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ON THE SPECTRUM OF A NONLINEAR OPERATOR

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INTRODUCTION

The aim of this paper is to investigate the eigenvalues of nonlinear operators. Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be an operator in the real Euclidean n-space. We shall suppose that the operator A is *real analytic*, this is the essential assumption in the whole work. A number λ is said to be an eigenvalue of the operator A, if there exists a vector $x \in \mathbf{R}^n$, $x \neq 0$ such that $Ax = \lambda x$. The spectrum of the operator A is the set of all its eigenvalues. It is well known that the spectrum of a linear symmetric operator is a finite set. Our aim is to prove an analogous assertion also for nonlinear operators.

For $n = 1$, $Ax = x^4$, the spectrum of A is the whole real line. Hence it is clear that in the above form the assertion on the discreteness of the spectrum does not hold. We must hence distinguish "discrete" and "continuous" components of the spectrum in such a way that we restrict the set of those $x \in \mathbf{R}^n - \{0\}$ among which we seek eigenvectors. It will be seen that it is suitable to consider the sets

$$M_r(f) = \{x \in \mathbf{R}^n \mid f(x) = r\},$$

where $r > 0$, $f(x) = \frac{1}{2}|x|^2$. The connection of the function f with our problem is obvious, f is the potential of identity operator.

Let us set

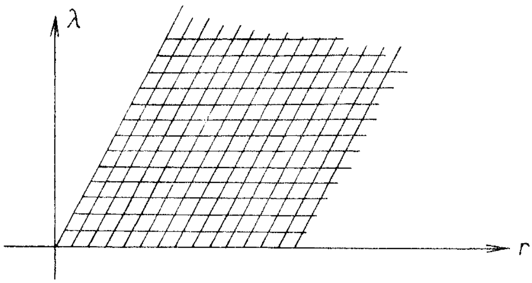
$$G = \{[r, \lambda] \in \mathbf{R}^2 \mid \text{there exists } x \in \mathbf{R}^n \text{ such that } Ax = \lambda x,\ f(x) = r\},$$

there is a question, if the set $G_{r_0} = \{[r, \lambda] \in G \mid r = r_0\}$ is finite.

First, nothing can be done without assumption that the operator A is *potential*. For example, we can choose

$$A : [x, y] \mapsto [x^3, x^2 y].$$

Then clearly every vector from $\boldsymbol{R}^2$ is an eigenvector, $\lambda(x, y) = x^2$ and the set G has the following shape:



Similarly, we can choose arbitrarily the function $\lambda(x)$ on $\boldsymbol{R}^n$ and define the mapping

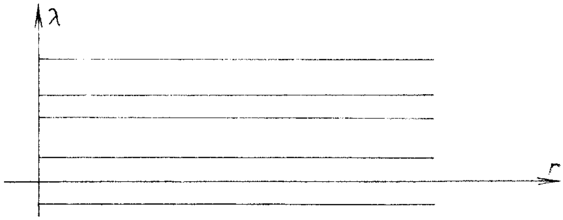
$$A : [x_1, \ldots, x_n] \mapsto [\tfrac{1}{2}x_1\, \lambda(x), \ldots, \tfrac{1}{2}x_n\, \lambda(x)] .$$

Then again every vector from $\boldsymbol{R}^n$ is an eigenvector and the function $\lambda(x)$ can be quite arbitrary. (On the contrary, if A is a potential operator and if every $x \in \boldsymbol{R}^n - \{0\}$ is eigenvector, then $\lambda(x)$ is constant on $M_r(f)$ for all $r > 0$, i.e. the set G_{r_0} is a one-point set for all $r_0 > 0$.)

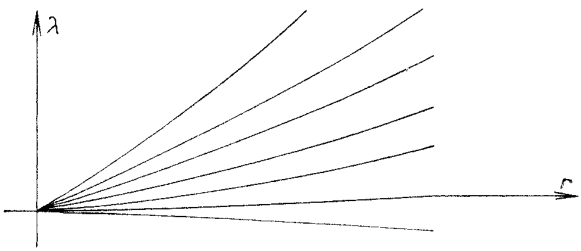
In the particular case when the operator A is b-homogeneous, $b > 0$, it was proved in [FNSS 1–6] that the set $G_{r_0} = G \cap \{[r, \lambda] \mid r = r_0\}$ is at most finite for all $r_0 > 0$. The sets G_r correspond to one another by the relation

$$G_r = r^{b-1} G_1 .$$

For a linear operator, the set G has the following shape:

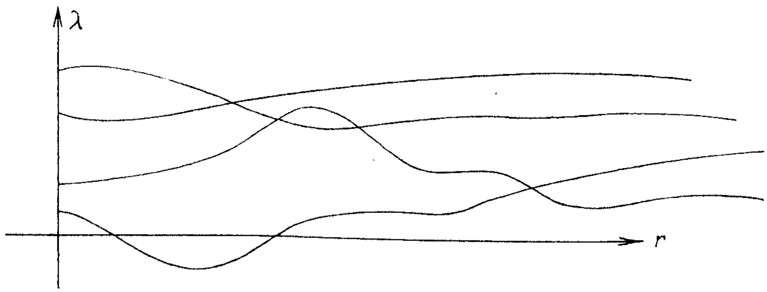


and, for a homogeneous operator A:



In these figures, the discrete and the continuous components of the set G are seen very clearly: we obtain the set G if we take the set G_r, $r = 1$ and change it continuously together with the number r (a multiple of an eigenvector is again an eigenvector). The proper aim of this work is to show that also in the nonhomogeneous case the "discrete" and "continuous" components of the spectrum can be obtained by means of the system $M_r(f)$.

In the nonhomogeneous case there is generally no relation between the sets G_r and G_1, hence it is natural to suppose that the parabolas from the graph corresponding to a homogeneous operator will change into some analytic curves, for example



It will be seen that this idea is roughly correct, but somewhat too simple. Let us give some example which show, what can happen.

1) The curves in the figure need not be generally the graph of a function. Let

$$g(x, y) = \tfrac{1}{5}x^5 - \tfrac{1}{3}x^3 + (x^3 - x)(y - 1)$$

i.e.

$$\nabla g(x, y) = [x^4 - x^2 + (3x^2 - 1)(y - 1),\ x^3 - x]$$

and the equations for eigenvectors have the form

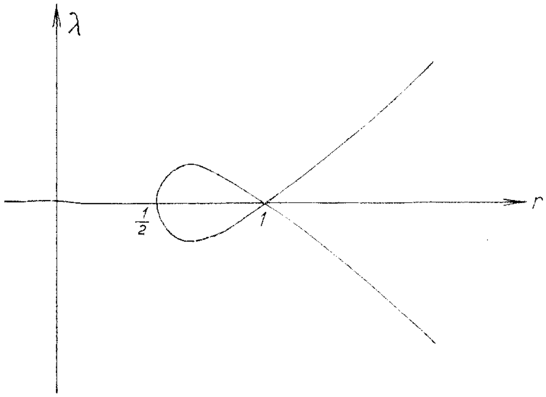
$$x^4 - x^2 + (3x^2 - 1)(y - 1) = \lambda x$$

$$x^3 - x = \lambda y\ .$$

It is clear that each point of the line $B_1 = \{[x, y] \mid y = 1\}$ is an eigenvector and it holds

$$\lambda(x, 1) = x(x^2 - 1)\,, \quad f(x, 1) = \tfrac{1}{2}(x^2 + 1)\ .$$

The set $G_1 = [f, \lambda](B_1)$ has the following shape:



2) The curves in the figure can be closed curves and need not be smooth. Let

$$g(x, y) = x(\tfrac{1}{6}y^6 + \tfrac{80}{21}x^6 - \tfrac{64}{5}x^4 + 16x^2 - \tfrac{32}{3}),$$

then the equations for eigenvector have the form

$$\tfrac{1}{6}y^6 + \tfrac{80}{3}x^6 - 64x^4 + 48x^2 - \tfrac{32}{3} = \lambda x\,, \quad xy^5 = \lambda y\,.$$

It is easy to see that each point of the ellipse B_1, $B_1 = \{[x, y] \mid x^2 + \frac{1}{4}y^2 = 1\}$ is an eigenvector of this problem. For $[x, y] \in B_1$ it holds

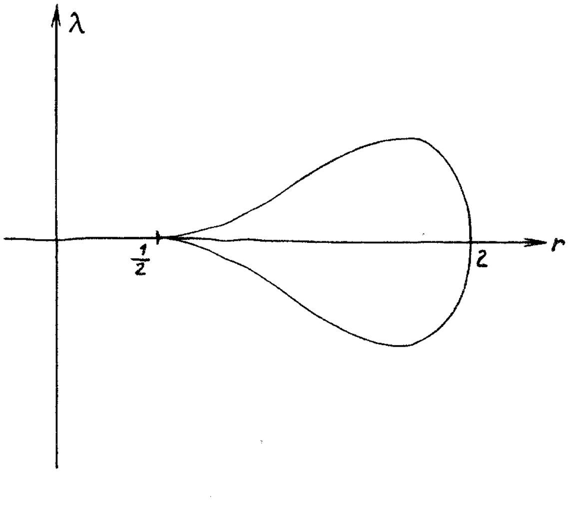
$$f = \tfrac{1}{2}(4 - 3x^2) = \tfrac{1}{2}(1 + \tfrac{3}{4}y^2)$$

$$x^2 = \tfrac{1}{3}(4 - 2f)\,, \quad y^2 = \tfrac{4}{3}(2f - 1)\,,$$

hence

$$\lambda(x, y) = \pm\tfrac{16}{9}(2f - 1)^2 \sqrt{\tfrac{1}{3}(4 - 2f)}\,.$$

The corresponding set $G_1 = [f, \lambda](B_1)$ has then the following shape:



3) We cannot hope to prove that the sets G_r are finite as in the homogeneous case, for fixed $r_0 > 0$ the set G_{r_0} can contain an interval. Let

$$g(x, y) = x(x^2 + y^2 - 1)$$

then equations for the eigenvectors have the form

$$3x^2 + y^2 - 1 = 2x\lambda\,, \quad 2xy = 2y\lambda\,.$$

Then there are two possibilities.

a) If $y = 0$, then it follows from the first equation that

$$\lambda = \frac{3x^2 - 1}{2x}$$

hence for all $x \in \mathbf{R}$, $x \neq 0$ the vector $[x, 0]$ is an eigenvector and for $f(x, 0) = x^2$, we have

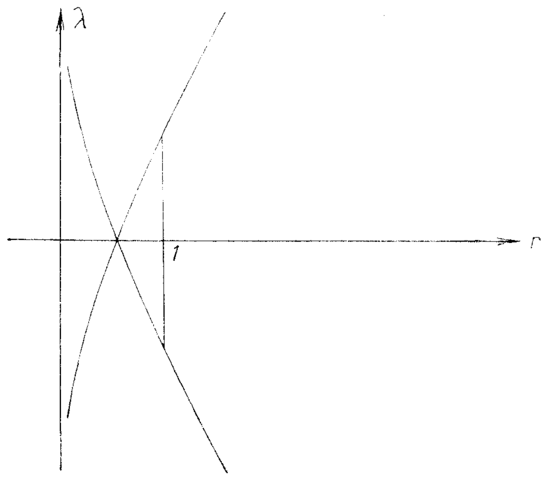
$$\lambda = \pm \frac{3r - 1}{2\sqrt{r}}\,.$$

b) If $y \neq 0$, then $x = \lambda$ and

$$3x^2 + y^2 - 1 = 2x^2$$

hence $x^2 + y^2 = 1$. The second branch of eigenvectors is hence the unit circle.

The set G has the following shape:



Generally, if $\varphi(x)$ is a real analytic function on $\mathbf{R}^n$ and if we choose

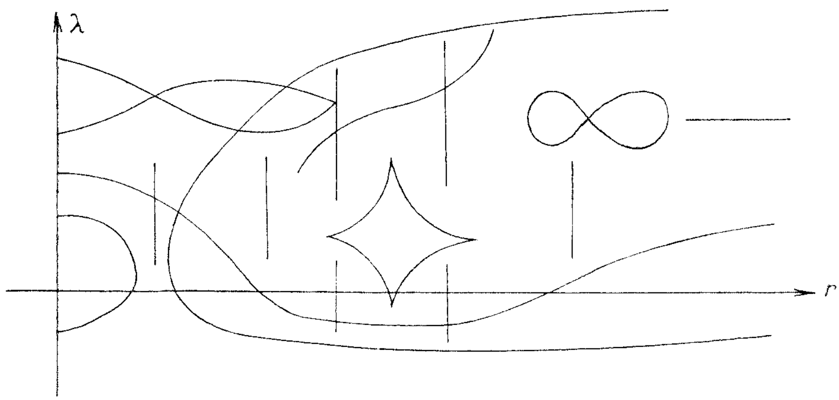
$$g(x) = \varphi(x)\,[f(x) - r]$$

then clearly for every point $x \in M_r(f)$ it holds

$$\nabla g(x) = \varphi(x)\,\nabla f(x) + \nabla\varphi(x)\,[f(x) - r] = \varphi(x)\,\nabla f(x)\,.$$

Hence the whole sphere $M_r(f)$ lies in the set of eigenvectors B. Values of the function $\lambda(x)$ for $x \in M_r(f)$ are then equal to $\varphi(x)$. Hence if we take a sphere $M_r(f)$ and prescribe the values of function $\lambda(x)$ on it so that $\lambda(x) = \varphi(x)|_{M_r(f)}$, where $\varphi(x)$ is an real analytic function in $\mathbf{R}^n$ then we can choose a function $g(x)$ in such a way that the points $x \in M_r(f)$ are eigenvectors of this problem and that the corresponding eigenvalues $\lambda = \lambda(x)$ are equal to the prescribed number.

Hence the parabolas from the homogeneous case change into parametric curves, they can have isolated singularities, moreover, they can include vertical line segment. We can imagine that the set G can have e.g. the following shape:



The sets G_r generally need not be finite for all $r > 0$, but it holds:

There exists a discrete set $R \subset (0, \infty)$ such that for all $r \in (0, \infty) - R$ the set G_r is at most finite and for $r \in R$ the set G_r consists of at most finite number of (possibly degenerated) closed line segments.

The essential fact is that the set G cannot be of the following shape:

i.e. it does not contain, roughly speaking, two-dimensional parts (in patricular, for example, vertical line segments form a locally finite collection).

In this work we consider a slightly more general case, when instead of the function $f(x) = \frac{1}{2}|x|^2$ we can take a general nonhomogeneous, nonquadratic function such that its gradient is similar (in some sense) to the identity operator. The discreteness of the sets G_r is proved for homogeneous operators in [FNSS 1–7] also for the case of operators in infinite dimensional spaces and extensive applications to integral, differential and integrodifferential equations are also given. The method of those papers is different from that used here. In the homogeneous case it is sufficient to show that the set of all critical levels is a discrete set, since there exists a simple relation between the set of critical levels and the set of eigenvalues. However, this relation does not hold for nonhomogeneous operators. The results which are proved here for finite dimensional spaces can be transferred to infinite Hilbert spaces, too. This can be done by the methods used in [FNSS 2] under similar assumptions and the results can be again applied to differential and integral equations. There are only technical difficulties if we attempt to transfer the result to operators in Banach spaces (as in [FNSS 1–7]), but the method is the same. We intend to devote further papers to this problem.

In Chapter I we introduce notions and theorems which we shall use in the following. Especially, as this article is intended mostly to the readers which are not specialists in the theory of functions of several complex variables, we recall basic notions and some theorems from the theory of analytic sets.

In Chapter II an analogue of Morse-Sard theorem for holomorphic functions on an analytic set is proved. We use these results in the following chapter. For the algebraic case a similar theorem was proved in [M], for holomorphic functions in $\mathbf{C}^n$, this theorem was proved in [S–S].

In Chapter III we prove the basic Theorem 3.10. The method of the proof is the following. The analytic set B of eigenvectors can be divided into irreducible branches and the collection of the branches again into two groups. The first group consists of the branches which are contained in some $M_r(f)$. The discrete set R is then formed by those r for which $M_r(f)$ contains such a branch while the corresponding graph in G is formed by vertical line segments. For the branches from the other group, we shall prove, (see Theorem 3.8), roughly speaking:

If the point $x \in B$ moves in B in such a way that $f(x)$ is constant, then $\lambda(x)$ is also constant.

