even}, \end{cases}$$

where $\|\cdot\|_2$ is the spectral norm and Q_n denotes the nth Pell-Lucas number.

Proof Since $A = C_r(Q_0^2, Q_1^2, \ldots, Q_{n-1}^2)$ is a r-circulant matrix, if the matrices $B = C_r(Q_0, Q_1, \ldots, Q_{n-1})$ and $C = C(Q_0^2, Q_1^2, \ldots, Q_{n-1}^2)$ we get $A = B \circ C$; thus, we obtain

$$\|A\|_2 \leq \begin{cases} n|r|\frac{Q_{2n-1}+6}{2}, & n \text{ odd}, \\ n|r|\frac{Q_{2n-1}+2}{2}, & n \text{ even}. \end{cases}$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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