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On the spectrum and stiffness of an elastic body with surface stresses

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Key words Surface stresses, eigenfrequencies, energy spaces of Sobolev's type, Rayleigh variational principle, Courant's maximum-minimum principle.

A mathematical investigation of the eigenvalue problems for elastic bodies including surface stresses is presented. Weak setup of the problems is based on the Rayleigh variational principle. Certain spectral properties are established for the problems under consideration. In particular, bounds for the eigenfrequencies of an elastic body with surface stresses are presented. These bounds demonstrate increases in both the rigidity of the body and of the eigenfrequencies over those of the body with surface stresses neglected.

Introduction

The development of nanomechanics has generated interest in the theory of elasticity with surface stresses [6]. Surface effects can explain the abnormal properties of nanomaterials. In particular, it has been shown [7] that by taking into account surface elasticity, we obtain the stiffness increase of nanoporous materials in comparison with their usual models. In certain proposed models for surface stresses [10,12,22,25,26], the influence of surface elasticity on the natural vibrations of micro- and nano-sized beams is analysed. The theory of elasticity with surface stresses can be found in, e.g., [11, 18, 19, 23]. In this theory, an elastic body with surface stresses is considered as a linear elastic body having an elastic membrane glued to all or part of its surface. Existence of a weak solution and related problems in linear elasticity with surface stresses are investigated in [2]. Existence of classical solutions to boundary value problems with surface reinforcements is established in [20, 21].

In this paper we show that the ordered eigenfrequencies of an elastic body with a non-fixed boundary portion can only increase if we consider the action of surface stresses on this portion. That is, for a bounded elastic body with a fixed boundary portion and surface stresses acting on the rest of the boundary, the ordered eigenfrequencies are greater than or equal to the corresponding eigenfrequencies of the body when surface stresses are neglected. Surface elasticity does not affect the density of the body; this means that by adding the action of surface stresses, we increase the stiffness of the body. Moreover, we prove that the ordered eigenfrequencies of an elastic body with surface stresses can only increase if we fix the whole boundary. Hence we obtain lower and upper bounds for the eigenfrequencies of an elastic body with surface stresses.

1 Basic relations of linear elasticity with surface stresses

Suppose an elastic body occupies a bounded volume $V \subset \mathbb{R}^3$ with a piecewise smooth boundary Ω. We consider a problem with mixed boundary conditions. Let a nonempty portion Ω_1 of Ω be fixed so that the displacement $\boldsymbol{u} = \boldsymbol{0}$ at each point of Ω_1. Surface stresses $\boldsymbol{\tau}$ act over the remainder of the boundary $\Omega_2 = \Omega \setminus \Omega_1$. The boundary value problem is given by the relations [6, 7, 11, 18]

$$\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{u}}, \quad \boldsymbol{x} \in V, \tag{1}$$

$$\boldsymbol{u}|_{\Omega_1} = \boldsymbol{0}, \quad \boldsymbol{n} \cdot \boldsymbol{\sigma}|_{\Omega_2} = \nabla_S \cdot \boldsymbol{\tau}, \quad \boldsymbol{x} \in \Omega, \tag{2}$$

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where $\boldsymbol{\sigma}$ is the stress tensor, ∇ is the three-dimensional nabla operator, ρ is the material density, $\boldsymbol{n}$ is the exterior unit normal to Ω, $\boldsymbol{\tau}$ is the surface stress tensor, $\nabla_S = \nabla - \boldsymbol{n}\,\partial/\partial z$ is the surface nabla operator, and z is the coordinate along $\boldsymbol{n}$. An overdot denotes differentiation with respect to time t. The dot symbol "·" between variables stands for the dot product in $\mathbb{R}^3$. For simplicity, we consider a homogeneous boundary value problem without volume forces or external surface forces.

In the case of an isotropic material, Eqs. (1)–(2) should be supplemented as follows. Inside V we require

$$W = W(\boldsymbol{\varepsilon}) \equiv \frac{1}{2}\lambda \mathrm{tr}^2 \boldsymbol{\varepsilon} + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \tag{3}$$

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \equiv 2\mu\boldsymbol{\varepsilon} + \lambda \mathbf{I}\,\mathrm{tr}\,\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \equiv \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right), \tag{4}$$

where W is the volume energy density, $\boldsymbol{\varepsilon}$ is the volume strain tensor, λ, μ are Lamé's moduli for the bulk material, and the symbol ":" stands for the inner product in the space of second order tensors. Because the displacement vector for points of Ω is the continuation of $\boldsymbol{u}$ in the volume, it is also denoted by $\boldsymbol{u}$. On Ω_2 we require

$$U = U(\boldsymbol{\epsilon}) \equiv \frac{1}{2}\lambda_S \mathrm{tr}^2 \boldsymbol{\epsilon} + \mu_S \boldsymbol{\epsilon} : \boldsymbol{\epsilon}, \tag{5}$$

$$\boldsymbol{\tau} = \frac{\partial U}{\partial \boldsymbol{\varepsilon}} \equiv 2\mu_S \boldsymbol{\epsilon} + \lambda_S \mathbf{A}\,\mathrm{tr}\,\boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{u}) \equiv \frac{1}{2}\left(\nabla_S \boldsymbol{u}\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla_S\boldsymbol{u})^T\right), \tag{6}$$

where U is the energy density of the surface deformation, $\boldsymbol{\epsilon}$ is the surface strain tensor, $\mathbf{A} = \mathbf{I} - \boldsymbol{n}\otimes\boldsymbol{n}$ is the surface unit tensor, $\mathbf{I}$ is the three-dimensional unit tensor, and λ_S, μ_S are the surface elastic Lamé's moduli.