A trouble arises if the point $x \in B \cap M_r(f)$ at which we investigate the relation between $B \cap M_r(f)$ and B is a singular point of B. Hence we must use the theorem on local parametrization of an analytic set and Lemma 3.4 on tangent vectors for analytic sets.

In Chapter IV we prove an assertion analogous to that in Chapter III for operators in Hilbert spaces. In the proof we use the usual method from the theory of bifurcation, which uses the Fredholm property of the operator $\lambda \nabla f - \nabla g$. For purposes of this chapter, it is necessary to prove a generalization of Theorem 3.8.

Chapter I.

A. NOTATION, CRITICAL POINTS

Let us denote $\mathbf{R}^n$ (resp. $\mathbf{C}^n$) the n-fold Cartesian product of the field of real (resp. complex) numbers. We denote by $\mathscr{Y}$ the involution $\mathscr{Y} : z \to \bar{z}$ on $\mathbf{C}^n$. The set $\mathbf{R}^n$ will be imbedded into $\mathbf{C}^n$ by means of the mapping $[x_1, \ldots, x_n] \mapsto [x_1 + \mathrm{i}0, \ldots \ldots, x_n + \mathrm{i}0]$ and the set $\mathbf{C}^k$ $(0 < k < n)$ into $\mathbf{C}^n$ by

$$[z_1, \ldots, z_k] \mapsto [z_1, \ldots, z_k, 0, \ldots, 0] .$$

For a polydisc $\Delta(0, r) = \{z \in \mathbf{C}^n \mid |z_j| < r_j;\ 1 \leqq j \leqq n\}$ we shall denote by $\Delta_k(0, r)$ the set

$$\Delta_k(0, r) = \Delta(0, r) \cap \mathbf{C}^k = \{z \in \mathbf{C}^n \mid |z_j| < r_j,\ j = 1, \ldots, k;\ z_{k+1} = \ldots = z_n = 0\} .$$

Further, we denote by Π the projection $\Pi : \mathbf{C}^n \to \mathbf{C}^n$:

$$[z_1, \ldots, z_n] \mapsto [z_1, \ldots, z_k, 0, \ldots, 0]$$

and by $\nabla f(z)$ the gradient of f at the point z. The symbol Ω will be reserved for open subsets of $\mathbf{R}^n$(or $\mathbf{C}^n$).

Let us recall that the real function f is said to be $\mathbf{R}$-analytic in $\Omega \subset \mathbf{R}^n$ if each point $x \in \Omega$ has a neighborhood U, $x \in U \subset \Omega$ such that the function f has a power series expansion in $\mathscr{U}$.

Let $q = [q_1, \ldots, q_n]$ be a holomorphic (resp. $\mathbf{R}$-analytic) mapping of a neighborhood of p in $\mathbf{C}^n$ into $\mathbf{C}^n$ (resp. of a neighborhood of p in $\mathbf{R}^n$). The set $\{q_i\}_1^n$ is said to be a coordinate set at p, if $q_i(p) = 0$; $i = 1, \ldots, n$ and if $\det J_q(p) \neq 0$ (where $J_q(p)$ is the Jacobian matrix of the mapping q at the point p).

A subset M of $\mathbf{C}^n$ (resp. $\mathbf{R}^n$) is called a *complex* (*resp. real*) *submanifold*, if to every $p \in M$ there corresponds a coordinate set $q = \{q_i\}_1^n$ at p such that in some neighborhood $\mathscr{U}$ of p

$$M \cap \mathscr{U} = \{z \in \mathscr{U} \mid q_{k+1}(z) = \ldots = q_n(z) = 0\}$$

for a positive integer k. This number k is independent of the choice of the coordinate set and is called the dimension of M at p.

If f is holomorphic (**R**-analytic) function on a submanifold $M \subset \mathbf{C}^n$ (ev. $\mathbf{R}^n$), the point p will be called *a critical point* of f on M if

$$\frac{\partial[f(z(q))]}{\partial q_i}(q(z_0)) = 0\,; \quad i = 1, \ldots, k\,,$$

(where $\{q_i\}_1^n$ is the corresponding coordinate set from the definition of the submanifold M).

It is clear that the property of being a critical point is independent of the choice of the coordinate set q.

Suppose further that f is complex (or real) function on a set $V \subset \mathbf{C}^n$ (or $V \subset \mathbf{R}^n$). The point $a \in V$ will be called *a metric critical point* of f on V, if

$$\lim_{n\to\infty} \frac{f(y_n) - f(a)}{|y_n - a|} = 0$$

for all sequence $\{y_n\}$ such that

$$y_n \in V\,, \quad y_n \to a\,.$$

It is easy to see that if $M \subset \Omega \subset \mathbf{C}^n$ is a (complex) submanifold and f is holomorphic in Ω, then

$a \in M$ is a critical point of f on M iff a is a metric critical point of f on M.

Of course, the same is true in the real case.

Theorem A 1. *Suppose that $\Omega \subset \mathbf{R}^n$, $\Omega^* \subset \mathbf{C}^n$ are (open) sets such that*

$$\Omega = \Omega^* \cap \mathbf{R}^n\,.$$

Let $M^* \subset \Omega^*$ be a (complex) submanifold, the dimension of which is at every point equal to p. Suppose further that M^* is symmetric, i.e. $\mathscr{Y}(M^*) = M^*$.

Let f^* be a holomorphic function on Ω^* such that

$$f = f^*|_\Omega$$

is a real function.

Then

$$M = M^* \cap \Omega$$

is a real submanifold the dimension of which at every point is p and if we denote

$$N^* = \{z \in M^* \mid z \text{ is a critical point of } f^* \text{ on } M^*\}\,,$$

$$N = \{x \in M \mid x \text{ is a critical point of } f \text{ on } M\}\,,$$

then

$$N = N^* \cap \Omega .$$

Proof. Let $z_0 \in M$ be fixed.

1) We can choose a coordinate set $\{q_i^*\}$ in a symmetric neighborhood $\mathcal{O}^* \subset \Omega^*$ such that

$$M^* \cap \mathcal{O}^* = \{z \in \mathcal{O}^* \mid q_{p+1}^*(z) = \ldots = q_n^*(z) = 0\} .$$

We shall show that we can suppose that $q_j^*|_{\Omega \cap \mathcal{O}^*}$ are real functions. Indeed, if this is not the case, we can define functions

$$\varphi_j(z) = q_j^*(z) + \overline{q_j^*(\bar{z})} ; \quad j = 1, \ldots, n ,$$
$$\psi_j(z) = \mathrm{i}[q_j^*(z) - \overline{q_j^*(\bar{z})}] .$$

These functions are holomorphic in $\mathcal{O}^*$ and $\varphi_j|_{\Omega \cap \mathcal{O}^*}$, $\psi_j|_{\Omega \cap \mathcal{O}^*}$ are real functions. The set M^* is symmetric, hence the functions $\varphi_{p+1}, \psi_{p+1}, \ldots, \varphi_n, \psi_n$ are equal to zero on M^*. It holds

$$\text{(1,1)} \qquad \operatorname{rank} \frac{\partial(\varphi_{p+1}, \psi_{p+1}, \ldots, \varphi_n, \psi_n)}{\partial(z_1, \ldots, z_n)}(z_0) \geqq n - p$$

for we can write $q_j^* = \frac{1}{2}[\varphi_j - \mathrm{i}\psi_j]$, $j = p + 1, \ldots, n$ and $\{q_i^*\}_1^n$ is the coordinate set at z_0.

Further, the function φ_{p+1} is a combination of the functions $q_{k+1}^*, \ldots, q_n^*$ (in a small neighborhood), i.e.

$$\varphi_{p+1}(z) = \sum_{j=p+1}^{n} h_j(z)\, q_j^*(z) ,$$

where h_j are holomorphic functions. This follows immediately from the Taylor expansion of the function φ_{p+1} in the variables $q_1^*, \ldots, q_n^*$ and from the fact that φ_{p+1} is equal to zero on M^*. The same is true for the functions $\psi_{p+1}, \ldots, \varphi_n, \psi_n$, hence

$$\operatorname{rank} \frac{\partial(\varphi_{p+1}, \psi_{p+1}, \ldots, \varphi_n, \psi_n)}{\partial(z_1, \ldots, z_n)}(z_0) \leqq n - p .$$

From this and from (1,1) it follows that we can choose functions $\eta_{p+1}, \ldots, \eta_n$ from the functions $\varphi_{p+1}, \psi_{p+1}, \ldots, \varphi_n, \psi_n$ such that

$$\operatorname{rank} \frac{\partial(\varphi_{p+1}, \psi_{p+1}, \ldots, \varphi_n, \psi_n)}{\partial(z_1, \ldots, z_n)}(z_0) \leqq n - p .$$

In the same way as the assertion (1,1) we can show that

$$\operatorname{rank} \frac{\partial(\varphi_1, \ldots, \psi_n)}{\partial(z_1, \ldots, z_n)}(z_0) = n ;$$

hence we can choose functions $\eta_1, \ldots, \eta_p$ from the functions $\varphi_1, \psi_1, \ldots, \varphi_n, \psi_n$ in such a way that $\{\eta_i\}_1^n$ is the coordinate set at z_0.

It is sufficient now to show that in a small neighborhood of z_0 it holds

$$M^* = \{z \mid \eta_{p+1}(z) = \ldots = \eta_n(z) = 0\} .$$

However, $\{\eta_j\}_1^n$ is the coordinate set at z_0, hence in a small neighborhood of z_0 the set $\{z \mid \eta_{p+1}(z) = \ldots = \eta_n(z) = 0\}$ is a submanifold of the dimension p. Clearly this submanifold contains the submanifold M^* the dimension of which at z_0 is also p. Hence in a small neighborhood of z_0 these two submanifolds are equal (this assertion is contained – for analytic sets – in Theorem B 5). Hence we can suppose that $q_j = q_j^*|_{\Omega \cap \mathcal{O}^*}$ are real functions.

Then $\mathcal{O} = \mathcal{O}^* \cap \Omega$ is a neighborhood of z_0 in $\mathbf{R}^n$,

$$M \cap \mathcal{O} = \{x \in \mathcal{O} \mid q_{p+1}(x) = \ldots = q_n(x) = 0\} \tag{1,2}$$

and the mapping $q = \{q_i\}_1^n$ is nonsingular at p. Hence M is a real submanifold of $\mathbf{R}^n$ the dimension of which at z_0 is p.

2) Further, the function f^* being holomorphic, we can write $f^*(q_i^*, \ldots, q_n^*) = \sum_{i_1, \ldots, i_n \geqq 0} a_{i_1, \ldots, i_n} q_1^{*i_1} \ldots q_n^{*i_n}$, where $a_{i_1, \ldots, i_n}$ are real numbers.

It follows immediately from the definition of critical points that

$$z_0 \in N^* \Leftrightarrow a_i = 0 \quad \text{for} \quad i = [1, 0, \ldots, 0], \ldots, [0, \ldots, \underset{p}{1}, \ldots, 0] \Leftrightarrow z_0 \in N .$$

Hence $N = N^* \cap \Omega$.

B. ANALYTIC SETS IN C^n

A collection $\{A_i\}_{i \in I}$ of subsets of $\mathbf{C}^n$ (or $\mathbf{R}^n$) is said to be *locally finite*, if for each point $z \in \mathbf{C}^n$ (or $z \in \mathbf{R}^n$) there exists a neighborhood $\mathcal{U}_z$ such that

$$A_i \cap \mathcal{U}_z \neq \emptyset$$

holds only for a finite number of indices $i \in I$.

A function f is said to be *locally constant* on a set $X \subset \mathbf{C}^n$ (or $\mathbf{R}^n$) if each point $x \in X$ has a neighborhood $\mathcal{U}_z$ such that f is constant on $\mathcal{U} \cap X$.

A subset $V \subset \Omega \subset \mathbf{C}^n$ is said to be *an analytic set* (complex) if for every z in Ω there are a neighborhood $\mathcal{U}_z$ and functions $f_1, \ldots, f_s$ holomorphic in $\mathcal{U}_z$ such that

$$V \cap \mathcal{U}_z = \{y \in \mathcal{U}_z \mid f_1(y) = \ldots = f_s(y) = 0\} = V(f_1, \ldots, f_s) .$$

This definition implies immediately that finite union and finite intersection of analytic sets is again an analytic set. It is also clear that if $\{V_i\}_{i\in I}$ is a locally finite collection of analytic sets, then also $\bigcup_{i\in I} V_i$ is an analytic set. Further, it can be proved (see [$G-R$], p. 86, Theorem III.E.3) that the intersection of any collection of anlytic sets is again an analytic set.

Suppose $X \subset \mathbf{C}^n$, $z \in \mathbf{C}^n$, then the germ of the set X at z will be denoted by $[X]_z$. It is clear that intersection, union of two germs, as well as inclusion betwee n them, are well defined operations on germs.

Similarly, if f is defined in a neighborhood of z, then the germ of the function f at z will be denoted by $[f]_z$. The set ${}_n\mathcal{O}_z$ of all germs of holomorphic functions at z form an integral domain (${}_n\mathcal{O}_z$ inherits its algebraic structure from the global objects from which it is derived).

We say that $[f]_z$ vanishes on $[X]_z$, if $[X]_z \subset [V(f)]_z$.

A germ $[X]_z$ is said to be the *germ of an analytic set*, if there are elements $[f_1]_z, \ldots, [f_t]_z \in {}_n\mathcal{O}_z$ such that

$$[X]_z = [f_1]_z \cap \ldots \cap [f_n]_z .$$

The collection of all germs of analytic sets at z will be denoted by ${}_nG_z$. This collection is clearly closed with respect to finite unions and finite intersections (see [G – R], p. 87, Proposition II.E.7).

Let $[V]_z \in {}_nG_z$. The *ideal of* $[V]_z$ is defined to be the set

$$\operatorname{id}[V]_z = \{[f]_z \in {}_n\mathcal{O}_z \mid [f]_z \text{ vanishes on } [V]_z\} .$$

Let $A \subset {}_n\mathcal{O}_z$. The *locus of* A is defined to be

$$\operatorname{loc} A = \bigcap [V(f)]_z ,$$

where the intersection is taken over all $[f]_z \in A$.

It holds that id $[V]_z$ is an ideal of ${}_n\mathcal{O}_z$, loc A is a well defined germ of an analytic set and loc id $[V]_z = [V]_z$ (see [G – R], Theorems II.E.9, II.E.11(v)).

A germ $[V]_z \in {}_nG_z$ is said to be *irreducible*, if $[V]_z = [V_1]_z \cup [V_2]_z$ for $[V_1]_z, [V_2]_z \in {}_nG_z$ implies that either $[V]_z = [V_1]_z$ or $[V]_z = [V_2]_z$. The germ $[V]_z$ is irreducible if and only if id $[V]_z$ is prime (see [G – R], p. 89, Theorem II.E.13).

Suppose that $\mathscr{P}$ is a prime ideal in ${}_n\mathcal{O}_0$. A coordinate set $z_1, \ldots, z_k, \ldots, z_n$ is said to be a *regular system of coordinates for the ideal* $\mathscr{P}$, if the following conditions are satisfied:

(i) $\mathscr{P} \cap {}_k\mathcal{O}_0 = \{0\}$, where ${}_k\mathcal{O}_0$ is the collection of all germs of functions depending only on variables $z_1, \ldots, z_k$,

(ii) the factorring ${}_n\mathcal{O}_0/\mathscr{P}$ is integral over ${}_k\mathcal{O}_0$,

(iii) if $[z_{k+1}] \in {}_n\mathcal{O}_0/\mathcal{P}$ is the element corresponding to the element $[z_{k+1}]_0 \in {}_n\mathcal{O}_0$ and if $\mathcal{F}_n$, $\mathcal{F}_k$ are quotient fields of ${}_n\mathcal{O}_0/\mathcal{P}$ and ${}_k\mathcal{O}_0$, respectively, then $\mathcal{F}_n$ is generated over $\mathcal{F}_k$ by the single element $[z_{k+1}]$.

