ON THE SPECTRUM OF A LINEAR OPERATOR

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In the definition of the spectrum of a linear operator, it is customary to assume that the underlying space is complete. However there are occasions for which it is neither desirable nor necessary to assume completeness in order to obtain a spectral theory for an operator; for example, completeness is not needed in the Riesz theory of a compact operator (see e.g. [**1**: XI. 3]). Several non-equivalent definitions for the spectrum of an operator on normed spaces have appeared in the literature. We shall discuss the relationship among these definitions and some of the difficulties that arise in applying these definitions to obtain a spectral theory.

In the following, X will denote a complex normed linear space and $\hat{X}$ will denote the completion of X. Let $L(X)$ be the space of continuous linear operators with domain X and range contained in X, and let I be the identity operator on X. For any $T \in L(X)$, let $\hat{T}$ be the unique continuous extension of T to $\hat{X}$. Finally let $\mathbf{C}$ denote the complex plane and $\hat{\mathbf{C}}$ the one point compactification of $\mathbf{C}$; all topological considerations (like closure etc.) will be taken with respect to $\hat{\mathbf{C}}$.

We begin with three different definitions of the spectrum of a bounded linear operator on X.

DEFINITION 1. Let $T \in L(X)$. The resolvent set $\rho_1(T)$ of T is the set of $\lambda \in \mathbf{C}$ such that

(i) $(\lambda I - T)X$ is dense in X, and
(ii) $(\lambda I - T)^{-1}$ exists and is continuous.

The spectrum $\sigma_1(T)$ is defined by $\sigma_1(T) = \mathbf{C} \setminus \rho_1(T)$.

DEFINITION 2. Let $T \in L(X)$. The resolvent set $\rho_2(T)$ of T is the set of $\lambda \in \mathbf{C}$ such that

(i) $(\lambda I - T)X = X$, and
(ii) $(\lambda I - T)^{-1}$ exists and is continuous.

The spectrum $\sigma_2(T)$ is defined by $\sigma_2(T) = \mathbf{C} \setminus \rho_2(T)$.

DEFINITION 3. Let $T \in L(X)$. The resolvent set $\rho_3(T)$ of T is the set of $\lambda \in \hat{\mathbf{C}}$ for which there exists a neighborhood V_λ of λ in $\hat{\mathbf{C}}$ and a function $\mu \to R_\mu$ defined on $V_\lambda \cap \mathbf{C}$ with values in $L(X)$ satisfying, for each $\mu \in V_\lambda \cap \mathbf{C}$, the conditions

(i) $R_\mu(\mu I - T) = (\mu I - T)R_\mu = I$,
(ii) the set $\{R_\mu : \mu \in V_\lambda \cap \mathbf{C}\}$ is bounded in $L(X)$.

$\sigma_3(T)$ is then defined by

$$\sigma_3(T) = \hat{\mathbf{C}} \setminus \rho_3(T).$$

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Definitions 1 and 2 are applied to operators on complex normed spaces by Taylor [3] and Dieudonné [1] respectively. Definition 3 was originally considered by Waelbroeck (see [4]) for complete locally convex spaces. It is used in not necessarily complete spaces by Neubauer [2]. Some of our results may be found in these four references. We remark that if the underlying space is complete, then all three of these definitions coincide (see e.g. [3]); in this case we shall denote the spectrum simply by $\sigma(T)$ and the resolvent set by $\rho(T)$.

Proposition 1. *Let $T \in L(X)$. Then*

(a) $\sigma_1(T) \subset \sigma_2(T) \subset \sigma_3(T)$;
(b) closure $\sigma_2(T) = \sigma_3(T)$.

Proof. Part (a) is clear from the definitions. The fact that $\sigma_3(T)$ is closed is also direct from the definition, and hence $\mathrm{cl}\,[\sigma_2(T)] \subset \sigma_3(T)$. To see the reverse inclusion, let $\lambda (\neq \infty)$ be an element of the interior of $\rho_2(T)$. Then there exists a neighbourhood V_λ of λ such that $(\mu I - T)^{-1}$ is defined and continuous on X for all $\mu \in V_\lambda$. Using the resolvent equation (see [3, p. 257]), we obtain, for $\mu \in V_\lambda$,

$$(\mu I - T)^{-1} = (\lambda I - T)^{-1} + (\lambda - \mu)(\mu I - T)^{-1}(\lambda I - T)^{-1}.$$

Using the triangle inequality and solving for $\|(\mu I - T)^{-1}\|$, we get

$$\|(\mu I - T)^{-1}\| \leqq \frac{\|(\lambda I - T)^{-1}\|}{1 - |\lambda - \mu| \, \|(\lambda I - T)^{-1}\|},$$

for $|\lambda - \mu| < \|(\lambda I - T)^{-1}\|^{-1}$. Thus, for

$$\mu \in V_\lambda \cap \{\mu : |\lambda - \mu| < \tfrac{1}{2}\|(\lambda I - T)^{-1}\|^{-1}\},$$

we have that $\|(\mu I - T)^{-1}\|$ is bounded; and hence $\lambda \in \rho_3(T)$.