We assume that W and U are positive definite quadratic forms with respect to their arguments:

$$W(\boldsymbol{\varepsilon}) \geq c_1 \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}, \quad U(\boldsymbol{\epsilon}) \geq c_2 \boldsymbol{\epsilon} : \boldsymbol{\epsilon}, \tag{7}$$

where c_1, c_2 are positive constants. From (7) it follows that

$$3\lambda + 2\mu > 0, \quad \mu > 0, \qquad \lambda_S + \mu_S > 0, \quad \mu_S > 0.$$

When $\lambda_S = \mu_S = 0$, Eqs. (1)–(2) reduce to the mixed boundary value problem of classical linear elasticity:

$$\nabla\cdot\boldsymbol{\sigma} = \rho\ddot{\boldsymbol{u}}, \quad \boldsymbol{x}\in V; \qquad \boldsymbol{u}|_{\Omega_1} = \mathbf{0}, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}|_{\Omega_2} = \mathbf{0}. \tag{8}$$

When $\lambda_S, \mu_S \to \infty$, because $\boldsymbol{u} = \mathbf{0}$ on the boundary of Ω_2, Eqs. (1)–(2) reduce to the dynamic equations of classical elasticity for a body with fixed boundary:

$$\nabla\cdot\boldsymbol{\sigma} = \rho\ddot{\boldsymbol{u}}, \quad \boldsymbol{x}\in V; \qquad \boldsymbol{u}|_{\Omega} = \mathbf{0}. \tag{9}$$

These relations for the problems allow us to compare their ordered sets of eigenfrequencies.

2 Eigenoscillations

Let us seek the displacement field in the form $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x})\exp(i\omega t)$. Eqs. (1)–(2) reduce to

$$\nabla\cdot\boldsymbol{\sigma} = -\rho\omega^2\boldsymbol{w}, \quad \boldsymbol{x}\in V; \qquad \boldsymbol{w}\big|_{\Omega_1} = \mathbf{0}, \quad (\boldsymbol{n}\cdot\boldsymbol{\sigma} - \nabla_S\cdot\boldsymbol{\tau})\big|_{\Omega_2} = \mathbf{0}. \tag{10}$$

These, when supplemented with the appropriately transformed relations (3)–(6), constitute the eigenfrequency problem for a body V with surface stresses. We refer to it as *Problem* P_{ss}.

The spectrum of Problem P_{ss} will be compared with the spectra of two problems obtained by substituting $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{w}(\boldsymbol{x})\exp(i\omega t)$ into (8) and (9). For (8), the eigenoscillation equations are

$$\nabla\cdot\boldsymbol{\sigma} = -\rho\omega^2\boldsymbol{w}, \quad \boldsymbol{x}\in V; \qquad \boldsymbol{w}|_{\Omega_1} = \mathbf{0}, \quad \boldsymbol{n}\cdot\boldsymbol{\sigma}|_{\Omega_2} = \mathbf{0}. \tag{11}$$

Supplemented with the correspondingly transformed relations (4), they constitute *Problem* P_f. Finally, the equations

$$\nabla\cdot\boldsymbol{\sigma} = -\rho\omega^2\boldsymbol{w}, \quad \boldsymbol{x}\in V; \qquad \boldsymbol{w}|_{\Omega} = \mathbf{0} \tag{12}$$

supplemented with the correspondingly transformed relations (4), constitute eigenfrequency *Problem* P_0 for a body V with fixed boundary.

Note that the operators of the boundary value problems for elastic bodies with surface stresses have properties similar to those for the operators of linear elasticity; these are well established, cf. for example [4, 9, 15]. Weak setup of boundary value problems in the theory of elasticity with surface stresses is studied in [2]. We present the necessary definitions.

For Problem P_{ss}, we introduce the energy space $\boldsymbol{E}$ with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{E}} = \int_V [\lambda \mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{v}) + 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})] \, dV + \int_{\Omega_2} [\lambda_S \mathrm{tr}\, \boldsymbol{\epsilon}(\boldsymbol{u}) \mathrm{tr}\, \boldsymbol{\epsilon}(\boldsymbol{v}) + 2\mu_S \boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})] \, d\Omega \tag{13}$$

and the norm

$$\|\boldsymbol{u}\|^2_{\boldsymbol{E}} = \langle \boldsymbol{u}, \boldsymbol{u} \rangle_{\boldsymbol{E}}.$$

Roughly speaking, an element $\boldsymbol{u} \in \boldsymbol{E}$ belongs to both $(W^{1,2}(V))^3$ and $(W^{1,2}(\Omega_2))^3$.

An eigenpair $(\omega_k, \boldsymbol{w}_k)$, where $\boldsymbol{w}_k \neq \boldsymbol{0}$ and ω_k is an eigenfrequency, satisfies

$$\langle \boldsymbol{w}_k, \boldsymbol{v} \rangle_{\boldsymbol{E}} = \omega_k^2 \int_V \rho \boldsymbol{w}_k \cdot \boldsymbol{v} \, dV$$

for any $\boldsymbol{v} \in \boldsymbol{E}$. It has been shown [2] that the set of eigenfrequencies $\{\omega_k\}$, $0 < \omega_{\min} = \omega_1 \leq \omega_2 \leq \omega_3, \ldots$, of Problem P_{ss} is countable, that its only point of accumulation is infinite, and that to each ω_k there corresponds an eigenmode $\boldsymbol{w}_k$ which can be selected so that

$$\int_V \rho \boldsymbol{w}_k \cdot \boldsymbol{w}_m \, dV = \delta_{km}, \quad \text{and} \quad \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_{\boldsymbol{E}} = 0 \quad \text{for} \quad i \neq j, \tag{14}$$

where δ_{km} is Kronecker's symbol. Moreover, the set of $\boldsymbol{w}_k$, $k = 1, 2, \ldots$, is complete in $(L^2(V))^3$ and $\boldsymbol{E}$.

For Problems P_f and P_0, the results for the eigensolutions are similar to those stated above; cf. [14, 15] among others. For Problem P_f, the energy space $\boldsymbol{H}$ is the completion of the set of vector functions $\boldsymbol{u} \in (C^{(2)}(\overline{V}))^3$ that vanish on Ω_1. The norm $\|\cdot\|_{\boldsymbol{H}}$ is induced by the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\boldsymbol{H}} = \int_V [\lambda \mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) \mathrm{tr}\, \boldsymbol{\varepsilon}(\boldsymbol{v}) + 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})] \, dV. \tag{15}$$

To enable the use of Sobolev's embedding theorems [1], we suppose V has a piecewise smooth boundary that satisfies the cone condition. The boundary of Ω_2 should be piecewise smooth and should also satisfy the cone condition on the plane of its inner coordinates.