The integer k is called the dimension of $\mathcal{P}$. It can be seen that it is independent of the coordinate choice. The regular system of coordinates for $\mathcal{P}$ can be, for example, found as follows:

Theorem B 1. *Suppose that $[V]_0$ is an irreducible germ of an analytic set and that* dim id $[V]_0 = k$. *Suppose further that*

$$[V]_0 \cap [\{z \mid z_1 = \ldots = z_k = 0\}]_0 = [\{0\}]_0 .$$

Then there exists a linear transformation

$$z'_j = \sum_{i>k} a_{ij} z_i , \quad j > k$$

such that $z_1, \ldots, z_k, z'_{k+1}, \ldots, z'_n$ is the regular system of coordinates for id $[V]_0$.

(For the proof see [G – R], Theorems III.C.5 and III.C.7.)

Let X, $\tilde{X}$ be topological spaces, $p : \tilde{X} \to X$ continuous mapping. The mapping p is said to be a *simple covering map*, if $p^{-1}(X)$ is a disjoint union of open subsets of $\tilde{X}$ such that the restriction of p to each of them is a homeomorphism onto X.

A map $p : \tilde{X} \to X$ is said to be a *covering map*, if every point $x \in X$ has a neighborhood $\mathcal{U}$ such that p restricted on $p^{-1}(\mathcal{U})$ is a simple covering map.

A map $p : \tilde{X} \to X$ is said to be a *proper mapping*, if for every compact set $K \subset X$ the set $p^{-1}(K)$ is again a compact set.

The following theorem is of basic importance for the investigation of local geometry of analytic sets.

Theorem B 2. (Local parametrization of an analytic set.)

Let $[V]_0 \in {}_nG_0$ be an irreducible germ. Suppose that $z_1, \ldots, z_k, \ldots, z_n$ form a regular coordinate set for id $[V]_0$.

Then there exist (arbitrarily small) $r = [r_1, \ldots, r_n]$ and a function $d = d(z_1, \ldots \ldots, z_k)$ holomorphic in $\Delta(0, r)$ such that (if we denote $D = \{z \in \Delta(0, r) \mid d(z) = 0\}$)

(i) *the mapping*

$$\pi : V \cap \Delta(0, r) \to \Delta_k(0, r)$$

is a proper mapping,

(ii) *the mapping*

$$\pi : (V \cap \Delta) \setminus D \to \Delta_k \setminus D$$

is a covering map,

(iii) *each point* $a \in \Delta_k - D$ *has a neighborhood* $\mathcal{U}_a \subset \Delta_k \setminus D$ *such that*

$$\pi : V \cap \pi^{-1}(\mathcal{U}_a) \to \mathcal{U}_a$$

is a simple covering map and each of the disjoint sheets of the set $V \cap \pi^{-1}(\mathcal{U}_a)$ *can be described by a holomorphic mapping*

$$G : \mathcal{U}_a \to \pi^{-1}(\mathcal{U}_a)\,,$$
$$G : [z_1, \ldots, z_k] = z' \mapsto [z_1, \ldots, z_k, g_{k+1}(z'), \ldots, g_n(z')]\,.$$

(Proof of this theorem follows from the assertion of Lemma III.A.9 and Theorem III.A.10 of [G – R], since the ideal id $[V]_0$ is prime.)

Suppose that $[V]_0$ is an irreducible germ of an analytic set and that $z_1, \ldots, z_k, \ldots, z_n$ form a regular coordinate set for id $[V]_0$. Then the number k is independent of the choice of regular coordinate sets (see [G – R], remark preceding Theorem III.C.1). This number is called the *dimension of* $[V]_0$ and it will be denoted by

$$\dim [V]_0 = k\,.$$

By translation into origin we can define similarly

$$\dim [V]_z$$

if the germ $[V]_z$ is irreducible.

Further, let us consider a germ $[V]_z \in {}_nG_z$ and suppose that

$$[V]_z = [V_1]_z \cap \ldots \cap [V_s]_z$$

is the decomposition of $[V]_z$ into its irreducible branches. Then we denote

$$\dim [V]_z = \max_{1 \leq i \leq s} \dim [V_i]_z\,.$$

The following theorem on the dimension of germs of varieties will be useful in the sequel:

Theorem B 3. *Let* $[V]_z \in {}_nG_z$ *be an irreducible germ and* $[f]_z \in {}_n\mathcal{O}_z$. *Suppose that* $f(z) = 0$ *and* $[f]_z \notin \mathrm{id}\,[V]_z$.

Then

$$\dim [V \cap \{z \mid f(z) = 0\}]_z = \dim [V]_z - 1\,.$$

(For the proof see [G – R], Theorem III.C.14.)

Let $V \subset \Omega \subset \mathbf{C}^n$ be an analytic set. We denote

$$\dim V = \sup_{z \in V} \dim [V]_z\,.$$

A point $z \in V$ is said to be a *regular point of* V, if there exists a neighborhood $\mathcal{U}_z$ such that $V \cap \mathcal{U}_z$ is a complex submanifold of $\mathbf{C}^n$.

A point $z \in V$ is said to be a *singular point of* V, if it is not regular.

The set of all regular points of V will be denoted by $\mathcal{R}(V)$. It can be proved that $\mathcal{R}(V)$ is a dense subset of V (see [H], Theorem 5, p. 107).

The set $\mathcal{S}(V)$ of all singular points of V is again an analytic set (in Ω) and

$$\dim \mathcal{S}(V) < \dim V$$

(see [H], Theorem 6, p. 117).

We say that analytic set $V \subset \Omega \subset \mathbf{C}^n$ is *irreducible*, if it cannot be written as the union of two analytic sets, both different from it.

It can be shown that a germ $[V]_z$ is irreducible if there exists a fundamental system of neighborhoods $\{\mathcal{U}_i\}_{i \in I}$ of the point z such that $V \cap \mathcal{U}_i$ is irreducible (in $\mathcal{U}_i$) for all $i \in I$ (see [H], Corollary 2., p. 124).

For the reader's convenience, we introduce a theorem, which will be used in the following.

Theorem B 4. *Let $V, W \subset \Omega \subset \mathbf{C}^n$ be analytic sets. Suppose that $S \subset \mathcal{R}(V)$ is an open connected subset of $\mathcal{R}(V)$ such that S is not contained in W.*

Then $S \setminus W$ is connected.

Proof. See [H], Assertion 4, p. 64.

Theorem B 5. *Let $V \subset \Omega \subset \mathbf{C}^n$ be an irreducible analytic set. Then*

(i) *V is connected,*

(ii) *$\dim [V]_z$ is constant for all $z \in V$,*

(iii) *let $W \subset \Omega \subset \mathbf{C}^n$ be an analytic set and let for a point $z \in V$ the germ $[W]_z$ contain one of the irreducible components of the germ $[V]_z$, then*

$$W \supset V.$$

Particularly, if f is holomorphic in Ω and for a point $z \in V$ the germ $[f]_z$ vanishes on $[V]_z$, then

$$f \equiv 0 \quad on \quad V.$$

(iv) *for each analytic set $W \subset \Omega$ the set $V \setminus W$ is an irreducible analytic set in $\Omega \setminus W$ and the set $V \setminus W$ is either empty or dense in V.*

(v) *if $W \subset \Omega$ is an analytic set and if $W \subset V$, $W \neq V$, then*

$$\dim [W]_z < \dim [V]_z$$

for all $z \in V$.

Proof. See [H]. Corollary 3, p. 124.

Theorem B 6. (The decomposition into irreducible branches.)

Let $V \subset \Omega \subset \boldsymbol{C}^n$ be an analytic set. Let us denote by $\{W_n\}$ the (countable) collection of connected components of the set $\mathscr{R}(V)$ of all regular points and by $\overline{W}_n$ the closure of W_n in Ω.

Then $\{\overline{W}_n\}$ is a locally finite collection of irreducible varieties and

$$V = \bigcup_n \overline{W}_n\,, \quad \overline{W}_n \not\subset \bigcup_{n' \neq n} \overline{W}_{n'}\,.$$

The sets $\overline{W}_n$ will be called irreducible branches of V.

Proof. See [H], Theorem 8, p. 126.

Suppose that $V \subset \Omega \subset \boldsymbol{C}^n$ is an analytic set. If we denote $M_1 = \mathscr{R}(V)$, we can write

$$V = M_1 \cup \mathscr{S}(V)$$

and since $\mathscr{S}(V)$ is again an analytic set (in Ω), the dimension of which is less then the dimension of V, we can repeat the same procedure with $\mathscr{S}(V)$ and by induction we obtain a (disjoint) partition of V into submanifolds:

$$(1,2) \qquad V = M_1 \cup \ldots \cup M_s\,, \qquad \dim M_1 > \ldots > \dim M_s\,,$$

where M_i are submanifolds of $\boldsymbol{C}^n$.

We can divide every submanifold M_i into its connected components $M_{i_1}, M_{i_2}, \ldots$ and in this way we obtain another (disjoint) partition of V into connected submanifolds

$$(1,3) \qquad V = \bigcup_{i=1}^{s} [M_{i_1} \cup M_{i_2} \cup \ldots]\,.$$

We say that a (disjoint) partition

$$V = \bigcup_{i \in I} M_i$$

is *a strict partition*, if $\{M_i\}$ is a locally finite collection and if for all $i \in I$ the sets (closure are considered in Ω)

$$\overline{M}_i,\ \overline{M}_i \setminus M_i$$

are anlytic sets in Ω.

It can be easily shown (see [W], p. 536) that the partitions (1,2) and (1,3) are strict.

Moreover, if V is a symmetric set, i.e. $\mathscr{Y}(V) = V$, then it is easy to see that $\mathscr{R}(V)$ (and hence also $\mathscr{S}(V)$) are symmetric sets. From this it follows immediately that the partition (1,2) is a partition of V into symmetric submanifolds.

Suppose $V \subset \Omega \subset \mathbf{R}^n$.

The set V is said to be a (real) analytic set in Ω, if for each point $x_0 \in \Omega$ there exists a neighborhood $\mathcal{U}_{x_0} \subset \Omega$ and functions $f_1, \ldots, f_t$, which are $\mathbf{R}$-analytic in $\mathcal{U}_{x_0}$ and such that

$$U_{x_0} \cap V = \{x \in U_{x_0} \,|\, f_1(x) = \ldots = f_t(x) = 0\} .$$

The set V is said to be a *$\mathbf{C}$-analytic set* in, Ω if there exist functions $f_1, \ldots, f_t$ which are $\mathbf{R}$-analytic in the whole Ω and such that

$$V = \{x \in \Omega \,|\, f_1(x) = \ldots = f_t(x) = 0\} .$$

Hence every $\mathbf{C}$-analytic set in Ω is a (real) analytic set in Ω, the converse not being true.

For any (real) analytic set $V \subset \Omega \subset \mathbf{R}^n$, a point x is said to be a *regular point of V*, if there exists a neighborhood $\mathcal{U}_x$ of x such that

$$\mathcal{U}_x \cap V$$

is a (real) submanifold of $\mathbf{R}^n$. The set of all regular points of V will be denoted by $\mathcal{R}(V)$.

It can be shown (see [W–B], Prop. 2, p. 141) that every analytic set $V \subset \Omega \subset \mathbf{R}^n$ is locally arcwise connected (i.e. every point $x \in V$ has a fundamental system of neighborhoods $\{\mathcal{U}_i\}_{i \in I}$ of x such that for all $i \in I$, any points $y_1, y_2 \in V \cap \mathcal{U}_i$ can be joined by a continuous curve in $V \cap \mathcal{U}_i$).

A great advantage of $\mathbf{C}$-analytic sets is the possibility to define global decomposition (i.e. decomposition in Ω) into irreducible branches, while for (real) analytic sets such decomposition can be done only locally (i.e. on compacts). There is also the following characterization of $\mathbf{C}$-analytic sets:

Theorem C 1. *A set $V, V \subset \Omega \subset \mathbf{R}^n$ is a $\mathbf{C}$-analytic set iff there exists a neighborhood Ω^* of the set Ω in $\mathbf{C}^n$ and a complex analytic set V^* in Ω^* such that*

$$\Omega = \Omega^* \cap \mathbf{R}^n \quad \text{and} \quad V = V^* \cap \mathbf{R}^n .$$

Moreover, if V is $\mathbf{C}$-analytic set in Ω, then the set V^* can be chosen in such a way that V^* is symmetric.

Proof. See [W–B], p. 153.

A $\mathbf{C}$-analytic set V in Ω is said to be *$\mathbf{C}$-irreducible*, if V cannot be written as a union of two $\mathbf{C}$-analytic sets in Ω, which are different from V.

Again, some theorems on $\mathbf{C}$-analytic sets, useful in the following, will be recalled.

Theorem C 2. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a $\mathbf{C}$-irreducible $\mathbf{C}$-analytic set and let Ω^* be a neighborhood of Ω in $\mathbf{C}^n$ such that $\Omega = \Omega^* \cap \mathbf{R}^n$ and that there exists a complex analytic set $V^* \subset \Omega^*$ for which $V = V^* \cap \mathbf{R}^n$ holds.*

Let us define

$$W^* = \bigcup V^*$$

where the intersection is taken over all (complex) analytic sets V^* in Ω^*, which contain V.

Then W^* is an irreducible analytic set in Ω^*.

Proof. The set W^* is an analytic set, being the intersection of analytic sets. If

$$W^* = W_1^* \cup W_2^*$$

where W_1^*, W_2^* are analytic sets in Ω^*, different from W^*, then denoting

$$W_1 = W_2^* \cap \mathbf{R}^n, \quad W_2 = W_2^* \cap \mathbf{R}^n$$

we have $V = W_1 \cap W_2$. And, since W^* is the least analytic set in Ω^* containing V, it is true that W_1 and W_2 are different from V which is a contradiction.

Theorem C 3. *For every $\mathbf{C}$-analytic set V in Ω there exists a locally finite collection of $\mathbf{C}$-irreducible $\mathbf{C}$-analytic sets $\{V_i\}_{i \in I}$ such that*

$$V = \bigcup_{i \in I} V_i \quad \text{and} \quad V_i \not\subset V_j \quad \text{for} \quad i \neq j\,.$$

The sets V_i are called $\mathbf{C}$-irreducible branches of V.

Proof. See [W – B], p. 155, Proposition 11.

Theorem C 4. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a $\mathbf{C}$-irreducible $\mathbf{C}$-analytic set. Then V is connected.*

Proof. It follows from [W – B], Corol. 2, p. 148 and Remark on p. 146 that there exists a continuous mapping, which maps a connected set onto V, hence V is also connected.

Chapter II.

MORSE – SARD THEOREM

Let f be a real function defined on $\mathbf{R}^n$. Let us denote

$$N = \{x \in \mathbf{R}^n \mid \nabla f(x) = 0\}$$

the set of all critical points of f. A well known problem in analysis is the problem how large the set $f(N)$ can be. If the function f is n-times continuously differentiable, then the (Lebesgue) measure of $f(N)$ is zero (see [S]). When increasing the smoothness of f, the set $f(N)$ becomes smaller (which can be measured by Hausdorff measures, see [K]).

In the case of an **R**-analytic function f, the set $f(N)$ is at most countable, moreover, the function f is locally constant on N (hence $f(N \cap K)$ is a finite set for all compact sets K, see [S–S]).

The generalization of the Morse-Sard theorem for the set N of all critical points (in some sense) of f with respect to an analytic set V is very useful for the investigation of analytic functions on analytic sets. For algebraic case similar theorem was proved in [M], p. 23. In this chapter we prove such a theorem for analytic case. First we prove this theorem for the complex case. And then use the result to prove the real case.

Theorem 2.1. *Let $V \subset \Omega \subset \mathbf{C}^n$ be an analytic set and let f be holomorphic in Ω. Let us denote*

$$N = \{z \in \mathscr{R}(V) \mid z \text{ is a critical point of } f \text{ on } \mathscr{R}(V)\}.$$

Then $\overline{N}$ (closure in Ω) is an analytic set in Ω and f is locally constant on $\overline{N}$.

First we prove this theorem in a special case:

Lemma 2.2. *Let $M \subset \Omega \subset \mathbf{C}^n$ be a connected submanifold of $\mathbf{C}^n$ and let $V = \overline{M}$ (closure in Ω) and $V \setminus M$ be analytic sets in Ω. Suppose that f is holomorphic in Ω and let us denote*

$$N = \{z \in M \mid z \text{ is a critical point of } f \text{ on } M\}.$$

Then the set $\overline{N}$ is an analytic set in Ω and the function f is locally constant on $\overline{N}$. Moreover, it holds that $\overline{N} \cap M = N$.