Now, if $\sigma_2(T)$ is bounded in $\mathbf{C}$, then $(\lambda I - T)^{-1} \to 0$ as $\lambda \to \infty$ since $(\lambda I - T)^{-1} = (\lambda \hat{I} - \hat{T})^{-1}|_X$ and $(\lambda \hat{I} - \hat{T})^{-1} \to 0$ as $\lambda \to \infty$. It thus follows that $\infty \in \rho_3(T)$. Conversely, if $\sigma_2(T)$ is unbounded in $\mathbf{C}$, then it results directly from the definition that $\infty \in \sigma_3(T)$. Thus $\mathrm{cl}\,[\sigma_2(T)] = \sigma_3(T)$ with the closure taken in $\hat{\mathbf{C}}$.

Proposition 2. *Let $T \in L(X)$. Then $\sigma_1(T) = \sigma(\hat{T})$, and hence $\sigma_1(T)$ is compact.*

Proof. Let $\lambda \in \rho(\hat{T})$. We claim that $(\lambda I - T)X$ is dense in X. For let $x \in X$ and let $y \in \hat{X}$ be such that $(\lambda \hat{I} - \hat{T})y = x$. Choose $(x_n) \subset X$ such that $x_n \to y$; then $(\lambda I - T)x_n \to x$. Since any restriction of $(\lambda \hat{I} - \hat{T})^{-1}$ is continuous, we conclude that $\lambda \in \rho_1(T)$.

Conversely, let $\lambda \in \rho_1(T)$. By elementary arguments similar to the above, it can be shown that $\lambda \hat{I} - \hat{T}$ is 1-1 and onto $\hat{X}$. Hence, by the Banach open mapping theorem, $(\lambda \hat{I} - \hat{T})^{-1}$ is continuous, and so $\lambda \in \rho(\hat{T})$.

From the above, it follows that $\sigma_1(T)$ and $\sigma_3(T)$ are closed sets. However $\sigma_2(T)$ need not be closed; in fact, as the next example shows, $\sigma_2(T)$ may be virtually any subset of the plane. The example also shows that $\sigma_1(T)$, $\sigma_2(T)$ and $\sigma_3(T)$ may all be distinct.

Example 1. Let $D \subset \mathbf{C}$ be such that $D \cap [0, 1] = \emptyset$. Let X be the subspace of $C[0, 1]$

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consisting of all functions of the form

$$f(t) = p(t) \Big/ \prod_{i=1}^{n} (\lambda_i - t)^{k_i},$$

where p is a polynomial, $\lambda_i \in D$ and k_i is a non-negative integer for $i = 1, \ldots, n$ (Here n may depend on f). Then X is a normed space under the supremum norm and $\hat{X} = C[0, 1]$. Define $T: X \to X$ by $Tf(t) = tf(t)$ for $t \in [0, 1]$.

For $\lambda \in D$ we have $(\lambda I - T)^{-1} f(t) = f(t)(\lambda - t)^{-1} \in X$; for $\lambda \notin D$, $(\lambda I - T)X \neq X$. Hence $\rho_2(T) = D$. We therefore have $\sigma_1(T) = \sigma(\hat{T}) = [0, 1]$; $\sigma_2(T) = \mathbf{C} \backslash D$; and $\sigma_3(T) = \mathrm{cl}(\mathbf{C} \backslash D)$.

If, in the above example, we take D to be a set which is not open, then $\sigma_1(T)$, $\sigma_2(T)$ and $\sigma_3(T)$ are all distinct. If D is chosen to be open, but $D \neq \mathbf{C} \backslash [0, 1]$, then $\sigma_2(T) = \sigma_3(T)$, but both are larger than $\sigma_1(T)$. However, if $\sigma_1(T) = \sigma_2(T)$, it then follows that $\sigma_1(T) = \sigma_3(T)$ since $\sigma_1(T)$ is closed. West [5] calls an operator for which $\sigma_1(T) = \sigma_2(T)$, an operator with single spectrum; in his paper he develops a spectral theory for these operators. In the next proposition we give some conditions to ensure that $\sigma_1(T) = \sigma_2(T) = \sigma_3(T)$.

Proposition 3. *Let* $T \in L(X)$.

(a) $\sigma_1(T) = \sigma_2(T) = \sigma_3(T)$ *if and only if* $(\lambda \hat{I} - \hat{T})^{-1} X \subset X$ *for all* $\lambda \in \rho(\hat{T})$.

(b) *If* $\hat{T}(\hat{X}) \subset X$, *then* $\sigma_1(T) = \sigma_2(T) = \sigma_3(T)$.

(c) *If* closure $[T(X)]$ *is complete, then* $\sigma_1(T) = \sigma_2(T) = \sigma_3(T)$.