The weak setup of the eigenvalue Problem P_f is given by

$$\langle \boldsymbol{w}_k, \boldsymbol{v} \rangle_{\boldsymbol{H}} = (\omega_k^f)^2 \int_V \rho \boldsymbol{w}_k \cdot \boldsymbol{v} \, dV \tag{16}$$

which holds for all $\boldsymbol{v} \in \boldsymbol{H}$. Here ω_k^f is an eigenfrequency. We retain the notation $\boldsymbol{w}_k$ for eigenmodes, although these are in general distinct for different problems. To formulate the results for Problem P_f from those presented above for Problem P_{ss}, we change the space $\boldsymbol{E}$ to $\boldsymbol{H}$ and the inner product to (15). The set of ordered eigenmodes of Problem P_f is complete in $\boldsymbol{H}$ and $(L^2(V))^3$; furthermore,

$$\int_V \rho \boldsymbol{w}_k \cdot \boldsymbol{w}_m \, dV = \delta_{km} \quad \text{and} \quad \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_{\boldsymbol{H}} = 0 \quad \text{for} \quad i \neq j. \tag{17}$$

A particular case of Problem P_f is Problem P_0. Here the elements $\boldsymbol{w}$ vanish on Ω. The energy space $\boldsymbol{H}_0$, a subspace of $\boldsymbol{H}$, is the completion of the set of vector functions $\boldsymbol{u} \in (C_0^{(2)}(\overline{V}))^3$. $\boldsymbol{H}_0$ inherits the norm $\|\cdot\|_{\boldsymbol{H}}$ of $\boldsymbol{H}$. The equation for the eigensolutions coincides with (16), which must hold for all $\boldsymbol{v} \in \boldsymbol{H}_0$. The spectral properties are the same as those of Problem P_f, including the relations (17).

We recall that $\boldsymbol{u} \in \boldsymbol{H}$ means that $\boldsymbol{u} \in (W^{1,2}(V))^3$ and, by the trace theorem [1], $\boldsymbol{u} \in (W^{1/2,2}(\Omega_2))^3$. The relation $\boldsymbol{u} \in \boldsymbol{H}_0$ means that $\boldsymbol{u} \in (W_0^{1,2}(V))^3$. We may consider $\boldsymbol{E}$ as a subset of $\boldsymbol{H}$, and $\boldsymbol{H}_0$ as a subset of $\boldsymbol{E}$. We will use this in what follows.

3 Rayleigh quotient and some properties of the spectrum

3.1 The least eigenfrequency

The estimation of the least eigenfrequency is an important problem from an engineering standpoint. For solids with surface stresses, the eigenvalue Problem P_{ss} can be formulated using Rayleigh's variational principle [2].

Let $R(\boldsymbol{w})$ be Rayleigh's quotient for the body with surface stresses:

$$R(\boldsymbol{w}) = \frac{\|\boldsymbol{w}\|^2_{\boldsymbol{E}}}{K(\boldsymbol{w})}, \quad K(\boldsymbol{w}) = \int_V \rho \boldsymbol{w} \cdot \boldsymbol{w} \, dV.$$

The squared least eigenfrequency ω_{min} is determined as the infimum of $R(\boldsymbol{w})$:

$$\omega^2_{\text{min}} = \inf_{\boldsymbol{w} \in \boldsymbol{E}} R(\boldsymbol{w}).$$

For Problems P_f and P_0, Rayleigh's quotient is the same as in linear elasticity, i.e.,

$$R_0(\boldsymbol{w}) = \frac{\|\boldsymbol{w}\|^2_{\boldsymbol{H}}}{K(\boldsymbol{w})}.$$

For Problem P_f, the squared least eigenfrequency ω^f_{min} is the infimum of $R_0(\boldsymbol{w})$ over $\boldsymbol{H}$. For Problem P_0, the squared least eigenfrequency $\omega^\circ_{\text{min}}$ is the infimum of $R_0(\boldsymbol{w})$ over $\boldsymbol{H}_0$.

$R(\boldsymbol{w})$ is determined on $\boldsymbol{E}$. Let us formally extend its domain to $\boldsymbol{H}$ as follows. An element of $\boldsymbol{E}$ can be considered as an element of $\boldsymbol{H}$, and so for this element we define the value of R on $\boldsymbol{w} \in \boldsymbol{E}$. When $\boldsymbol{w} \in \boldsymbol{H}$ but $\boldsymbol{w}$ lacks the smoothness on Ω_2 needed to associate it with an element of $\boldsymbol{E}$ and $\|\boldsymbol{w}\|_{\boldsymbol{E}}$ is undefined, then we formally put $\|\boldsymbol{w}\|_{\boldsymbol{E}} = +\infty$. By the positivity of the energy function, it is easy to see that $R_0(\boldsymbol{w}) \le R(\boldsymbol{w})$ on $\boldsymbol{H}$, and $R_0(\boldsymbol{w}) = R(\boldsymbol{w})$ if $\boldsymbol{w} \in \boldsymbol{H}_0$.

The properties of the spaces $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{H}_0$ and the functionals R, R_0 allow us to prove the following

Theorem 3.1. *The least eigenfrequency of a bounded elastic body with surface stresses (Problem P_{ss}) is no less than the least eigenfrequency for the same body with free boundary Ω_2 (Problem P_f), and it is no greater than the least eigenfrequency for the same body with fixed boundary (Problem P_0):*

$$\omega^f_{\text{min}} \le \omega_{\text{min}} \le \omega^\circ_{\text{min}}. \tag{18}$$

Proof. The proof follows from two inequality chains:

$$(\omega^f_{\text{min}})^2 = \inf_{\boldsymbol{w} \in \boldsymbol{H}} R_0(\boldsymbol{w}) \le \inf_{\boldsymbol{w} \in \boldsymbol{E} \subset \boldsymbol{H}} R_0(\boldsymbol{w}) \le \inf_{\boldsymbol{w} \in \boldsymbol{E}} R(\boldsymbol{w}) = (\omega_{\text{min}})^2,$$

since $\boldsymbol{E}$ constitutes a subset of $\boldsymbol{H}$, and

$$(\omega_{\text{min}})^2 = \inf_{\boldsymbol{w} \in \boldsymbol{E}} R(\boldsymbol{w}) \le \inf_{\boldsymbol{w} \in \boldsymbol{H}_0 \subset \boldsymbol{E}} R(\boldsymbol{w}) = \inf_{\boldsymbol{w} \in \boldsymbol{H}_0} R_0(\boldsymbol{w}) = (\omega^\circ_{\text{min}})^2.$$

□

Two results follow from the proof of Theorem 3.1.