Proof. We must prove that for every point $z \in \Omega$ there exists a neighborhood $\mathscr{U}_z \subset \Omega$ such that $\overline{N} \cap \mathscr{U}_z$ is an analytic set in $\mathscr{U}_z$ and that f is constant on $\overline{N} \cap \mathscr{U}_z$. If $z \notin \overline{N}$, then this is trivial.

Suppose that $z_0 \in \overline{N}$ is fixed.

The theorem on the coherence of the sheaf of ideals of an analytic set (see [G–R], Corollary IV.D.3) implies that there exist functions $g_1, \ldots, g_s$ is a neighborhood $\mathscr{U}_{z_0}$ such that the germs $[g_1]_y, \ldots, [g_s]_y$ generate id $[V]_y$ for each point $y \in \mathscr{U}_{z_0}$. Clearly it holds

$$M \subset \mathscr{R}(V) \subset V = \overline{M},$$

hence $\mathscr{R}(V)$ is also connected and by Theorem B6 V is irreducible in Ω.

It follows from Theorem B5 (ii) that $\dim [V]_y$ is constant for $y \in V$ provided $\dim V \equiv j$. Further, the sets M and $\mathscr{R}(V)$ are open dense subsets of V, hence M and $\mathscr{R}(V)$ are equal in a small neighborhood of any point from M.

First part of proof. Let $z_1 \in \mathscr{U}_{z_0}$, $z_1 \in M$. There exists a neighborhood $\mathscr{U}_{z_1} \subset \subset \mathscr{U}_{z_0}$ of the point z_1 and a coordinate set $w = \{w_i\}_1^n$ at z_1 such that

$$\mathscr{U}_{z_1} \cap M = \mathscr{U}_{z_1} \cap \mathscr{R}(V) = \{z \in \mathscr{U}_{z_1} \mid w_{j+1}(z) = \ldots = w_n(z) = 0\} .$$

Let $\mathscr{V} = w(\mathscr{U}_{z_1})$. For the sake of brevity we shall denote the functions $g_i(z(w))$, $f(z(w))$ shortly by $g_i = g_i(w)$, $f = f(w)$. Since the matrix

$$\frac{\partial(g_1, \ldots, g_s)}{\partial(w_1, \ldots, w_n)} (0)$$

is the product of the matrices

$$\frac{\partial(g_1, \ldots, g_s)}{\partial(z_1, \ldots, z_n)} (z_1)$$

and

$$\frac{\partial(z_1, \ldots, z_n)}{\partial(w_1, \ldots, w_n)} (0)$$

and since $J_w\, z(0) \neq 0$, it holds

$$\operatorname{rank} \frac{\partial(g_1, \ldots, g_s)}{\partial(z_1, \ldots, z_n)} (z_1) = \operatorname{rank} \frac{\partial(g_1, \ldots, g_s)}{\partial(w_1, \ldots, w_n)} (0)$$

and by the same argument we obtain

$$\operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(z_1, \ldots, z_n)} (z_1) = \operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(w_1, \ldots, w_n)} (0) \tag{2,1}$$

The functions $g_i(z)$ vanish on $V \cap \mathscr{U}_{z_0}$, hence in $\mathscr{V}$

$$g_i(w_1, \ldots, w_j, 0, \ldots, 0) = 0\,, \quad i = 1, \ldots, s$$

and also

$$\frac{\partial g_i}{\partial w_k} (0) = 0\,; \quad i = 1, \ldots, s\,; \quad k = 1, \ldots, j\,.$$

The matrix

$$\frac{\partial(g_1, \ldots, g_s, f)}{\partial(w_1, \ldots, w_n)} (0)$$

has the form:

$$\begin{matrix} 0, & \ldots, & 0, & \dfrac{\partial g_1}{\partial w_{j+1}}(0), & \ldots, & \dfrac{\partial g_1}{\partial w_n}(0) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & \ldots, & 0, & \dfrac{\partial g_s}{\partial w_{j+1}}(0), & \ldots, & \dfrac{\partial gs}{\partial w_n}(0) \\ \dfrac{\partial f}{\partial w_1}(0), & \ldots, & \dfrac{\partial f}{\partial w_j}(0), & \dfrac{\partial f}{\partial w_{j+1}}(0), & \ldots, & \dfrac{\partial f}{\partial w_n}(0). \end{matrix}$$

We have

$$\text{(2,2)} \qquad \operatorname{rank} \frac{\partial(g_1, \ldots, g_s)}{\partial(w_1, \ldots, w_n)}(0) = n - j$$

since it follows immediately from the form of this matrix that its rank is less or equal to $n - j$ while the converse inequality follows from the fact that $[w_{j+1}]_{z_1}, \ldots$ $\ldots, [w_n]_{z_1}$ vanishes on $[V]_{z_1}$; hence we can write in a neighborhood of the point z_1

$$w_e(z) = \sum_{k=1}^{s} h_k(z)\, g_k(z) \quad \text{i.e.} \quad w_e = \sum_{k=1}^{s} h_k(z(w))\, g_k(w)\,; \quad e = j + 1, \ldots, n\,,$$

where $h_k(z)$ are holomorphic functions.

Consequently

$$\frac{\partial w_e}{\partial w_m} = \sum_{k=1}^{s} \frac{\partial h_k}{\partial w_m} g_k + \sum_{k=1}^{s} \frac{\partial g_k}{\partial w_m} h_k\,, \quad e = j + 1, \ldots, n\,, \quad m = 1, \ldots, n\,,$$

but $g_k(0) = 0$, $k = 1, \ldots, s$ and therefore

$$n - j = \operatorname{rank} \frac{\partial(w_{j+1}, \ldots, w_n)}{\partial(w_1, \ldots, w_n)}(0) \leqq \operatorname{rank} \frac{\partial(g_1, \ldots, g_s)}{\partial(w_1, \ldots, w_n)}(0)\,.$$

Now we have

$$\begin{aligned} z_1 \in N &\Leftrightarrow \frac{\partial f}{\partial w_1}(0) = \ldots = \frac{\partial f}{\partial w_j}(0) = 0 \\ &\Leftrightarrow \operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(w_1, \ldots, w_n)}(0) = n - j \quad (\text{by (2,2)}) \\ &\Leftrightarrow \operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(w_1, \ldots, w_n)}(0) \leqq n - j \quad (\text{by (2,1)}) \\ &\Leftrightarrow \operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(z_1, \ldots, z_n)}(z_1) \leqq n - j\,. \end{aligned}$$

Hence in the whole neighborhood $\mathscr{U}_{z_0}$

$$N \cap \mathscr{U}_{z_0} = \left\{ z \in M \mid \operatorname{rank} \frac{\partial(g_1, \ldots, g_s, f)}{\partial(z_1, \ldots, z_n)}(z) \leqq n - j \right\}.$$

Let us define the analytic set W in $\mathscr{U}_{z_0}$ by

$$W = \{ z \in \mathscr{U}_{z_0} \mid z \in V,\ D_\alpha(z) = 0 \},$$

where D_α are all $k \times k$ subdeterminants of the matrix

$$\frac{\partial(g_1, \ldots, g_s, f)}{\partial(z_1, \ldots, z_n)}(z)$$

where $k \geqq n - j$. From the above reasoning it follows that in $\mathscr{U}_{z_0}$.

$$W \cap M = N .$$

Second part of proof. Let us denote $H = V \setminus M$; then H is an analytic set in Ω. By Theorem B6 the set W can be decomposed in $\mathscr{U}_{z_0}$ into its irreducible branches

$$W = \bigcap_{i=1}^{\infty} W_i ,$$

where $\{W_i\}_1^\infty$ is a locally finite collection of irreducible analytic sets in $\mathscr{U}_{z_0}$.

We can suppose that

$$z_0 \in W_i \quad \text{for} \quad i = 1, \ldots, l ,$$

$$z_0 \notin W_i \quad \text{for} \quad i \geqq l + 1$$

and that (for some t, $0 \leqq t \leqq l$)

$$[W_i]_{z_0} \not\subset [H]_{z_0} \quad \text{for} \quad i,\ 1 \leqq i \leqq t , \tag{2,4}$$

$$[W_i]_{z_0} \subset [H]_{z_0} \quad \text{for} \quad i,\ t + 1 \leqq i \leqq l$$

and denote $W' = \bigcup_{1 \leqq i \leqq t} W_i$ (i.e. the set W' is empty if $t = 0$).

For every component W_i, $i = 1, \ldots, t$ it holds in $\mathscr{U}_{z_0}$:

$$\mathscr{R}(W_i) \not\subset H ;$$

hence by Theorem B4 (where we put $V = W_i$, $S = \mathscr{R}(W_i)$, $W = H$ and $\Omega = \mathscr{U}_{z_0}$) the set $\mathscr{R}(W_i) \setminus H$ is connected. In $\mathscr{U}_{z_0}$ we have further (by (2,3))

$$\mathscr{R}(W_i) \setminus H \subset N \subset M ; \quad i = 1, \ldots, t . \tag{2,5}$$

Let $z_0 \in \mathscr{R}(W_i) \setminus H$ be fixed. We can choose a coordinate set $\{q_i\}$ in a neighborhood $\mathscr{U}$ of z such that it holds in $\mathscr{U}$:

$$\mathscr{R}(W_i) \setminus H = \{y \in \mathscr{U} \mid q_{k+1}(y) = \ldots = q_n(y) = 0\} ,$$
$$M = \{y \in \mathscr{U} \mid q_{j+1}(y) = \ldots = q_n(y) = 0\} ,$$

where $k \leqq j$.

The set N is the set of critical points of f on M, hence

$$\frac{\partial f(z(q))}{\partial q_i} (q_1, \ldots, q_k, 0, \ldots, 0) = 0 ; \quad i = 1, \ldots, j$$

for $[q_1, \ldots, q_k, 0, \ldots, 0]$ from a sufficiently small polydisc Δ_k. Hence the function $f(z(q))$ is constant on Δ_k. Since the point z can be arbitrary, the function f is locally constant on the connected set $\mathscr{R}(W_i) \setminus H$ and hence f is constant on $\mathscr{R}(W_i) \setminus H$.

Further, if we put $\Omega = \mathscr{U}_{z_0} \setminus \mathscr{S}(W_i)$, $V = \mathscr{R}(W_i)$, $W = H \cap \mathscr{U}_{z_0}$ in Theorem B5 (iv), we conclude that the set $\mathscr{R}(W_i) \setminus H$ is dense in $\mathscr{R}(W_i)$ (for i, $1 \leqq i \leqq t$). However the set $\mathscr{R}(W_i)$ of all regular points of W_i is dense in W_i, therefore (closure is taken in $\mathscr{U}_{z_0}$)

$$\overline{\mathscr{R}(W_i) \setminus H} = W_i$$

and f is constant on W_i.

By (2,4) and by Theorem B5 (iii) we have $W_i \subset H$ for i, $t < i \leqq l$. The collection $\{W_i\}$ is locally finite, hence we can choose a neighborhood $\mathscr{U}'_{z_0} \subset \mathscr{U}_{z_0}$ such that

$$W = \bigcap_{i=1}^{l} W_i$$

in $\mathscr{U}'_{z_0}$. This implies by (2,3) that

(2,6) $$N = M \cap W = M \cap W' \quad \text{in} \quad \mathscr{U}'_{z_0}$$

where $W' = \bigcap_{i=1}^{t} W_i$.

From (2,6) it follows that (closures in $\mathscr{U}'_{z_0}$)

$$\overline{N} = \overline{M \cap W'} \subset V \cap W' = W' \quad (\text{in } \mathscr{U}'_{z_0}) .$$

However, on the other hand, (2,5) implies (closures in $\mathscr{U}'_{z_0}$)

$$\overline{\mathscr{R}(W_i) \setminus H} = W_i \subset \overline{N} \quad (\text{in } \mathscr{U}'_{z_0}), \quad i = 1, \ldots, t ,$$

hence also

$$W' = \bigcap_{l=1} W_i \subset \overline{N} \quad (\text{in } \mathscr{U}'_{z_0}) .$$

So we have

$$W' = \overline{N} \quad (\text{in } \mathscr{U}'_{z_0})$$

and the function f is constant on W', since f is constant on all W_i and $z_0 \in W_i$ for $i = 1, \ldots, t$.

Proof of Theorem 2.1. If V is irreducible, then $\mathscr{R}(V)$ is a connected submanifold $\mathbf{C}^n$ and we can use Lemma 2.2 for $M = \mathscr{R}(V)$.

In the opposite case, by Theorem B6 we can write the decomposition of V into irreducible branches in Ω:

$$V = \bigcup_{i \in I} V_i\,, \quad V_i = \overline{K}_i\,,$$

where K_i are connected components of $\mathscr{R}(V)$ and $\overline{K}_i$ are their closures in Ω.

Let us denote by N and N_i $(i \in I)$ the set of all critical points of f on $\mathscr{R}(V)$ and on K_i, respectively.

Clearly it holds

$$N = \bigcup_{i \in I} N_i\,. \tag{2,7}$$

The collection $\{\overline{N}_i\}_{i \in I}$ is locally finite, since $\overline{N}_i \subset V_i$, $i \in I$ (closure in Ω). Hence it follows immediately from (2,7) that also (closure in Ω)

$$\overline{N} = \bigcup_{i \in I} \overline{N}_i\,. \tag{2,8}$$

Further we shall show that

$$V_i \setminus K_i = V_i \cap \mathscr{S}(V)\,.$$

The inclusion $V_i \cap \mathscr{S}(V) \subset V_i \setminus K_i$ follows from the definition of K_i. On the other hand, if $z \in V_i \setminus K_i$ then $z \in \mathscr{S}(V)$, for if $z \in \mathscr{R}(V)$ then there exists $i_0 \in I$ such that $z \in K_{i_0}$, but $z \in V_i = \overline{K}_i$, $z \notin K_i$ which contradicts the fact that K_i, K_{i_0} are two different connected components of $\mathscr{R}(V)$.

Hence V_i and $V_i \setminus K_i$ are analytic sets in Ω and Lemma 2.2 implies that $\overline{N}_i$, $i \in I$ are analytic sets in Ω and that f is locally constant on $\overline{N}_i$.

The collection $\{\overline{N}_i\}_{i \in I}$ is locally finite, hence $\overline{N}$ is also an analytic set in Ω and f is locally constant on $\overline{N}$.

Theorem 2.3. *Let $V \subset \Omega \subset \mathbf{C}^n$ be an analytic set in Ω and let f be holomorphic in Ω.*

Suppose that $V = \bigcup_{i \in I} M_i$ is a strict partition of V into submanifolds of $\mathbf{C}^n$.

Let us denote

$$N_i = \{x \in M_i \mid x \text{ is critical point of } f \text{ on } M_i\} .$$

Then f is locally constant on the analytic set $\bigcup_{i \in I} \overline{N}_i$.

Proof. A) Suppose first that all submanifolds M_i are connected. From the definition of strict partition it follows that (closures in Ω) $\overline{M}_i$ and $\overline{M}_i \setminus M_i$ are analytic sets in Ω. From Lemma 2.2 it follows that $\overline{N}_i$, $i \in I$ are analytic sets in Ω and that f is locally constant on $\overline{N}_i$. The collection $\{\overline{N}_i\}$ is locally finite, hence $\bigcup_{i \in I} \overline{N}_i$ is an analytic set in Ω and f is constant on this set.

B) If some of M_i are not connected, then we can write these M_i as countable union of connected submanifolds

$$M_i = \bigcup_{k \in I_i} M_{ik} .$$

We shall show that the partition

$$V = \left[\bigcup_{k \in I_1} M_{1k}\right] \cap \left[\bigcup_{k \in I_2} M_{2k}\right] \cup \ldots$$

is a strict partition of V into connected submanifolds.