Proof. (a) The statement $\sigma_1(T) \supset \sigma_2(T)$ is equivalent to the statement: if $\lambda \in \rho_1(T) = \rho(\hat{T})$, then $(\lambda I - T)X = X$. This in turn is equivalent to the statement: if $\lambda \in \rho(\hat{T})$, then $(\lambda \hat{I} - \hat{T})^{-1} X \subset X$.

(b) Let $0 \neq \lambda \in \rho_1(T) = \rho(\hat{T})$ and let $x \in X$. Then there exists $y \in \hat{X}$ such that $(\lambda \hat{I} - \hat{T})y = x$, and therefore $y = (1/\lambda)x + (1/\lambda)Ty \in X$. Hence $(\lambda I - T)X = X$ and $\lambda \in \rho_2(T)$.

Part (c) follows directly from (b).

Corollary. *If* $T \in L(X)$ *is compact, then* $\sigma_1(T) = \sigma_2(T) = \sigma_3(T)$. (See also [**1**].)

It may be remarked that the converse to Proposition 3(b) is not true. (See [**5**, Example 3].)

We now give two examples and make some comments concerning spectral sets and functions of the operator. One of the principal results of spectral theory on Banach spaces is the decomposition of the operator T if the spectrum is disconnected; this states that, if $\sigma(T) = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$, where A_1 and A_2 are both open and closed in $\sigma(T)$, then $X = X_1 \oplus X_2$ with $T(X_i) \subset X_i$ and $\sigma(T | X_i) = A_i$, for $i = 1, 2$. The following two examples show that, if the underlying space is not complete, then the above decomposition fails for each of the spectra considered.

Example 2. Let X be the set of polynomials on R restricted to the domain $[0, 1] \cup [2, 3]$, and equip X with the supremum norm. Then $\hat{X} = C[0, 1] \oplus C[2, 3]$. Let $T: X \to X$ be defined by $Tp(t) = tp(t)$. Then $\sigma_1(T) = \sigma(\hat{T}) = [0, 1] \cup [2, 3]$. Corresponding to the spectral sets, we have the decomposition $\hat{X} = \hat{X}_1 \oplus \hat{X}_2$ for T, where

$$\hat{X}_1 = \{(f, 0) : f \in C[0, 1]\}$$

and

$$\hat{X}_2 = \{(0, g) : g \in C[2, 3]\}.$$

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However there cannot exist a decomposition of the form $X = X_1 \oplus X_2$ for which $T(X_i) \subset X_i$, $\sigma_1(T|X_1) = [0, 1]$ and $\sigma_1(T|X_2) = [2, 3]$. For, let $X_1 \subset X$ with $\sigma_1(T|X_1) = [0, 1]$ and $TX_1 \subset X_1$. Then, for each $\lambda \in [2, 3]$, $(\lambda I - T)X_1$ is dense in X_1, so that, if $p(t) \in X_1$, then $p(t)$ can be uniformly approximated by functions of the form $(\lambda - t)p_n(t)$ with $p_n(t) \in X_1$ and, since such functions vanish at $t = \lambda$, it follows that $p(\lambda) = 0$. This implies that $p(t) = 0$ for $t \in [2, 3]$ and hence that $p(t) \equiv 0$. Thus $X_1 = 0$ and this is a contradiction.

Example 3. Let $D = \mathbf{C} \backslash ([0, 1] \cup \{2\})$, and let X and T be as in Example 1. Then $\sigma_2(T) = \sigma_3(T) = [0, 1] \cup \{2\}$. However, for every non-trivial invariant subspace $Y \subset X$ for T, $\sigma_2(T|Y) \cap [0, 1] \neq \emptyset$. Hence there is no spectral set decomposition of X relative to either $\sigma_2(T)$ or $\sigma_3(T)$.

Examples 2 and 3 also show that, if f is a function which is analytic in a neighbourhood of $\sigma_i(T)$ $(i = 1, 2, 3)$, then $f(T)$ need not be defined. (Take f to be the function which is 1 on one part of the spectrum and 0 on the other part.) However, for any polynomial p, $p(T)$ is always defined, and one can prove the spectral mapping theorem: $p(\sigma_i(T)) = \sigma_i(p(T))$ for each of $i = 1, 2, 3$. For $\sigma_1(T)$, this follows from the fact that $\sigma_1(T) = \sigma(\hat{T})$; for $\sigma_2(T)$ and $\sigma_3(T)$, the proof is the same as in the complete case (see e.g. [3]).

In conclusion, the authors would like to state that they believe that Definition 3 will probably be the most useful one for consideration of spectral theory on incomplete spaces. Definition 2 suffers from the topological defect that the spectrum need not be closed; while Definition 1 has the drawback that it does not distinguish between an operator T and its extension $\hat{T}$ (or among any of the other operators S for which $\hat{S} = \hat{T}$).

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