Corollary 1. *The equality $\omega_{\text{min}} = \omega^f_{\text{min}}$ holds if and only if $U(\boldsymbol{\epsilon}(\boldsymbol{w}_{\text{min}})) = 0$ on Ω_2.*

By positive definiteness, $U = 0$ if and only if $\boldsymbol{\epsilon}(\boldsymbol{w}_{\text{min}}) = \boldsymbol{0}$ on Ω_2. Hence the displacement $\boldsymbol{w}_{\text{min}}$ of Ω_2 describes an infinitesimal isometric deformation of Ω_2. In particular, $\boldsymbol{\epsilon} = \boldsymbol{0}$ if $\boldsymbol{w}_{\text{min}}$ describes a rigid body motion.

Corollary 2. *The equality $\omega_{\text{min}} = \omega^\circ_{min}$ holds if and only if $\boldsymbol{w}_{\text{min}} = \boldsymbol{0}$ on Ω_2, which is when $\boldsymbol{w}_{\text{min}} \in \boldsymbol{H}_0$.*

Cases when $\boldsymbol{\epsilon}(\boldsymbol{w}_{\text{min}}) = \boldsymbol{0}$ or $\boldsymbol{w}_{\text{min}} = \boldsymbol{0}$ on Ω_2 should be rare for an elastic body of general shape and with general boundary conditions. Hence, in general we can expect the strict inequalities

$$\omega^f_{\text{min}} < \omega_{\text{min}} < \omega^\circ_{\text{min}}.$$

This extends the inequality $\omega^f_{\text{min}} < \omega^\circ_{\text{min}}$, well known in the theory of elasticity and mathematical physics [5, 24].

The least eigenfrequency ω_{min} depends on λ_S and μ_S. An increase in the surface elastic moduli implies an increase in the least eigenfrequency of Problem P_{ss}. Indeed, let us consider two bodies of equal shape and equal internal moduli λ and μ, but with different values of λ_S and μ_S. Denote the surface moduli of the bodies by $\lambda^{(1)}_S$, $\mu^{(1)}_S$ and $\lambda^{(2)}_S$, $\mu^{(2)}_S$, respectively. Denote the least eigenfrequencies of the bodies by $\omega^{(1)}_{\text{min}}$ and $\omega^{(2)}_{\text{min}}$, respectively.

Theorem 3.2. *Let*

$$0 < \mu_S^{(1)} \le \mu_S^{(2)}, \quad 0 < \lambda_S^{(1)} + \mu_S^{(1)} \le \lambda_S^{(2)} + \mu_S^{(2)}. \tag{19}$$

Then

$$\omega_{\min}^{(1)} \le \omega_{\min}^{(2)}. \tag{20}$$

Proof. The proof follows immediately from the inequality $R_{(1)}(\boldsymbol{w}) \le R_{(2)}(\boldsymbol{w})$, where $R_{(\alpha)}(\boldsymbol{w})$, $\alpha = 1, 2$, are Rayleigh's quotients for the bodies. Indeed, from (19) it follows that $U_{(1)} \le U_{(2)}$ where

$$U_{(\alpha)} = \frac{1}{2}\lambda_S^{(\alpha)} \operatorname{tr}^2 \boldsymbol{\epsilon} + \mu_S^{(\alpha)} \boldsymbol{\epsilon} : \boldsymbol{\epsilon}, \quad \alpha = 1, 2.$$

The corresponding energy spaces $\boldsymbol{E}_{(1)}$ and $\boldsymbol{E}_{(2)}$ for the problems coincide up to the form of the energy norms, which are equivalent, and so the infimum is taken over the same set of elements $\boldsymbol{E}_{(1)} = \boldsymbol{E}_{(2)} = \boldsymbol{E}$. Hence

$$\left(\omega_{\min}^{(1)}\right)^2 = \inf_{\boldsymbol{w} \in \boldsymbol{E}} R_{(1)}(\boldsymbol{w}) \le \inf_{\boldsymbol{w} \in \boldsymbol{E}} R_{(2)}(\boldsymbol{w}) = \left(\omega_{\min}^{(2)}\right)^2.$$

□

Now we demonstrate that the least eigenfrequency depends continuously on the values of λ_S and μ_S.

Theorem 3.3. *For any number $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|\omega_{\min}^{(1)} - \omega_{\min}^{(2)}| \le \varepsilon$ whenever $|\mu_S^{(1)} - \mu_S^{(2)}| \le \delta$ and $|\lambda_S^{(1)} - \lambda_S^{(2)}| \le \delta$.*

Proof. We get

$$\begin{aligned} R_{(1)}(\boldsymbol{w}) - R_{(2)}(\boldsymbol{w}) &= \frac{\int_{\Omega_2} \left[\left(\lambda_S^{(1)} - \lambda_S^{(2)}\right) \operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w}) + 2\left(\mu_S^{(1)} - \mu_S^{(2)}\right) \boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\right] d\Omega}{K(\boldsymbol{w})} \\ &\le \frac{|\lambda_S^{(1)} - \lambda_S^{(2)}| \int_{\Omega_2} \operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w})\, d\Omega + 2|\mu_S^{(1)} - \mu_S^{(2)}| \int_{\Omega_2} \boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\, d\Omega}{K(\boldsymbol{w})} \\ &\le \delta \frac{\int_{\Omega_2} \left[\operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w}) + 2\boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\right] d\Omega}{K(\boldsymbol{w})}. \end{aligned}$$

Taking the infimum over $\boldsymbol{E}$ of both sides of the inequality, we have

$$\left(\omega_{\min}^{(1)}\right)^2 - \left(\omega_{\min}^{(2)}\right)^2 \le \delta \inf_{\boldsymbol{w} \in \boldsymbol{E}} \frac{\int_{\Omega_2} \left[\operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w}) + 2\boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\right] d\Omega}{K(\boldsymbol{w})},$$

which completes the proof. □

From Theorem 3.3 we deduce an important

Corollary 3. *The least eigenfrequency of a bounded elastic body with surface stresses tends to the least eigenfrequency for the same body with free boundary Ω_2:*

$$\omega_{min} \to \omega_{min}^f \quad \text{as} \quad \lambda_S \to 0 \quad \text{and} \quad \mu_S \to 0.$$