The sets $\overline{M}_i$ and $S_i = \overline{M}_i \setminus M_i$ (closures in Ω) are by the assumptions analytic sets in Ω. The set $\mathscr{R}(\overline{M}_i)$ can be also written as union of its connected components

$$\mathscr{R}(\overline{M}_i) = \bigcup_{l \in J_i} K_l .$$

By Theorem B6 the sets $\overline{K}_l$ (closure in Ω) are analytic sets in Ω. Clearly $M_i \subset \mathscr{R}(\overline{M}_i)$, hence for each $k \in I_i$ there exists only one $l_k \in J_i$ such that

$$M_{ik} \subset K_{l_k} ;$$

moreover we have

$$M_{ik} \subset K_{l_k} \setminus S_i \subset M_i .$$

However, by Theorem B4 (where we put $S = K_{l_k}$, $V = \overline{M}_i$, $W = S_i$) the set $K_{l_k} \setminus S_i$ is connected. The set M_{ik} is a connected component of M_i, hence

$$M_{ik} = K_{l_k} \setminus S_i .$$

Further, Theorem B5 (iv) (where we put $\Omega = \Omega \setminus \mathscr{S}(\overline{M}_i)$, $V = K_{l_k}$, $W = S_i \setminus \mathscr{S}(\overline{M}_i)$) implies that $K_{l_k} \setminus S_i$ is dense in K_{l_k}; hence (closure in Ω)

$$\overline{M}_{ik} = \overline{K}_{l_k}$$

and therefore $\overline{M}_{ik}$ and $\overline{M}_{ik} \setminus M_{ik}$ are analytic sets in Ω.

By Theorem B6 the collection $\{K_l\}$ is locally finite, so that also $\{M_{ik}\}$ is a locally finite collection.

Let us denote now

$$N_{ik} = \{z \in M_{ik} \mid z \text{ is critical point of } f \text{ on } M_{ik}\} .$$

Clearly

$$N_i = \bigcup_{k \in I_i} N_{ik} \quad \text{and} \quad \overline{N}_i = \bigcup_{k \in I_i} \overline{N}_{ik} .$$

Hence from the previous part of the proof we conclude that the function f is locally constant on the analytic set

$$\bigcup_{i \in I} \bigcup_{k \in I_i} \overline{N}_{ik} = \bigcup_{i \in I} \overline{N}_i .$$

Theorem 2.4. *Let $V \subset \Omega \subset \mathbf{C}^n$ be an analytic set and let f be holomorphic in Ω. Let us denote*

$$N = \{z \in V \mid z \textit{ is a metric critical point of } f \textit{ on } V\} .$$

Then f is locally constant on $\overline{N}$.

Proof. Let

$$V = \bigcup_{i \in I} M_i$$

be an arbitrary strict partition of V into submanifolds (we can take for example the partition (1,2) from Chapter I). Further let us denote

$$N_i = \{z \in M_i \mid z \text{ is a critical point of } f \text{ on } M_i\} .$$

If $z \in N$, then there exists $i_0 \in I$ such that $z \in M_{i_0}$, but z is also a metric critical point of f on M_{i_0}, hence $z \in N_i$.

We have

$$\overline{N} \subset \bigcup_{i \in I} \overline{N}_i$$

and it is sufficient to use Theorem 2.3.

Remark. From the above theorems it is clear that the set N of all critical points of f on $\mathcal{R}(V)$ is contained in the set of all metric critical points and this again is contained in the set N, defined by an arbitrary partition of V into submanifolds. Further, the finer is the partition of V the greater is the set N. Hence the best information can be obtained by means of Theorem 2.3. However, it has no sense to consider very fine partitions of V, for we can take, for example, an arbitrary isolated set in V and to consider its point as new submanifolds M_i. This points will be then in N, but we obtain no new information about the function f.

Theorem 2.5. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a (real) analytic set. Let f be an* **R**-*analytic function in Ω. Let us denote*

$$N = \{x \in V \mid x \text{ is metric critical point of } f \text{ on } V\} .$$

Then f is locally constant on $\bar{N}$ (closure in Ω).

Proof. Let $z_0 \in V$ be fixed. It is sufficient to show that there exists a neighborhood $\mathcal{U}_{z_0}$ of z_0 such that f is locally constant on $\overline{\mathcal{U}_{z_0} \cap N}$ (closure in $\mathcal{U}_{z_0}$).

By definition there exists a neighborhood $\mathcal{U}_{z_0}$ of z_0 such that $\mathcal{U}_{z_0} \cap V$ is a **C**-analytic set in $\mathcal{U}_{z_0}$.

Let us denote

$$\mathcal{U}_{z_0} \cap V = V' , \quad \mathcal{U}_{z_0} \cap N = N' .$$

There exists a neighborhood $\Omega^* \subset \mathbf{C}^n$ of the set $\mathcal{U}_{z_0}$ and a function f^* holomorphic in Ω^* such that $\Omega^* \cap \mathbf{R}^n = \mathcal{U}_{z_0}$ and that $f^*|_{\mathcal{U}_{Z_0}} = f$ (see [N], p. 13, Lemma 1.1.5).

Further, if we assume the neighborhood Ω^* sufficiently small, then there exists a (complex) analytic set

$$V^* \subset \Omega^* , \quad V^* \cap \mathbf{R}^n = V' .$$

We can suppose also that the sets V^*, Ω^* are symmetric. There exists a strict partition $V^* = \bigcup_{i \in I} M_i^*$ into symmetric submanifolds.

Let us denote

$$M_i = M_i^* \cap \mathcal{U}_{z_0} ,$$

$$N_i^* = \{z \in M_i^* \mid z \text{ is critical point of } f^* \text{ on } M_i^*\} ,$$

$$N_i = \{x \in M_i \mid x \text{ is critical point of } f \text{ on } M_i\} .$$

By Theorem A1 the sets M_i are (real) submanifolds of $\mathbf{R}^n$ and

$$N_i = N_i^* \cap \mathcal{U}_{z_0} .$$

By Theorem 2.3 the function f^* is locally constant on the analytic set $\bigcup_{i \in I} \bar{N}_i^*$ (closure in Ω^*), by (2,9) it holds (closures in $\mathcal{U}_{z_0}$)

$$\bigcup_{i \in I} \bar{N}_i \subset [\bigcup_{i \in I} \bar{N}_i^*] \cap \mathcal{U}_{z_0} ,$$

hence f is locally constant on $\bigcup_{i \in I} \bar{N}_i$.

Since $V = \bigcup_{i \in I} M_i$ is a partition of V for each point $x \in V$, there exists a submanifold M_{i_0} such that $x \in M_{i_0}$. Moreover, if $x \in N'$, then also $x \in N_{i_0}$. Hence $N \subset \bigcup_{i \in I} N_i$.

Since the collection $\{M_i^*\}_{i\in I}$ is locally finite, the collections $\{N_i^*\}_{i\in I}$, $\{N_i\}_{i\in I}$ are also locally finite and it holds (closure in $\mathscr{U}_{z_0}$):

$$\bar{N} \subset \bigcup \bar{N}_i$$

and f is locally constant on $\bar{N}$.

Theorem 2.6. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a (real) analytic set. Let f be* **R***-analytic in Ω. Let $\mathscr{R}(V)$ be the set of regular points of V and $N = \{x \in \mathscr{R}(V) \mid x$ is critical point of f on $\mathscr{R}(V)\}$.*

Then f is locally constant on $\bar{N}$ (closure in Ω).

Proof. Let $x \in N$ be fixed. Then

$$\mathscr{U}_x \cap \mathscr{R}(V) = \mathscr{U}_x \cap V$$

in a neighborhood $\mathscr{U}_x \subset \Omega$, hence x is a metric point of f on V.

The set N is contained in the set of all metric critical points of f on V and we can use Theorem 2.5.

Chapter III.

THE STRUCTURE OF EIGENVALUES

Let us denote $\mathcal{O} = \mathbf{R}^n - \{0\}$. Let $f, g : \mathcal{O} \to \mathbf{R}$ be $\mathbf{R}$-analytic functions and suppose that

$$\lim_{|x|\to\infty} f(x) = +\infty\,, \quad f(0) = 0\,, \quad f(x) > 0\,, \quad \nabla f(x) \neq 0 \quad \text{for} \quad x \neq 0\,.$$

(A typical example is $f(x) = \frac{1}{2}\Sigma x_i^2 = \frac{1}{2}|x|^2$.)

In this chapter we shall investigate the set of eigenvectors B and the set of eigenvalues Λ of mappings F, G where $F = \nabla f$, $G = \nabla g$. Hence we denote

$$B = \{x \in \mathcal{O} \mid \text{there exists } \lambda \in \mathbf{R} \text{ such that } \lambda\, F(x) = G(x)\}\,.$$

Since $\nabla f(x) \neq 0$ in $\mathcal{O}$, the eigenvalue λ corresponding to an eigenvector x is determined uniquely and we shall denote it by $\lambda(x)$. Further,

$$\Lambda = \{\lambda \in \mathbf{R} \mid \text{there exists } x \in B \text{ such that } \lambda = \lambda(x)\}\,.$$

It will be seen that we can get a very good picture of the structure of the sets B and Λ investigating the set

$$G = [f, \lambda](B) = \{[r, \lambda] \in \mathbf{R}^2 \mid \text{there exists } x \in B \text{ such that } r = f(x),\ \lambda = \lambda(x)\}\,.$$

The aim of this chapter is to discuss the structure and properties of the set G.

Lemma 3.1. *The set B is $\mathbf{C}$-analytic in $\mathcal{O}$.*

Proof. Let us denote by (x, y) the scalar product in $\mathbf{R}^n$.
The function

$$\lambda(x) = \frac{(\nabla g(x), \nabla f(x))}{|\nabla f(x)|^2}$$

is $\mathbf{R}$-analytic in $\mathcal{O}$. If $x \in B$ is an eigenvector, then clearly $\lambda(x)$ is the corresponding eigenvalue. Hence

$$B = \{x \in \mathcal{O} \mid \lambda(x) \nabla f(x) = \nabla g(x)\} .$$

By Theorem C3 we can divide B in $\mathcal{O}$ into $\mathbf{C}$-irreducible branches:

$$B = \bigcup_{\alpha \in \mathrm{U}} B_\alpha$$

where B_α are $\mathbf{C}$-irreducible $\mathbf{C}$-analytic sets in $\mathcal{O}$ and the collection $\{B_\alpha\}$ is locally finite.

The behavior of the function $\lambda(x)$ on B_α depends on the fact whether $f \equiv \text{const}$ on B_α or not. Both these cases will be dealt with separately in Theorems 3.2 and 3.9. The summary of both cases as well as the information concerning the set G is contained in Theorem 3.10.

Theorem 3.2. *Let B_α be a $\mathbf{C}$-irreducible component of B such that $f \not\equiv \text{const}$ on B_α.*

Then for all $r > 0$ there exists only a finite number of λ such that $[r, \lambda] \in G_\alpha$, where

$$G_\alpha = [f, \lambda] (B_\alpha) .$$

In other words, if we denote

$$M_r(f) = \{x \in \mathcal{O} \mid f(x) = r\} ,$$

then for all $r > 0$ the set

$$\Lambda_{\alpha r} = \lambda(B_\alpha \cap M_r(f))$$

is at most finite.

The essential steps of the proof of this theorem will be done in the following lemmas. All these lemmas deal with the case when f, g are complex functions of complex variables.

Notations and assumptions. In the next lemmas 3.3—3.7 we shall suppose that:

(1) f, g are holomorphic functions in $\Omega \subset \mathbf{C}^n$,

(2) $\sum_{i=1}^{n} \left(\frac{df}{\partial z_i}\right)^2 \neq 0$ in Ω,

(3) $V \subset \Omega$ is an analytic set and

$$\lambda(z)\, \nabla f(z) = \nabla g(z)$$

for all $z \in V$, where the function $\lambda(z)$ is defined by

$$\lambda(z) = \frac{\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(z) \frac{\partial g}{\partial z_i}(z)}{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial z_i}\right)^2 (z)},$$

(4) f is not constant on V.

Lemma 3.3. *Let the assumptions (1)–(4) be fulfilled. Moreover, let V and $M = V \cap \{z \in \Omega \mid f(z) = 0\}$, be submanifolds of $\mathbf{C}^n$, V being connected.*

Then every point $z \in M$ is a critical point of λ on M.

Proof. Let $z_0 \in M$ be fixed. Let us denote $\dim_{z_0} V = k$. By Theorem B3 we have $\dim_{z_0} M = \dim_{z_0} V - 1$. The cases $k = 0, 1$ being trivial, we shall suppose $2 \leqq k \leqq n$. We can choose a neighborhood $\mathscr{U}_{z_0}$ and a coordinate set $q = \{q_i\}_1^n$ in $\mathscr{U}_{z_0}$ such that

$$\mathscr{U}_{z_0} \cap V = \{z \in \mathscr{U}_{z_0} \mid q_{k+1}(z) = \ldots = q_n(z) = 0\},$$
$$\mathscr{U}_{z_0} \cap M = \{z \in \mathscr{U}_{z_0} \mid q_k(z) = \ldots = q_n(z) = 0\}.$$

Moreover, we can assume that for some $\varepsilon > 0$ it holds

$$\mathscr{U}_{z_0} = \{z \in \mathbf{C}^n \mid |q_i(z)| < \varepsilon,\ i = 1, \ldots, n\},$$

i.e. $q(\mathscr{U}_{z_0}) = \Delta(0, \varepsilon) \subset \mathbf{C}^n_{q_1,\ldots,q_n}$. Then

$$q(\mathscr{U}_{z_0} \cap V) = \Delta_k(0, \varepsilon) = \Delta_k,$$
$$q(\mathscr{U}_{z_0} \cap M) = \Delta_{k-1}(0, \varepsilon) = \Delta_{k-1}.$$

On $\Delta = \Delta(0, \varepsilon)$ we shall define functions of variables $q_1, \ldots, q_n$ (we shall denote them again by f, g, λ)

$$f(q) = f(z(q)), \quad g(q) = g(z(q)), \quad \lambda(q) = \lambda(z(q)).$$

For the functions f, g it holds (for $q \in \Delta$)

$$\nabla_q f(q) = \nabla_z f(z(q)) \circ \frac{\partial(z_1, \ldots, z_n)}{\partial(q_1, \ldots, q_n)} \quad \text{etc.},$$

where

$$\nabla_q = \left(\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}\right), \quad \nabla_z = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right)$$

and since

$$\lambda(z(q)) \,.\, \nabla_z f(z(q)) = \nabla_z g(z(q))$$

for all $q \in \Delta_k$, it holds also

$$\lambda(q) \, \nabla_q f(q) = \nabla_q g(q) \tag{3,1}$$

for all $q \in \Delta_k$.

Let i, $1 \leqq i \leqq k - 1$ be fixed, we have to prove that

$$\left.\frac{\partial \lambda}{\partial q_i}(q)\right|_{q=0} = 0 \,.$$

From (3,1) it follows that

$$\lambda(q) \frac{\partial f}{\partial q_k}(q) = \frac{\partial g}{\partial q_k}(q), \quad q \in \Delta_k$$

and

$$\lambda(q) \frac{\partial f}{\partial q_i}(q) = \frac{\partial g}{\partial q_i}(q), \quad q \in \Delta_k$$

and differentiating these equations with respect to q_i and q_k we obtain

$$\frac{\partial \lambda}{\partial q_i} \frac{\partial f}{\partial q_k} + \lambda \frac{\partial^2 f}{\partial q_k \, \partial q_i} = \frac{\partial^2 g}{\partial q_k \, \partial q_i},$$

$$\frac{\partial \lambda}{\partial q_k} \frac{\partial f}{\partial q_i} + \lambda \frac{\partial^2 f}{\partial q_i \, \partial q_k} = \frac{\partial^2 g}{\partial q_i \, \partial q_k}.$$

However, the second derivatives can be interchanged, which yields

$$\frac{\partial \lambda}{\partial q_i}(q) \frac{\partial f}{\partial q_k}(q) = \frac{\partial \lambda}{\partial q_k}(q) \frac{\partial f}{\partial q_i}(q) \tag{3,2}$$

for all $q \in \Delta_k$.

The function f is holomorphic in Δ_k, hence we can write its Taylor expansion at $0 \in \Delta_k$:

$$f(q_1, \ldots, q_k, 0, \ldots, 0) = \sum_{i_1, \ldots, i_k \geqq 0} a_{i_1, \ldots, i_k} q_1^{i_1} \ldots q_k^{i_k} \,.$$

Further, it holds $f \equiv 0$ on Δ_{k-1}, but $f \not\equiv 0$ on Δ_k. For if $f \not\equiv 0$ on Δ_k, then $f(z) \equiv 0$ on $\mathscr{U}_{z_0} \cap V$ and by Theorem B5 (iii) it holds that $f(z) \equiv 0$ on V, which contradicts the assumptions.

Hence $a_{i_1,\ldots,i_{k-1},0} = 0$ for all $i_1, \ldots, i_{k-1} \geqq 0$ and there exists a positive integer $j_0 > 0$ such that

$$f(q_1, \ldots, q_k, 0, \ldots, 0) = q_k^{j_0} F(q_1, \ldots, q_k), \quad q \in \Delta_k,$$

where F is holomorphic on Δ_k and

$$F(q_1, \ldots, q_{k-1}, 0) \not\equiv 0 \quad \text{on} \quad \Delta_{k-1}.$$

Then (3,2) implies that

$$\frac{\partial \lambda}{\partial q_i}(q)\, j_0 q_k^{j_0 - 1} F(q) + \frac{\partial \lambda}{\partial q_i}(q)\, q_k^{j_0} \frac{\partial F}{\partial q_k}(q) = \frac{\partial \lambda}{\partial q_k}(q)\, q_k^{j_0} \frac{\partial F}{\partial q_i}(q).$$

Dividing by $q_k^{j_0-1}$ and setting $q_k = 0$, we obtain

$$\frac{\partial \lambda}{\partial q_i}(q)\, j_0\, F(q) \equiv 0$$

on Δ_{k-1} (i.e. for $q_k = 0$) but $F(q) \not\equiv 0$ on Δ_{k-1}, hence $\partial\lambda/\partial q_i(q) \equiv 0$ on Δ_{k-1}.

Consequently, we have also

$$\left.\frac{\partial \lambda}{\partial q_i}(q)\right|_{q=0} = 0.$$

In the next we shall need the following

Lemma 3.4 *Let V be an analytic set in Ω, $0 \in V$. Suppose that $[V]_0$ is an irreducible germ and that $z_1, \ldots, z_k, \ldots, z_n$ form a regular coordinate set for* id $[V]_0$ (see Chapter I). *Suppose further that the function $d = d(z_1, \ldots, z_k) \not\equiv 0$, holomorphic in polydisc $\Delta(0, r)$, satisfies the assertions* (i), (ii), (iii) *of Theorem* B2 *on local parametrization of an analytic set.*

Further, let $a = [a_1, \ldots, a_{k-1}, 0, \ldots, 0]$ be such a point that

(i) *there exists $\varrho = [\varrho, \ldots, \varrho]$ such that $\Delta_k(a, \varrho) \subset \Delta_k(0, r)$ and that in $\Delta_k(a, \varrho)$ it holds*

$$D \subset \{z \mid z_k = 0\}$$

where $D = \{z \in \Delta(0, r) \mid d(z) = 0\}$;

(ii) *in $\Delta(0, r)$ it holds*

$$V \cap \{z \mid z_k = 0\} = \{z \mid z_k = z_{k+1} = \ldots = z_n = 0\}.$$

Let us denote

$$\Delta^+ = \{z \in \Delta_k(a, \varrho) \mid z_k = x_k + iy_k,\ y_k > 0\}.$$

Then

$$\pi : V \cap \Delta(0, r) \cap \pi^{-1}(\Delta^+) \to \Delta^+$$

is a simple covering map and if $G : \Delta^+ \to V$

$$z' = [z_1, \ldots, z_k] \mapsto [z_1, \ldots, z_k, g_{k+1}(z'), \ldots, g_n(z')]$$

is a homeomorphism of Δ^+ *on any sheet of this simple covering map, then the functions* $g_{k+1}, \ldots, g_n$ *are holomorphic in* Δ^+ *and*

$$\lim_{\substack{z_k \to 0 \\ z_k > 0}} \frac{\partial g_j}{\partial z_i}(a_1, \ldots, a_{k-1}, z_k) = 0, \quad j = k+1, \ldots, n, \quad i = 1, \ldots, k-1.$$

Proof. A) First we prove that the functions $g_{k+1}, \ldots, g_n$ are holomorphic. By assumption (i) it holds that

$$\Delta^+ \subset \Delta_k(a, \varrho) \setminus D.$$

Hence by Theorem B2,

$$\pi : V \cap \pi^{-1}(\Delta^+) \to \Delta^+$$

is a covering map and since the set Δ^+ is clearly homotopically equivalent to a point, π is a simple covering map (see [Sp], Theorem 2 and 3, p. 89).

Hence there exists a continuous mapping G of Δ^+ on any sheet of this covering. Let $b \in \Delta^+$ be fixed. By Theorem B2 (iii) there exists a holomorphic mapping $G_b :$ $: \Delta_k(b, \sigma) \to V$ in a polydisc $\Delta_k(b, \sigma)$ such that $G_b(b) = G(b)$. And since the mapping π from (3,3) is a simple covering map, we have

$$G_b(z) = G(z) \quad \text{in } \Delta_k(b, \sigma) \cap \Delta^+.$$

Hence G is holomorphic in the whole Δ^+.

B) Further, we show that

$$\text{(3,4)} \qquad \forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in V[\pi(z) \in \Delta_k(a, \tfrac{1}{2}\varrho),\ |z_k| < \delta]$$
$$\forall j = k+1, \ldots, n \quad \text{is} \quad |z_j| < \varepsilon.$$

By Theorem B2 the mapping

$$\pi : V \cap \Delta(0, r) \to \Delta_k(0, r)$$

is a proper mapping. If (3,4) does not hold, then

$$\exists \varepsilon_0 > 0\ \forall n = 1, 2, \ldots\ \exists z^n \in V\left[\pi(z_n) \in \Delta_k(a, \tfrac{1}{2}\varrho),\ |z_k^n| < \frac{1}{n}\right]$$

such that for example $|z^n_{k+1}| \geqq \varepsilon_0$. We can suppose that the sequence $\{z^n\}$ is chosen in such a way that $\{\pi(z^n)\}$ is convergent. Moreover, since π is proper, we can also assume that $z^n \to z^0$, where $z^0 \in \Delta(0, r) \cap V$. But then $z^0_k = 0$, hence by the assumption (ii) $z^0_{k+1} = 0$, which is contradiction.

C) Let $\varepsilon > 0$ and $i, j, k + 1 \leqq j \leqq n, 1 \leqq i \leqq k - 1$ be fixed. Let us choose δ, $0 < \delta < \frac{1}{2}\varrho$ such that (3,4) holds. Then

$$|g_j(z_1, \ldots, z_{k-1}, z_k)| < \varepsilon$$

for all $0 < z_k < \delta$, $|z_j - a_j| < \frac{1}{2}\varrho, j = 1, \ldots, k - 1$ and from the Cauchy inequality for complex functions (see [N], it follows that

$$\left|\frac{\partial g_j}{\partial z_i}(a_1, \ldots, a_{k-1}, z_k)\right| \leqq \frac{2\varepsilon}{\varrho}.$$

Lemma 3.5. *Suppose that the assumptions (1)–(4) are fulfilled. Let $z_0 \in M \subset \subset V \subset \Omega \subset \mathbf{C}^n$, where*

$$M = V \cap \{z \in \Omega \mid f(z) = 0\}.$$

Suppose that V is an irreducible analytic set in Ω and that the germ $[V]_{z_0}$ is an irreducible germ. Suppose further that M is a submanifold of $\mathbf{C}^n$.

Then z_0 is a critical point of λ on M.

Proof. Let us denote $\dim [V]_{z_0} = k$. By Theorem B1 it holds that $\dim [M]_{z_0} = k - 1$ (where $\dim \emptyset = -1$). Since the cases $k = 0, 1$ are trivial, we can suppose $2 \leqq k \leqq n$.

First part of the proof. Suppose in addition to the assumptions of Lemma 3.5 that $z_0 = 0$, that $z_1, \ldots, z_k, \ldots, z_n$ form a regular coordinate set for id $[V]_0, f(z) = z_k$ and

$$M = \{z \in \Omega \mid z_k = z_{k+1} = \ldots = z_n = 0\},$$

i.e.

$$V \cap \{z \in \Omega \mid z_k = 0\} = M.$$

We have to prove

$$\frac{\partial \lambda}{\partial z_i}(z) = 0, \quad i = 1, \ldots, k - 1.$$

From Theorem B2 it follows that there exists a polydisc $\Delta(0, r) \subset \Omega$ and a function $d = d(z_1, \ldots, z_k)$, $d \not\equiv 0$ holomorphic in $\Delta(0, r)$ such that the assertion of the theorem holds.

Let us denote

$$D' = \{z' \in \Delta_k(0, r) \mid d(z') = 0\} = D \cap \mathbf{C}^k.$$

The set D' is an analytic set in $\Delta(0, r)$, hence there exists its decomposition into irreducible branches in $\Delta(0, r)$:

$$D' = \bigcup_{i \in I} D_i$$

where $\{D_i\}_{i \in I}$ is a locally finite collection. Let us denote

$$D'' = \bigcup_{D_i \not\subset \Delta_{k-1}(0,r)} D_i .$$

Then the set D'' is also an analytic set in $\Delta(0, r)$.

Let $i \in I$ be such that $D_i \not\subset \Delta_{k-1}(0, r)$. Suppose that $\Delta_{k-1}(0, r) \supset D_i$. Then by Theorem B5 (v) it holds that $\dim [\Delta_{k-1}(0, r)]_0 < \dim [D_i]_0$. But $\dim [\Delta_{k-1}(0, r)]_0 = k - 1$ and $\dim [D_i]_0 \leqq \dim [D]_0 \leqq k - 1$ which is a contradiction.

Hence, if $D_i \not\subset \Delta_{k-1}(0, r)$, then also $\Delta_{k-1}(0, r) \not\subset D_i$. By Theorem B5 (iv) the set $\Delta_{k-1}(0, r) \setminus D_i$ is an open dense subset of $\Delta_{k-1}(0, r)$. Further

$$\Delta_{k-1}(0, r) \setminus D'' = \bigcap_{D_i \not\subset \Delta_{k-1}(0,r)} \Delta_{k-1}(0, r) \setminus D_i .$$

Since the intersection of a finite number of open dense subsets of Δ_{k-1} is again an open dense subset of Δ_{k-1} and since the collection $\{D_i\}$ is locally finite, the set $\Delta_{k-1} \setminus D''$ is also an open, dense subset of Δ_{k-1}.

Let us denote

$$Q = \Delta_{k-1} \setminus D''$$

and let $a = [a_1, \ldots, a_{k-1}, 0, \ldots, 0] \in Q$ be fixed.

Then there exists $\varrho = [\varrho, \ldots, \varrho]$ such that $\Delta_k(a, \varrho) \subset \Delta(0, r)$ and $\Delta_k(a, \varrho) \cap D'' = \emptyset$. Hence in $\Delta_k(a, \varrho)$ it holds that $D \subset \Delta_{k-1}$. Hence all assumptions of Lemma 3.4 are satisfied. Let us denote

$$\Delta^+ = \{z \in \Delta_k(a, \varrho) \mid z_k = x_k + \mathrm{i}y_k,\ y_k > 0\}$$

and let $G : \Delta^+ \to V$ be a holomorphic mapping onto a sheet of a simple covering map

$$\pi : V \cap \pi^{-1}(\Delta^+) \to \Delta^+ ,$$

$$G : z' = [z_1, \ldots, z_k] \mapsto [z_1, \ldots, z_k, g_{k+1}(z'), \ldots, g_n(z')] .$$

The set $G(\Delta^+)$ is a connected submanifold and since it is one of the sheets of the set V over Δ^+, it is an analytic subset of the open set $\pi^{-1}(\Delta^+)$. Further, the set

$$G(\Delta^+) \cap \{z \mid z_k = \beta\} , \quad 0 < \beta < \varrho$$

is the image of the set $\{z \in \Delta^+ \mid z_k = \beta\}$ under the mapping G which is there nonsingular, hence it is also a submanifold of $\boldsymbol{C}^n$.

Now we can use Lemma 3.3 setting $\Omega = \pi^{-1}(\Delta^+)$, $f(z) = z_k - \beta$ (clearly z_k is not constant on $G(\Delta^+)$) and we obtain that $G(a_1, \ldots, a_{k-1}, \beta, 0, \ldots, 0)$ is a critical point of λ on $G(\{z \in \Delta^+ \mid z_k = \beta\})$. Hence for all $i = 1, \ldots, k-1$ it holds $(b' = (a_1, \ldots, a_{k-1}, \beta))$

$$0 = \frac{\partial}{\partial z_i} [\lambda(z', g_{k+1}(z'), \ldots, g_n(z'))] (b') =$$

$$= \frac{\partial \lambda}{\partial z_i} (b') + \sum_{j=k+1}^{n} \frac{\partial \lambda}{\partial z_j} (b', g_{k+1}(b'), \ldots, g_n(b')) \frac{\partial g_j}{\partial z_i} (b') .$$

Now, we can use Lemma 3.4 and obtain by the limit process $\beta \to 0$ that

$$\frac{\partial \lambda}{\partial z_i} (a_1, \ldots, a_{k-1}, 0, \ldots, 0) = 0 , \quad i = 1, \ldots, k-1 .$$

However, the point $a = [a_1, \ldots, a_{k-1}, 0, \ldots, 0] \in Q$ can be arbitrary and Q is dense in $\Delta_{k-1}(0, r) = M \cap \Delta(0, r)$. Hence each point from the set $M \cap \Delta(0, r)$ is a critical point of λ on M.

Second part of the proof. We shall show that we can choose a coordinate set at z_0 so that all assumptions from the 1st part of the proof are fulfilled.

The set M is a submanifold, $\dim [M]_{z_0} = k - 1$, hence there exist a neighborhood, $\mathscr{U}_{z_0} \subset \Omega$ and a coordinate set $\{q_i\}$ at z_0 such that

$$\mathscr{U}_{z_0} \cap M = \{z \in \mathscr{U}_{z_0} \mid q_k(z) = \ldots = q_n(z) = 0\} .$$

Since $f \equiv 0$ on M, each member in the Taylor expansion of the function $f(z(q))$ in varables $q_1, \ldots, q_n$ at 0 must contain a positive power of at least one from the variables $q_k, \ldots, q_n$, hence we can write (in a neighborhood of z_0):

$$f(z) = \sum_{i=k}^{n} h_i(z)\, q_i(z) \tag{3,6}$$

where $h_i(z)$ are holomorphic functions in this neighborhood.

Consequently, the vector $\nabla f(z_0)$ is a linear combination of vectors $\nabla g_k(z_0), \ldots$ $\ldots, \nabla g_n(z_0)$ (the members with ∇h_i vanish for $q_i(z_0) = 0$). The vector $\nabla f(z_0)$ being non zero, it holds (after event. relabeling)

$$\operatorname{rank} \frac{\partial(f, g_{k+1}, \ldots, g_n)}{\partial(z_1, \ldots, z_n)} (z_0) = n - k + 1 .$$

Hence the functions $\{q_1, \ldots, q_{k-1}, f, q_{k+1}, \ldots, q_n\}$ form a coordinate set in a (possibly smaller) neighborhood $\mathscr{U}_{z_0}$ and by (3,6) it holds in this neighborhood that the submanifold

$$M' = \{z \in \mathscr{U}_{z_0} \mid f(z) = q_{k+1}(z) = \ldots = q_n(z) = 0\}$$

contains M. Since the dimension of these submanifolds are equal and since we can suppose that M is connected, Theorem B5 (v) implies that $M = M'$ in $\mathscr{U}_{z_0}$.