Proof. Starting with $R(\boldsymbol{w}) - R_0(\boldsymbol{w})$ and repeating the transformations from the proof of Theorem 3.3, we obtain

$$0 \le R(\boldsymbol{w}) - R_0(\boldsymbol{w}) \le \delta \frac{\int_{\Omega_2} \left[\operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w}) + 2\boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\right] d\Omega}{K(\boldsymbol{w})},$$

where $\delta = \max\{|\lambda_S|, |\mu_S|\}$. The set $\boldsymbol{E}$ is dense in $\boldsymbol{H}$, as $\boldsymbol{E}$ and $\boldsymbol{H}$ are the completions of the same space $(C^{(2)}(\overline{V}))^3$ with respect to different norms. Therefore

$$\inf_{\boldsymbol{w} \in \boldsymbol{E}} R_0(\boldsymbol{w}) = \inf_{\boldsymbol{w} \in \boldsymbol{H}} R_0(\boldsymbol{w}) = (\omega_{\min}^f)^2,$$

and we get

$$0 \le (\omega_{\min})^2 - (\omega_{\min}^f)^2 \le \delta \inf_{\boldsymbol{w} \in \boldsymbol{E}} \frac{\int_{\Omega_2} \left[\operatorname{tr}^2 \boldsymbol{\epsilon}(\boldsymbol{w}) + 2\boldsymbol{\epsilon}(\boldsymbol{w}) : \boldsymbol{\epsilon}(\boldsymbol{w})\right] d\Omega}{K(\boldsymbol{w})} \to 0 \quad \text{as} \quad \delta \to 0.$$

□

Note that Theorems 3.1–3.3 hold when λ_S and μ_S are piecewise continuous on Ω_2. In other words, they hold for a body with nonhomogeneous surface properties.

3.2 Higher eigenfrequencies

The three eigenvalue Problems P_{ss}, P_f, and P_0 have discrete spectra, and the eigenmodes constitute complete orthogonal sets in the corresponding energy spaces and in the space $(L^2(V))^3$ [2, 14]. The eigenfrequencies of these problems can be compared using Courant's minimax principle [5].

Courant's minimax principle. On one hand, Courant's principle allows us to find all the eigenfrequencies for each of the problems. For each problem we must repeat the proof of the statement on pp. 406–407 of [5], using an appropriate form of Rayleigh's quotient and an appropriate energy space. Our situation is even better than that presented in [5], however, as we have an existence theorem for the eigenmodes. In [5], existence was implicitly assumed.

On the other hand, Courant's principle allows us to compare the values of the eigenfrequencies for the problems under consideration. To this end we must reformulate the principle in such a way that the eigenfrequencies are determined using the same space, $\boldsymbol{H}$. The main point is that we should define the eigenfrequencies using a set of elements $\boldsymbol{H}$ that is common to all three problems under consideration. Moreover, for each problem we will use the same constraint $K(\boldsymbol{w}) = 1$. Under the imposed constraint, $R(\boldsymbol{w}) = \|\boldsymbol{w}\|^2_{\boldsymbol{E}}$ and $R_0(\boldsymbol{w}) = \|\boldsymbol{w}\|^2_{\boldsymbol{H}}$. Here the situation differs somewhat from that presented in [5]. As we have said, we extend the domain of R by setting $R(\boldsymbol{w}) = +\infty$ for the elements $\boldsymbol{w} \in \boldsymbol{H}$ that cannot be regarded as elements of $\boldsymbol{E}$.

Again, for the Problem P_{ss} we can repeat the procedure stated on pp. 406-407 of [5], but on the space $\boldsymbol{H}$. This will yield the same set of eigenfrequencies that would be obtained from applying the procedure in $\boldsymbol{E}$.

The same is true for the eigenfrequencies of Problem P_0; we can define them using Courant's procedure in the space $\boldsymbol{H}$, while selecting from the orthogonal subspace $\boldsymbol{H}^{(k)}_{\perp}$ the elements that vanish on Ω.

We now describe in detail the first step of Courant's principle, which allows us to compare the eigenvalues for the three problems.

1. Denote by $\boldsymbol{H}^{(k)}$ the subspace of $\boldsymbol{H}$ spanned by $k-1$ arbitrary chosen elements $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$ of $\boldsymbol{H}$, $k > 1$. The space $\boldsymbol{H}^{(k)}_{\perp}$ is its "orthogonal" complement in $\boldsymbol{H}$:

$$\boldsymbol{H}^{(k)}_{\perp} = \{\boldsymbol{w} \in \boldsymbol{H} \mid \langle \boldsymbol{w}, \boldsymbol{v}_1\rangle_{\boldsymbol{L}} = \langle \boldsymbol{w}, \boldsymbol{v}_2\rangle_{\boldsymbol{L}} = \cdots = \langle \boldsymbol{w}, \boldsymbol{v}_{k-1}\rangle_{\boldsymbol{L}} = 0\},$$

where

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{L}} = \int_V \rho \boldsymbol{u} \cdot \boldsymbol{v}\, dV.$$

$\boldsymbol{H}^{(k)}_{\perp}$ is a closed subspace of $\boldsymbol{H}$. By $\widehat{\boldsymbol{H}}^{(k)}_{\perp}$ we denote the subset of elements of $\boldsymbol{H}^{(k)}_{\perp}$ with the constraint $\langle \boldsymbol{w}, \boldsymbol{w}\rangle_{\boldsymbol{L}} = 1$, i.e.,

$$\widehat{\boldsymbol{H}}^{(k)}_{\perp} = \{\boldsymbol{w} \in \boldsymbol{H}^{(k)}_{\perp} \mid \langle \boldsymbol{w}, \boldsymbol{w}\rangle_{\boldsymbol{L}} = 1\}.$$

2. Define

$$d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] = \inf_{\widehat{\boldsymbol{H}}^{(k)}_{\perp} \cap \boldsymbol{E}} \|\boldsymbol{w}\|^2_{\boldsymbol{E}},$$

$$d_f[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] = \inf_{\widehat{\boldsymbol{H}}^{(k)}_{\perp}} \|\boldsymbol{w}\|^2_{\boldsymbol{H}},$$

$$d_0[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] = \inf_{\widehat{\boldsymbol{H}}^{(k)}_{\perp} \cap \boldsymbol{H}_0} \|\boldsymbol{w}\|^2_{\boldsymbol{H}}.$$

3. Repetition of the proof of Courant's principle ([5], pp. 406–407) shows that by taking the supremum of these quantities over all possible combinations $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$ in $\boldsymbol{H}$, we obtain the following eigenfrequencies.

For Problem P_{ss} :

$$\omega_k^2 = \sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

For Problem P_f:

$$\omega_k^{f\,2} = \sup_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k-1}} d_f[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

For Problem P_0:

$$\omega_k^{\circ 2} = \sup_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k-1}} d_0[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

These maximum-minimum values are attained if the elements $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}$ coincide with the first $k-1$ eigenmodes of the corresponding eigenfrequency problem.