If $z \in V \cap \mathscr{U}_{z_0}$, $f(z) = 0$ then $z \in M$, hence $q_{k+1}(z) = \ldots = q_n(z) = 0$. From this it follows that

$$V \cap \{z \in \mathscr{U}_{z_0} \mid q_1(z) = \ldots = q_{k-1}(z) = f(z) = 0\} = \{0\} .$$

Hence by Theorem B1 there exists a linear transformation

$$w_j = \sum_{i=k+1}^{n} a_{ij} q_i , \quad j = k + 1, \ldots, n ; \quad w_j = q_j , \quad j = 1, \ldots, k$$

such that $\{w_i\}$ is a regular coordinate set for id $[w(V \cap \mathscr{U}_{z_0})]_0$. (Clearly $w(V \cap \mathscr{U}_{z_0})$ is an analytic set in $w(\mathscr{U}_{z_0})$ and the germ $[w(V \cap \mathscr{U}_{z_0})]_0$ is irreducible, since the same is true for $V \cap \mathscr{U}_{z_0} \subset \mathscr{U}_{z_0}$ and $[V]_{z_0}$.)

Since

$$w(M \cap \mathscr{U}_{z_0}) = \{w \in w(\mathscr{U}_{z_0}) \mid w_k = \ldots = w_n = 0\}$$

it holds that

$$w(M \cap \mathscr{U}_{z_0}) = w(V) \cap \{w \in w(\mathscr{U}_{z_0}) \mid w_k = 0\}$$

is a submanifold in $w(\mathscr{U}_{z_0})$. If we denote

$$\tilde{f}(w) = f(z(w)) \quad \text{etc.} ,$$

then the functions $\tilde{f}$, $\tilde{g}$, $\tilde{\lambda}$ are holomorphic in $w(\mathscr{U}_{z_0})$ and (as in the proof of Lemma 3.3) it holds that

$$\tilde{\lambda}(w) . \nabla \tilde{f}(w) = \nabla \tilde{g}(w) , \quad w \in w(V) .$$

Further, as $[w(V)]_0$ is an irreducible germ, there exists a neighborhood $\mathscr{V} \subset w(\mathscr{U}_{z_0})$ such that $w(V) \cap \mathscr{V}$ is an irreducible analytic set in $\mathscr{V}$.

Moreover, the function $\tilde{f}$ is not constant on $w(V) \cap \mathscr{V}$ since in the opposite case f would be constant on $V \cap w^{-1}(\mathscr{V})$ and then by Theorem B5 (iii) f would be constant on the whole V, which is a contradiction.

Now we can use the 1st part of the proof for the open set $\mathscr{V} \subset \mathbf{C}^n_{w_1, \ldots, w_n}$, the sets $w(V) \cap \mathscr{V}$ and $w(M) \cap \mathscr{V}$ and the functions $\tilde{f}$, $\tilde{g}$, $\tilde{\lambda}$. All assumptions are satisfied, hence 0 is a critical point of $\tilde{\lambda}$ on $w(M) \cap \mathscr{V}$ which implies immediately that z_0 is a critical point of λ on M.

Lemma 3.6. *Suppose that the assumptions* (1)–(4) *are fulfilled. Let* $V \subset \Omega \subset \mathbf{C}^n$ *be an irreducible analytic. set. Let us denote* (*for some* $z_0 \in V$)

$$W = V \cap \{z \in \Omega \mid f(z) = f(z_0)\} .$$

Then each point of $\mathscr{R}(W)$ is a critical point of λ on $\mathscr{R}(W)$.

Proof. Let $z_0 \in \mathscr{R}(W)$ be fixed. There exists a neighborhood $\mathscr{U}_{z_0} \subset \Omega$ such that

$$W \cap \mathscr{U}_{z_0} = \mathscr{R}(W) \cap \mathscr{U}_{z_0}$$

is a submanifold of $\mathbf{C}^n$.

If we denote $M = W \cap \mathscr{U}_{z_0}$, the germ $[M]_{z_0}$ is irreducible. Then there exists an irreducible branch $[V']_{z_0}$ of the germ $[V]_{z_0}$ containing $[M]_{z_0}$ (see [H], p. 53).

Further, we can find a neighborhood $\mathscr{U}'_{z_0} \subset \mathscr{U}_{z_0}$ such that $V' \cap \mathscr{U}'_{z_0}$ is an irreducible analytic set in $\mathscr{U}'_{z_0}$ (and clearly we can suppose $V' \subset V$ in $\mathscr{U}'_{z_0}$). The set $M \cap \mathscr{U}'_{z_0}$ is again a submanifold of $\mathbf{C}^n$ and f is not constant on $V' \cap \mathscr{U}'_{z_0}$ (by Theorem B5 (iii)).

From the preceding lemma (where we take $f(z) - f(z_0)$ instead of f and $\mathscr{U}'_{z_0}$ instead of Ω) it follows that z_0 is a critical point of λ on $\mathscr{R}(W)$.

Lemma 3.7. *Suppose* (1)–(4) *to be fulfilled.*

Let $V \subset \Omega \subset \mathbf{C}^n$ be an irreducible analytic set, $z_0 \in V$. Let us denote

$$W = V \cap \{z \in \Omega \mid f(z) = f(z_0)\} .$$

Then λ is locally constant on W.

Proof. By the preceding lemma each point from $\mathscr{R}(W)$ is a critical point of λ on $\mathscr{R}(W)$, hence by Theorem 2.2 λ is locally constant on W.

Theorem 3.8. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a $\mathbf{C}$-irreducible, $\mathbf{C}$-analytic set and let f, g be $\mathbf{R}$-analytic functions in Ω.*

Let us suppose that $\nabla f \neq 0$ in Ω and that $f \not\equiv$ const on V. Let us denote

$$\lambda(x) = \frac{(\nabla f(x), \nabla g(x))}{|\nabla f(x)|^2}$$

and suppose that $\lambda(x) \nabla f(x) = \nabla g(x)$, $x \in V$.

Then for every $x_0 \in \Omega$ the function λ is locally constant on the set

$$W = V \cap \{x \in \Omega \mid f(x) = f(x_0)\} .$$

Proof. There exists a neighborhood $\Omega^* \subset \mathbf{C}^n$ of the set Ω such that $\Omega^* \cap \mathbf{R}^n = \Omega$ and that there exist functions f^*, g^* holomorphic in Ω^* such that

$$f^*|_\Omega = f , \quad g^*|_\Omega = g .$$

By the assumptions it holds $\sum_{i=1}^{n} [(\partial f/\partial z_i)(z)]^2 \neq 0$ for $z \in \Omega$ and the set Ω^* can be chosen so that the same is true also for all $z \in \Omega^*$. If we denote

$$\lambda^*(z) = \frac{\sum_{i=1}^{n} \frac{\partial f^*}{\partial z_i}(z) \frac{\partial g^*}{\partial z_i}(z)}{\sum_{i=1}^{n} \left(\frac{\partial f^*}{\partial z_i}(z)\right)^2}$$

then also $\lambda^*|_\Omega = \lambda$.

Further, we can suppose that there exists a (complex) analytic set $V^* \subset \Omega^*$ such that

$$V^* \cap \Omega = V$$

and that V^* is the least such set, so that by Theorem C2 V^* is irreducible.

The set B^* of all eigenvectors of f^*, g^*, i.e.

$$B^* = \{z \in \Omega^* \mid \lambda^*(z)\, \nabla f^*(z) = \nabla g^*(z)\}$$

is an analytic set in Ω^* containing V, hence containing also V^*. Clearly $f^* \not\equiv$ const on V^*.

Let now $x_0 \in \Omega$. If $W \neq \emptyset$, we can suppose $x_0 \in V$. By Lemma 3.7 the function λ^* is locally constant on

$$W^* = V^* \cap \{z \in \Omega^* \mid f^*(z) = f^*(x_0)\}\,.$$

However, $W \subset W^*$ and $\lambda^*|_W = \lambda$.

Proof of Theorem 3.2. Let $r > 0$ be fixed. Let us denote

$$W = B_\alpha \cap \{x \in \mathcal{O} \mid f(x) = r\}\,.$$

Since $\lim_{x \to \infty} f(x) = +\infty$ and $f(0) = 0$, the set W is a bounded closed subset of $\mathbf{R}^n$, hence it is compact. By the preceding theorem λ is locally constant on W, hence $\lambda(W)$ is at most finite set.

In the case that the **C**-irreducible branch B_α of the set of eigenvectors B is such that it holds $f \equiv$ const on B_α, then it need not be true generally that λ is locally constant on B_α. Values of λ on this branch can form a closed interval.

Theorem 3.9. *Let B_α be a **C**-irreducible branch of B such that f is constant on B_α. Then*

$$\lambda(B_\alpha)$$

is a closed bounded connected (possibly degenerate) interval.

Proof. From Theorem C4 it follows that B_α is a connected set, moreover, it is locally arcwise connected. Further, B_α is a closed subset of $\mathcal{O}$, but $f(B_j) = \{r_0\}$ for some $r_0 > 0$, hence B_α is compact.

Supposing that the set of eigenvectors B is decomposed into $\mathbf{C}$-irreducible branches $\{B_\alpha\}_{\alpha\in\mathfrak{A}}$, we can divide the set of indices $\mathfrak{A}$ into two parts:

$$\mathfrak{A}' = \{\alpha \in \mathfrak{A} \mid f \text{ is constant on } B_\alpha\},$$
$$\mathfrak{A}'' = \{\alpha \in \mathfrak{A} \mid f \text{ is not constant on } B_\alpha\}.$$

Let us denote similarly

$$B' = \bigcup_{\alpha\in\mathfrak{A}'} B_\alpha, \quad B'' = \bigcup_{\alpha\in\mathfrak{A}''} B_\alpha$$

and

$$G' = [f, \lambda](B'), \quad G'' = [f, \lambda](B'').$$

Then the form of the set $G = G' \cap G''$ and hence also the structure of eigenvalues λ is described by the following theorem:

Theorem 3.10. *Let us denote $M_r(f) = \{x \in \mathcal{O} \mid f(x) = r\}$ for $r > 0$.*

1) *If we denote $A = f(B')$, then A is a countable, isolated subset of $(0, \infty)$ and the set $\lambda(B' \cap M_r(f))$ is*

a) *empty for $r \notin A$;*

b) *union of at most finite number of bounded connected, closed (possibly degerate) intervals for $r \in A$.*

2) *For all $r > 0$, the set*

$$\lambda(B'' \cap M_r(f))$$

is at most finite.

Corollary. *The set $\Lambda_r = \lambda(B \cap M_r(f))$ of all eigenvalues such that the corresponding eigenvectors are from $M_r(f)$ is at most finite for all $r \in (0, \infty) \setminus A$.*

Proof. 1) The collection $\{B_\alpha\}_{\alpha\in\mathfrak{A}}$ is locally finite, hence for all $x \in \mathcal{O}$ there exists a neighborhood $\mathcal{U}_x \subset \mathcal{O}$ such that

$$\mathcal{U}_x \cap B_\alpha \neq \emptyset$$

only for a finite number of $\alpha \in \mathfrak{A}'$; moreover, we can take the neighborhood $\mathcal{U}_x$ so small that if $\alpha \in \mathfrak{A}'$, $\mathcal{U}_x \cap B_\alpha \neq \emptyset$ then $x \in B_\alpha$. From this it follows that f is locally constant on B'. Let us denote $A = f(B')$.

The set A is countable and if there exists a sequence $r_n \in B'$ such that $r_n \to r_0 \in (0, \infty)$, $r_n \neq r_0$, then for every n there exists $x_n \in B'$ such that $f(x_n) = r_n$. Since $\lim_{\|x\|\to\infty} f(x) = \infty$, this sequence is bounded, hence we can choose a convergent subsequence $x_{n_k} \to x_0 \in \mathbf{R}^n$, but then $x_0 \neq 0$ and $x_0 \in B'$. Then $f(x_0) = r_0$ and there exists a neighborhood $\mathcal{U}_{x_0}$ such that f is constant on $B' \cap \mathcal{U}_{x_0}$. But then $r_n = r_0$ for n sufficiently large, which is a contradiction. Hence A is an isolated set in $(0, \infty)$.

If $r \in A$, then only a finite number of B_α, $\alpha \in \mathfrak{A}'$ intersect the compact set $M_r(f)$, hence it is sufficient to use Theorem 3.9.

2) Let $r > 0$. The set $M_r(f)$ is compact, hence

$$B_\alpha \cap M_r(f) \neq \emptyset$$

only for a finite number of indices from $\mathfrak{A}''$. Every set $\lambda(B_\alpha \cap M_r(f))$, $\alpha \in \mathfrak{A}''$ is by Theorem 3.2 finite, from which the assertion of Theorem follows.

Chapter IV

THE INFINITE DIMENSIONAL CASE

Notation. Let H_1, H_2 be real Hilbert spaces, $\Omega \subset H_1$ an open set. For an operator $F : \Omega \to H_2$, the usual Frechet derivative of F at x will be denoted by $DF(x, \cdot) = F'(x)$. The operator F is said to be *real analytic*, if:

(i) for every $x \in \Omega$ there exist Frechet derivatives of arbitrary orders $D^n F(x, \ldots)$,

(ii) for every $x \in \Omega$ there exists $\delta > 0$ such that for all $h \in H_1$, $\|h\| < \delta$ it is

$$F(x + h) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(x, h^n)$$

(the convergence being locally uniform and absolute).

For real analytic operators the implicit function theorem holds in the usual form, the composition of two real analytic mappings is a real analytic map (see e.g. [FNSS 2]).

For a mapping

$$F : \Omega \subset H_1 \times H_2 \to H_3\,,\ [x, y] \in \Omega \mapsto F(x, y)\,,$$

the partial derivative of F at $[x_0, y_0]$ with respect to y will be denoted by $F'_y(x_0, y_0)$ (hence $F'_y(x_0, y_0)$ is a linear, bounded mapping from H_2 into H_3).

An operator $F : \Omega \subset H_1 \to H_2$ is said to be *Fredholm operator* at x_0, if:

(i) there exists Frechet derivative $F'(x_0)$,

(ii) the kernel of this derivative $\{h \in H_1 \mid F'(x_0)\, h = 0\}$ has a finite dimension,

(iii) the image $F'(x_0)\,(H_1)$ is a closed subspace of H_2 and the factor space $H_2/F'(x_0)\,(H_1)$ is a finite dimensional one.

One can prove that if the derivative $F'(x_0)$ of an operator F can be written as $F'(x_0) = J + K$ where J is an isomorphism H_1 onto H_2 and K is bounded, linear, completely continuous, then F is a Fredholm operator at x_0 (see e.g. [G–R], Appendix B).

In this chapter we shall deal with the following problem:

Let H be a real Hilbert space.

Suppose that f, g are two real analytic functionals on H. Suppose further that

(i) g' is strongly continuous (i.e., it maps weakly convergent sequences onto strongly convergent ones),

(ii) f' is a bounded mapping,

(iii) $f(0) = 0, f(u) > 0$ for all $u \neq 0$, $\lim_{\|u\| \to \infty} f(u) = +\infty$,

(iv) there exists a continuous, nondecreasing function $C_1(t)$ for $t > 0$ such that for all $u, h \in H$

$$D^2 f(u; h, h) \geqq C_1(f(u)) \|h\|^2 ,$$

(v) there exists a continuous function $C_2(r)$, $r > 0$ such that

$$\inf_{\substack{x \in H \\ f(x) = r}} f'(x) x \geqq C_2(r) > 0 .$$

Let us denote (for $u \in H - \{0\}$)

$$\lambda(u) = \frac{g'(u) u}{f'(u) u} .$$

As in the finite dimensional case, we want to investigate the structure of the set of all eigenvectors

$$B = \{u \in H - \{0\} \mid \lambda(u) f'(u) = g'(u)\}$$

and the set $\Lambda = \lambda(B)$ of all eigenvalues by means of its image, i.e., by means of the set $G \subset \mathbf{R}^2$:

$$G = \{[r, \lambda] \in \mathbf{R}^2 \mid r = f(u),\ \lambda = \lambda(u),\ u \in B\} .$$

The method of generalizing the results from Chapter III to the infinite dimensional case is based on the Fredholm property of the operator $\lambda f' - g'$. There are several points to be observed:

If we want to write the assertion of the main theorem in the same form as in the finite dimensional case, then the exceptional set R is only a countable set (see Theorem 4.2). Hence we must consider the part G_ε of the graph G:

$$G_\varepsilon = \{[r, \lambda] \in G \mid |\lambda| \geqq \varepsilon\} ,$$

for which the exceptional set R_ε is isolated. This assertion is based on the fact that the set

$$B(\varepsilon; r_1, r_2) = \{u \in B \mid |\lambda(u)| \geqq \varepsilon;\ r_1 \leqq f(u) \leqq r_2\}$$

is a compact set.