We will prove item 3 for Problem P_{ss}. Recall that the eigenvalues are ordered as $0 < \omega_{\min} = \omega_1 \leq \omega_2, \ldots$ and that to each ω_k there corresponds a unique eigenmode $\boldsymbol{w}_k$. Moreover, the relations (14) hold for the eigenmodes. First we note that $d_{ss}[\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{k-1}] = \omega_k^2$. Indeed, for elements $\boldsymbol{w} \in \widehat{\boldsymbol{H}}_\perp^{(k)}$ that do not belong to $\boldsymbol{E}$, we obtain $\|\boldsymbol{w}\|_{\boldsymbol{E}}^2 = \infty$ so the infimum is sought on the set $\widehat{\boldsymbol{H}}_\perp^{(k)} \bigcap \boldsymbol{E}$. However, this is the procedure by which ω_k^2 was found in [2].

Now we show that for any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1} \in \boldsymbol{H}$ we have $d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] \leq \omega_k^2$. It suffices to find an element $\boldsymbol{v} \in \widehat{\boldsymbol{H}}_\perp^{(k)}$ for which $\|\boldsymbol{v}\|_{\boldsymbol{E}} \leq \omega_k^2$. Let us take a linear combination $\boldsymbol{v} = \sum_{m=1}^k c_m \boldsymbol{w}_m$, with constants $c_1, c_2, \ldots, c_k$, of the first k eigenmodes $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k$ that belongs to $\widehat{\boldsymbol{H}}_\perp^{(k)} \bigcap \boldsymbol{E}$. Such a combination exists because the elements $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k$ are linearly independent while the dimension of $\boldsymbol{H}^{(k)}$ spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1} \in \boldsymbol{H}$ is no more than $k-1$. By (14) we obtain

$$\begin{aligned}
\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\boldsymbol{L}} &= \left\langle \sum_{m=1}^k c_m \boldsymbol{w}_m, \sum_{n=1}^k c_n \boldsymbol{w}_n \right\rangle_{\boldsymbol{L}} \\
&= \sum_{m,n=1}^k c_m c_n \langle \boldsymbol{w}_m, \boldsymbol{w}_n \rangle_{\boldsymbol{L}} \\
&= \sum_{m=1}^k c_m^2.
\end{aligned}$$

Since $\boldsymbol{v} \in \widehat{\boldsymbol{H}}_\perp^{(k)}$ is normalized with $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_{\boldsymbol{L}} = 1$, we obtain the normalization condition for the constants:

$$\sum_{m=1}^k c_m^2 = 1.$$

Now we recall that $\langle \boldsymbol{w}_m, \boldsymbol{w}_n \rangle_{\boldsymbol{E}} = \omega_n^2 \delta_{mn}$. It follows that

$$\begin{aligned}
\|\boldsymbol{v}\|_{\boldsymbol{E}}^2 &= \left\langle \sum_{m=1}^k c_m \boldsymbol{w}_m, \sum_{n=1}^k c_n \boldsymbol{w}_n \right\rangle_{\boldsymbol{E}} \\
&= \sum_{m,n=1}^k c_m c_n \langle \boldsymbol{w}_m, \boldsymbol{w}_n \rangle_{\boldsymbol{E}} \\
&= \sum_{m,n=1}^k c_m c_n \omega_n^2 \delta_{mn} \\
&= \sum_{m=1}^k c_m^2 \omega_m^2 \\
&\leq \sum_{m=1}^k c_m^2 \omega_k^2 \\
&= \omega_k^2,
\end{aligned}$$

which completes the proof. □

In contrast to [5], we have not implicitly assumed the existence of the eigenmodes for the corresponding boundary value problems.

A process of comparison permits us to extend the inequalities (18) and (20) to higher eigenfrequencies. This is addressed in

Theorem 3.4. *Let ω_k be eigenfrequencies of a bounded elastic body with surface stresses enumerated in increasing order as $\omega_0 \le \omega_1 \le \omega_2, \ldots$, and let ω_k^f and ω_k° be correspondingly ordered eigenfrequencies of the elastic body with free boundary Ω_2 and with fixed boundary, respectively. Then*

$$\omega_k^f \le \omega_k \le \omega_k^\circ, \quad k = 1, 2, 3, \ldots. \tag{21}$$

Proof. First we prove the left-hand inequality of (21). Since $\|\boldsymbol{w}\|_{\boldsymbol{H}} \le \|\boldsymbol{w}\|_{\boldsymbol{E}}$, we have

$$d_f[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] \le \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)}} \|\boldsymbol{w}\|_{\boldsymbol{E}}^2 \le d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

Thus, for the greatest values of d_f and d_{ss} we obtain

$$\sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} d_f[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] \le \sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

We conclude that ${\omega_k^f}^2 \le \omega_k^2$. The right-hand inequality of (21) is proved in a similar fashion. When $\boldsymbol{w} \in \boldsymbol{H}_0$, we see that $\|\boldsymbol{w}\|_{\boldsymbol{E}} = \|\boldsymbol{w}\|_{\boldsymbol{H}}$. So we have

$$d_0[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] = \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)} \cap \boldsymbol{H}_0} \|\boldsymbol{w}\|_{\boldsymbol{H}}^2 = \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)} \cap \boldsymbol{H}_0} \|\boldsymbol{w}\|_{\boldsymbol{E}}^2.$$

But

$$\inf_{\widehat{\boldsymbol{H}}_\perp^{(k)} \cap \boldsymbol{H}_0} \|\boldsymbol{w}\|_{\boldsymbol{E}}^2 \ge \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)}} \|\boldsymbol{w}\|_{\boldsymbol{E}}^2.$$

So

$$d_0[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] \ge d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}].$$

Thus

$$\omega_k^2 = \sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} d_{ss}[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] \le \sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} d_0[\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}] = {\omega_k^\circ}^2.$$

□

In a similar way, using Courant's principle, inequality (20) can be extended to higher eigenfrequencies.

Theorem 3.5. *Let $\omega_k^{(1)}$ be eigenfrequencies of a bounded elastic body with moduli λ, μ and surface elastic moduli $\lambda_S^{(1)}, \mu_S^{(1)}$, ordered as $\omega_0^{(1)} \le \omega_1^{(1)} \le \omega_2^{(1)}, \ldots$. Let $\omega_k^{(2)}$ be the ordered eigenfrequencies for the elastic body with moduli λ, μ but with surface moduli $\lambda_S^{(2)}, \mu_S^{(2)}$. Let*

$$\mu_S^{(1)} \le \mu_S^{(2)}, \quad \lambda_S^{(1)} + \mu_S^{(1)} \le \lambda_S^{(2)} + \mu_S^{(2)}.$$

Then

$$\omega_k^{(1)} \le \omega_k^{(2)} \quad \textit{for} \quad k = 1, 2, 3, \ldots. \tag{22}$$

Proof. For the body with surface elastic moduli $\lambda_S^{(1)}, \mu_S^{(1)}$ we denote the energy space by $\boldsymbol{E}_{(1)}$ and assign the superscript (1) to all corresponding quantities. Similarly, $\boldsymbol{E}_{(2)}$ denotes the energy space for the body with surface elastic moduli and $\lambda_S^{(2)}, \mu_S^{(2)}$. The kth eigenfrequency of the αth body with surface stresses is given by

$$(\omega_k^{(\alpha)})^2 = \sup_{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{k-1}} \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)}} \|\boldsymbol{w}\|_{\boldsymbol{E}_{(\alpha)}}^2, \quad \alpha = 1, 2. \tag{23}$$

Although the expressions for the norms of $\boldsymbol{E}_{(1)}$ and $\boldsymbol{E}_{(2)}$ are distinct, the spaces contain the same elements. Moreover, $\|\boldsymbol{w}\|_{\boldsymbol{E}_{(1)}} \le \|\boldsymbol{w}\|_{\boldsymbol{E}_{(2)}}$. For the supremum we obtain

$$\sup_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k-1}} \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)}} \|\boldsymbol{w}\|^2_{\boldsymbol{E}_{(1)}} \le \sup_{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k-1}} \inf_{\widehat{\boldsymbol{H}}_\perp^{(k)}} \|\boldsymbol{w}\|^2_{\boldsymbol{E}_{(2)}},$$

which completes the proof. □

Theorems 3.1–3.5 can be proved for more general boundary conditions on Ω_1. For example, they hold true for boundary conditions of the form

$$\boldsymbol{w}|_{\Omega_1^{(1)}} = \mathbf{0}, \qquad \boldsymbol{n}\cdot\boldsymbol{\sigma}|_{\Omega_1^{(2)}} = \mathbf{0}, \qquad \boldsymbol{n}\cdot\boldsymbol{w}|_{\Omega_1^{(3)}} = 0, \qquad \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot(\mathbf{I}-\boldsymbol{n}\otimes\boldsymbol{n})|_{\Omega_1^{(3)}} = \mathbf{0},$$

where $\Omega_1 = \Omega_1^{(1)} \cup \Omega_1^{(2)} \cup \Omega_1^{(3)}$, $\Omega_1^{(1)} \cap \Omega_1^{(2)} = \varnothing$, and $\Omega_1^{(2)} \cap \Omega_1^{(3)} = \varnothing$.

These theorems can also be proved when $\Omega_1 = \varnothing$, i.e., when Problems P_f and P_{ss} are formulated for a body free from geometrical constraints. In this case the first six eigenfrequencies are equal to zero, and the corresponding eigenmodes constitute the basis for the translations and rotations of a rigid body.

4 Example: radial oscillations of an elastic sphere with surface stresses

To illustrate the spectral properties of Problems P_{ss}, P_f, and P_0, we consider the oscillations of an elastic sphere – a problem that admits analytical solution. The radial eigen-vibrations of an elastic sphere are treated in many textbooks, e.g., [16, 17]. For the radial vibrations, the displacement field is

$$\boldsymbol{w} = w(r)\boldsymbol{e}_r,$$

where r and $\boldsymbol{e}_r$ are the radius and the corresponding basis vector of spherical coordinates. By spherical symmetry it follows that

$$w(0) = 0. \tag{24}$$

For the radially symmetric problem, the point $r = 0$ corresponds to Ω_1. Substituting $w(r) = rf(r)$ into $(10)_1$, we reduce the equation to [16, 17]

$$f'' + \frac{4}{r}f' + \eta^2 f = 0, \quad \text{where} \quad \eta^2 = \frac{\rho\omega^2}{\lambda + 2\mu}. \tag{25}$$

The solution of (25) satisfying (24) is

$$f(r) = \frac{\eta r\cos\eta r - \sin\eta r}{r^3}. \tag{26}$$

Then the radial component of the stress tensor $\boldsymbol{\sigma}$ is

$$\sigma_r(r) = (2\mu + \lambda)w'(r) + 2\lambda\frac{w(r)}{r}.$$

Let us consider the surface stresses in the case of radial deformation. The boundary portion Ω_2 is $r = a$, where a is the radius of the ball. In spherical coordinates,

$$\nabla_S = \frac{1}{a}\left(\boldsymbol{e}_\theta\frac{\partial}{\partial\theta} + \boldsymbol{e}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

where ϕ, θ are the angular coordinate variables and $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ are their corresponding basis vectors [13]. So

$$\boldsymbol{\epsilon} = \frac{w(a)}{a}(\boldsymbol{e}_\theta\boldsymbol{e}_\theta + \boldsymbol{e}_\phi\boldsymbol{e}_\phi) \equiv \frac{w(a)}{a}\mathbf{A}.$$

Substituting this into (6), we obtain the surface stress tensor and its divergence:

$$\boldsymbol{\tau} = \tau\mathbf{A}, \quad \tau = 2(\mu_S + \lambda_S)\frac{w(a)}{a}, \quad \nabla_S\cdot\boldsymbol{\tau} = -\frac{2\tau}{a}\boldsymbol{n}.$$

The boundary conditions for the fixed boundary, the free boundary, and the boundary with surface stresses, i.e., for Problems P_0, P_f, and P_{ss}, reduce to

$$w(a) = 0, \tag{27}$$

$$\sigma_r(a) = 0, \tag{28}$$

$$\sigma_r(a) = -\frac{2\tau}{a}, \tag{29}$$

respectively. Using (26), we transform Eqs. (27)–(29) to the following transcendental equations in η:

$$\eta a \cos(\eta a) - \sin(\eta a) = 0, \tag{30}$$

$$(\lambda + 2\mu)\eta^2 a^2 \sin(\eta a) + 4\,\mu\,[\eta a \cos(\eta a) - \sin(\eta a)] = 0, \tag{31}$$

$$(\lambda + 2\mu)\eta^2 a^2 \sin(\eta a) + 4\,\mu\,[\eta a \cos(\eta a) - \sin(\eta a)] - \alpha\,(\cos(\eta a)\,\eta a - \sin(\eta a)) = 0, \tag{32}$$

where $\alpha = 4(\lambda_S + \mu_S)/\mu a$ is a dimensionless parameter. We denote the solutions of (30)–(32) by η_k°, η_k^f, and η_k, respectively. Since α in (32) depends on a explicitly, the solutions of (32) depend on the radius of the ball. Note that $\alpha \to 0$ as $a \to \infty$, while $\alpha \to \infty$ as $a \to 0$. With the given elastic moduli, the eigenfrequency depends on a, which is a so-called size effect. It is obvious that (32) reduces to (30) when $\alpha = 0$ and to (31) when $\alpha \to \infty$. It follows that $\eta_k = \eta_k^f$ when $\alpha = 0$ and $\eta_k \to \eta_k^\circ$ when $\alpha \to \infty$. So the bounds (21) cannot be strengthened, in general.

Table 1 Normalized eigenfrequencies of an elastic sphere for Problems P_f, P_{ss}, and P_0.

k	1	2	3	4	5	6	7
η_k^f/a	2.563434163	6.058670084	9.279861114	12.45884069	15.62235640	18.77840320	21.93025612
η_k/a	2.743707270	6.116764264	9.316615629	12.48593737	15.64386611	18.79625335	21.94551807
η_k°/a	4.493409458	7.725251837	10.90412166	14.06619391	17.22075527	20.37130296	23.51945250

The first seven eigenfrequencies are presented in Table 1, assuming $\lambda = \mu$ and $\alpha = 1$. For Problem P_{ss}, the dependencies of η_k on α are given in Fig. 1. Here the dashed and stroke-dashed lines correspond to η_k^f and η_k° for Problems P_f and P_0, respectively.

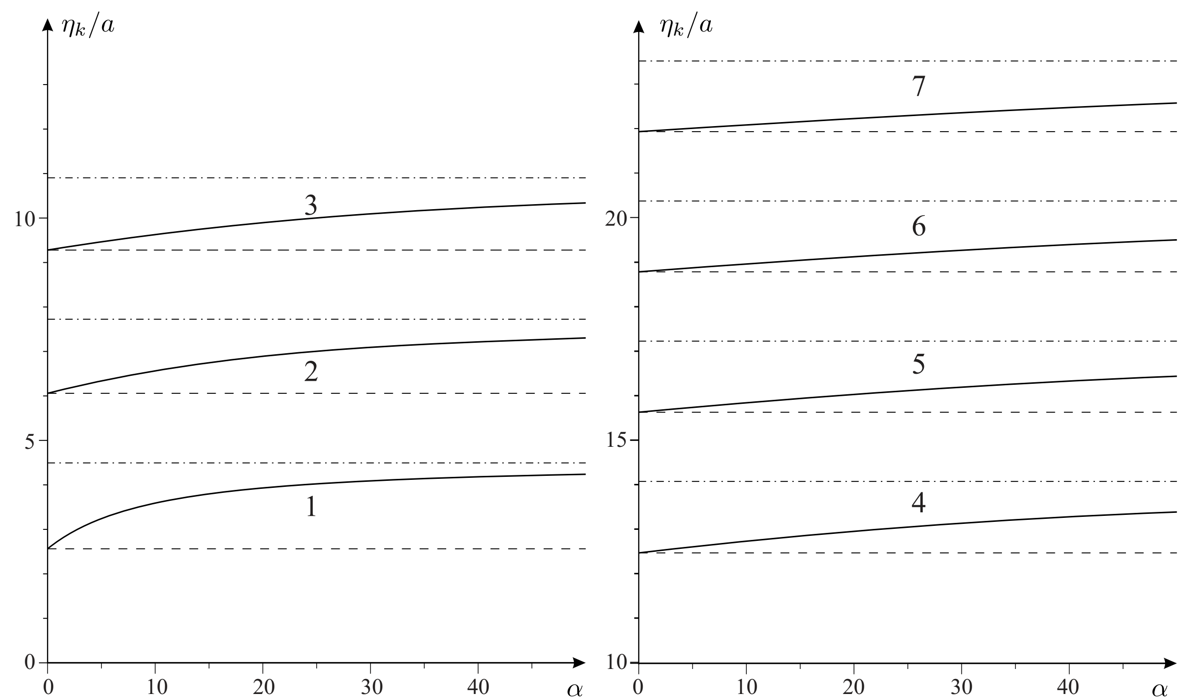

Fig. 1 Dependencies of the normalized eigenfrequencies η_k on α for $k = 1, 2, \ldots, 7$.

Conclusions

We established lower and upper bounds for the eigenfrequencies of an elastic body with surface stresses. These bounds cannot be improved. For the kth eigenfrequency, the lower bound is the kth eigenfrequency of the same body with free boundary, while the upper bound is the kth eigenfrequency of the same body with fixed boundary (the eigenfrequencies are numbered in increasing order, taking into account multiplicity of the modes). An increase in the values of surface elastic moduli implies an increase in the eigenfrequencies. The proof is based on Rayleigh's minimal principle and Courant's maximum-minimum principle. The classical formulation of Courant's principle [5] is modified to account for the peculiarities of the eigenvalue problems under consideration.

The increase in the eigenfrequencies for the elastic body with surface stresses, in comparison with the same body with free boundary, can be interpreted as the increase in the stiffness. This conclusion coincides qualitatively with the results in [6, 7] for nano-porous or nano-cellular media, and with the increase in stiffness parameters of nanosized plates and shells [3, 8]. The influence of the surface elasticity is more significant for higher eigenfrequencies and for bodies with surface imperfections. The stiffening effect is also more significant when the body sizes decrease, i.e., for nano-sized bodies. Note that for micro- and nano-sized solids, the initial (residual) surface stresses can significantly affect the apparent material properties [25, 27–29]. Residual surface stresses can increase or decrease the solid eigen-frequencies. From a mathematical viewpoint, the corresponding boundary-value problems are quite different from those we have considered; they are left as future work.

The general theory is illustrated by the free radial vibrations of an elastic sphere with surface stresses. For the vibrating sphere, the eigenfrequencies are obtained numerically and the influence of the surface elasticity properties is analyzed.

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