The following local lemma shows that the set $G_{\mathscr{U}}$ correponding to the set of eigenvectors $B \cap \mathscr{U}$ has the same property as in the finite dimensional case if the neighborhood $\mathscr{U}$ is sufficiently small.

Lemma 4.1. *Let $x_0 \in B$ be such a point that $\lambda(x_0) \neq 0$. Then there exists a neighborhood $\mathscr{U}$ of the point x_0 and a finite set $R \subset (0, \infty)$ such that the set*

$$\lambda(B \cap \mathscr{U} \cap \{u \mid f(u) = r\})$$

is at most finite set for all $r \notin R$.

Proof. First part. For the convenience of notation, we shall consider derivatives f', g' to be points in the Hilbert space H, i.e.

$$\langle f'(x), h \rangle = Df(x, h) .$$

First it will be shown that the operator

$$\Phi : H \to H , \quad \Phi(x) = \lambda(x) f'(x) - g'(x)$$

is a Fredholm operator at x_0. Indeed,

$$D\Phi(x_0, h) = D\lambda(x_0, h) f'(x_0) + \lambda(x_0)\, Df'(x_0, h) - Dg'(x_0, h) .$$

The operator g' is strongly continuous, hence its Frechet differential $Dg'(x_0, \cdot)$ is a linear bounded completely continuous operator (see [V], Chapter I, § 4). Further, the mapping $h \mapsto D\lambda(x_0, h) f'(x_0)$ has a one-dimensional range. Finally we have $\langle Df'(x_0, h), h \rangle \geqq C_1(f(x_0)) \|h\|^2$. Hence Φ is a Fredholm operator at x_0 (see [G – R], Appendix B).

Second part. Now we can suppose that $H = H_1 \oplus H_2$, where H_1 is the kernel of the operator $D\Phi(x_0, \cdot)$, and denote by K the range of this linear operator. From part 1 it follows that H_1 is a finite dimensional space and that K is a closed subspace of H of a finite codimension. We can suppose that there exist $\xi_1, \ldots, \xi_m \in H$ such that $K = \{x \in H \mid \langle \xi_i, x \rangle = 0,\ i = 1, \ldots, m\}$. Let us denote further by $P_K : H \to K$ the projection of H onto K. For the convenience of notation we shall identify the space H_1 with the Euclidean n-space $\mathbf{R}^n$. Hence we can write for every $x \in H : x = [y, z]$, where $y \in \mathbf{R}^n = H_1$, $z \in H_2$. Now $x \in B \Leftrightarrow \Phi(x) = 0 \Leftrightarrow P_K\, \Phi(x) = 0$ and $\langle \Phi(x), \xi_i \rangle = 0$, $i = 1, \ldots, m$. The Implicit function theorem applies to the equation $P_K \Phi([y, z]) = 0$ at the point $x_0 = [y_0, z_0]$, since if $h_2 \in H_2$ then

$$D[P_K \Phi]([y_0, z_0], h_2) = P_K\, D\Phi([y_0, z_0], h_2) = D\Phi([y_0, z_0], h_2) .$$

The mapping

$$D_z[P_K \Phi] : H_2 \to K$$

is therefore a continuous linear one-one mapping onto K, hence being a linear isomorphism. By Implicit function theorem there exist neighborhoods $\mathscr{U}_1$ and $\mathscr{U}_2$ of the point $[y_0, z_0]$ and a real analytic mapping $\omega : \mathscr{U}_1 \to \mathscr{U}_2$ such that for $[y, z] \in \in \mathscr{U}_1 \times \mathscr{U}_2$ it holds that

$$P_K \Phi([y, z]) = 0 \Leftrightarrow z = \omega(y) .$$

Clearly we can choose $\mathscr{U}_1, \mathscr{U}_2$ in such a way that $0 \notin \mathscr{U}_1 \times \mathscr{U}_2$. Hence we obtain (for $[y, z] \in \mathscr{U}_1 \times \mathscr{U}_2$) $x = [y, z] \in B \Leftrightarrow$ (i) $y \in B_1$, where $B_1 = \{y \in \mathscr{U}_1 \mid \langle \Phi([y, \omega(y)]), \xi_i \rangle = 0,\ i = 1, \ldots, m\}$, (ii) $z = \omega(y)$.

Third part. Now we shall consider the following finite dimensional eigenvalue problem. Let us define (in $\mathscr{U}_1 \subset \mathbf{R}^n$):

$$F(y) = f([y, \omega(y)]) , \quad G(y) = g([y, \omega(y)]) ,$$

$$\Lambda_1(y) = \lambda([y, \omega(y)]) ,$$

$$\Lambda_2(y) = \frac{(\nabla G(y), \nabla F(y))}{(\nabla F(y), \nabla F(y))} , \quad \text{if} \quad \nabla F(y) \neq 0 .$$

If $\nabla F(y) \equiv 0$ on $\mathscr{U}_1$, then $F(y) = F(y_0)$ for $y \in \mathscr{U}_1$, hence the set

$$B \cap (\mathscr{U}_1 \times \mathscr{U}_2) \cap \{u \mid f(u) = r\}$$

is empty for all $r \neq f(y_0)$ and we can choose $R = \{f(y_0)\}$. In the opposite case, the set

$$W = \{y \in \mathscr{U}_1 \mid \nabla F(y) = 0\}$$

is a **C**-analytic set in $\mathscr{U}_1$, $W \neq \mathscr{U}_1$. We shall prove that for every $y \in B_1 \setminus W$ it holds that $\Lambda_1(y) = \Lambda_2(y)$ and $\Lambda_2(y) \nabla F(y) = \nabla G(y)$. Let $y \in B_1 \setminus W$ be fixed. First, for all $h \in \mathbf{R}^n$

$$(4,1) \qquad (\nabla F(y), h) = \langle f'([y, \omega(y)]), [h, D\omega(y, h)] \rangle ,$$

$$(\nabla G(y), h) = \langle g'([y, \omega(y)]), [h, D\omega(y, h)] \rangle .$$

Further, since $[y, \omega(y)]$ is an eigenvector, it follows that

$$\lambda([y, \omega(y)]) f'([y, \omega(y)]) = g'([y, \omega(y)]) .$$

If we take $u = [h, D\omega(y, h)]$ for an arbitrary $h \in \mathbf{R}^n$, then

$$\Lambda_1(y) = \lambda([y, \omega(y)]) = \frac{\langle g'([y, \omega(y)]), u \rangle}{\langle f'([y, \omega(y)]), u \rangle}$$

implies by (4,1) that

$$A_1(y) = \frac{(\nabla G(y), h)}{(\nabla F(y), h)}, \quad h \in \mathbf{R}^n . \tag{4,2}$$

Hence $A_1(y) \nabla F(y) = \nabla G(y)$. If we take particularly $h_0 = \nabla F(y)$, then (4,2) implies $A_1(y) = A_2(y)$.

Fourth part. The set B_1 is clearly a **C**-analytic set in Ω and we can write

$$B_1 = \bigcup_{i \in I} C_i ,$$

its decomposition into **C**-irreducible branches (in $\mathcal{U}_1$). The family $\{C_i\}$ is locally finite, hence we can choose a smaller neighborhood $\mathcal{U}'_1 \subset \mathcal{U}_1$ of x_0 such that only a finite number of C_i, say $C_1, \ldots, C_s$ intersect $\mathcal{U}'_1$. Now the sets $C_1, \ldots, C_s$ can be divided into two classes: we can suppose (after relabeling if necessary) that

$$F \equiv \text{const. on } C_i , \quad i = 1, \ldots, t ,$$

$$F \not\equiv \text{const. on } C_i , \quad i = t + 1, \ldots, s .$$

From Lemma 4.3 it follows that for every C_i, $i = t + 1, \ldots, s$ we can choose a neighborhood $\mathcal{V}_i$ such that

$$\lambda(C_i \cap \mathcal{V}_i \cap \{x \in \mathcal{U}_1 \mid F(x) = r\}$$

is at most finite set for all $r \neq F(y_0) = f(x_0)$. Now, if we denote

$$\mathcal{U}''_1 = \mathcal{V}_{t+1} \cap \ldots \cap \mathcal{V}_n \cap \mathcal{U}'_1$$

then the assertion of the Lemma holds with

$$R = \{f(x_0), F(C_1), \ldots, F(C_t)\} \quad \text{and} \quad \mathcal{U} = \mathcal{U}''_1 \times \mathcal{U}_2 .$$

Theorem 4.2. *Let ε be a positive number. Under the above assumption ((i)–(v)) let us denote*

$$B_\varepsilon = \{x \in B \mid |\lambda(x)| \geqq \varepsilon\}$$

and

$$G_\varepsilon = \{[r, \lambda] \in \mathbf{R}^2 \mid r = f(u),\ \lambda = \lambda(u),\ u \in B_\varepsilon\} .$$

Then there exists an isolated set $R_\varepsilon \subset (0, \infty)$ such that the set $G_\varepsilon \cap \{[r, \lambda] \mid r = r_0\}$ is:

1) *at most finite, if $r_0 \notin R_\varepsilon$,*

2) *the union of at most finite number of closed (possibly degenerate) intervals if $r_0 \in R_\varepsilon$.*

Moreover, the set

$$R = \bigcup_{n=1}^{\infty} R_{1/n}$$

is at most countable and for all r, $r \notin R$ the set

$$G \cap \{[r, \lambda] \mid r = r_0\}$$

is either finite or a sequence of points converging to the point $[r_0, 0]$.

Proof. First part.

Let us denote

$$B(\varepsilon, r_1, r_2) = B \cap \{u \in H \setminus \{0\} \mid r_1 \leqq f(u) \leqq r_2,\ |\lambda(u)| \geqq \varepsilon\}\ .$$

We shall prove that this is a compact set.

Let $\{u_n\}$ be a sequence in B_ε. It follows from the assumption (iii) that B_ε is a bounded set, hence we can choose a subsequence (denoting it again by $\{u_n\}$) such that $u_n \rightharpoonup u_0$. The mapping g' is strongly continuous, hence $g'(u_n) \to g'(u_0)$, moreover, we have

$$|\lambda(u_n)| \geqq \varepsilon\ .$$

Then $\lambda(u_n) f'(u_n) = g'(u_n)$ implies that there exists $v \in H$ such that $f'(u_n) \to v$.

We can write

$$\langle f'(u_n) - f'(u_0),\ u_n - u_0\rangle =$$

$$= \int_0^1 \langle Df'(u_0 + t(u_n - u_0),\ u_n - u_0),\ u_n - u_0\rangle\,\mathrm{d}t \geqq K_n \|u_n - u_0\|^2$$

where $K_n = \int_0^1 C_1(f(u_0 + t(u_n - u_0)))\,\mathrm{d}t > 0$. Suppose that $\liminf_{n\to\infty} K_n = 0$. Then for a subsequence $\{K_{n_j}\}$ it holds $K_{n_j} \to 0$ and (since f is convex and continuous, hence also weakly lower semicontinuous)

$$0 = \lim_{j\to\infty} K_{n_j} \geqq \int_0^1 C_1(\liminf f(u_0 + t(u_n - u_0)))\,\mathrm{d}t \geqq C_1(f(u_0)) \geqq 0\ .$$

Thus $f(u_0) = 0$ and hence $u_0 = 0$.

On the other hand,

$$\langle f'(u_n) - f'(u_0),\ u_n - u_0\rangle = \langle f'(u_n), u_n\rangle \to 0$$

and this is a contradiction with (v) for $r_1 \leqq f(u_n) \leqq r_2$ for all n.

Second part. Let $r_1, r_2 > 0$, $r_1 \leqq r_2$ be fixed. Lemma 4.1 implies that for every $x \in B_\varepsilon$ there exists a neighborhood $\mathscr{U}_x$ and a finite set $R_x \subset (0, \infty)$ such that the set

$$G_{\mathscr{U}_x} \cap \{[r, \lambda] \mid r = r_0\}$$

is at most finite for every r_0, $r_0 \notin R_x$. Now, we can choose a finite covering $\mathscr{U}_{x_1}, \ldots, \mathscr{U}_{x_s}$ of $B(\varepsilon, r_1, r_2)$ and define

$$R(\varepsilon, r_1, r_2) = \bigcup_{i=1}^{s} R_{x_i} .$$

Further, let us denote

$$R_\varepsilon = \bigcup_{n=1}^{\infty} \left[R(\varepsilon, n, n + 1) \cup R\left(\varepsilon, \frac{1}{n + 1}, \frac{1}{n}\right) \right].$$

Then clearly for all r_0, $r_0 \notin R_\varepsilon$ it holds that the set

$$G_\varepsilon \cap \{[r, \lambda] \mid r = r_0\}$$

is at most finite and R_ε is an isolated subset of $(0, \infty)$.

Third part. If $r_0 \in R_\varepsilon$, then we shall consider the set

$$V = B \cap \{u \in H \mid f(u) = r_0\} .$$

This set is locally arcwise connected, for if $x_0 \in V$, then as in the 2nd part of the proof of Lemma 4.1 we can choose a neighborhood $\mathscr{U}_1 \times \mathscr{U}_2$ of $x_0 = [y_0, z_0]$ such that for $[x, y] \in \mathscr{U}_1 \times \mathscr{U}_2$ it holds that

$$\begin{aligned} [x, y] \in V \Leftrightarrow &\text{(i) } y \in B_1 = \{y \in \mathscr{U}_1 \mid \langle \Phi([y, \omega(y)]), \xi_i \rangle = 0,\ i = 1, \ldots, m\}, \\ &\text{(ii) } z = \omega(y), \\ &\text{(iii) } f([y, \omega(y)]) = F(y) = r_0. \end{aligned}$$

However, the set $B_1 \cap \{y \in \mathscr{U}_1 \mid F(y) = r_0\}$ is a **C**-analytic set in $\mathscr{U}_1$, hence it is locally connected and clearly the same is true also for $V \cap (\mathscr{U}_1 \times \mathscr{U}_2)$. Therefore, if $V = \bigcup_{i \in I} V_i$ is the decomposition of V into its connected components, then there is only a finite number of V_i, say $V_1, \ldots, V_s$, which intersect the set $B(\varepsilon, r_0, r_0)$ and we have

$$B(\varepsilon, r_0, r_0) = \bigcup_{i=1}^{s} (V_i \cap \{u \mid |\lambda(u)| \geqq \varepsilon\}) .$$

Now if $V_i \cap \{u \mid |\lambda(u)| \geqq \varepsilon\}$ is a connected set, then

$$\lambda(V_i \cap \{u \mid |\lambda(u)| \geqq \varepsilon\})$$

is a closed interval.

If the set $V_i \cap \{u \mid |\lambda(u)| \geqq \varepsilon\}$ has more than one connected components, then each of them must contain a point x for which $f(x) = \varepsilon$ (in the opposite case the component would be closed and open subset of V_i, different from V_i, which is impossible), hence its image under the function λ is the closed interval $\langle \varepsilon, \alpha \rangle$, $\alpha \geqq \varepsilon$. Now

$$\lambda(V_i \cap \{u \mid |\lambda(u)| \geqq \varepsilon\})$$

is the interval $\langle \varepsilon, \sup \alpha \rangle$.

Therefore, the set $\lambda(B_\varepsilon \cap \{u \mid f(u) = r_0\}$ is at most finite union of closed (possibly degenerate) intervals.

Remark. If the functions f, g satisfy some further assumptions, which are required in the Ljusternik-Schnirelmann theory (see [F–N]) then the set $G \cap \{[r, \lambda] \mid r = r_0\}$ is at least countable for all $r_0 > 0$. Hence together with the preceding theorem we obtain the following assertion:

There exists a countable set $R \subset (0, \infty)$ such that the set $G \cap \{[r, \lambda] \mid r = r_0\}$ is:

1) a sequence of points which converge to the point $[r_0, 0]$, if $r_0 \notin R$;

2) a sequence of closed (possibly degenerate) intervals which converge (in the obvious sense) to the point $[r_0, 0]$, if $r_0 \in R$.

Lemma 4.3. *Let $V \subset \Omega \subset \mathbf{R}^n$ be a $\mathbf{C}$-irreducible, $\mathbf{C}$-analytic set and let f, g, λ be $\mathbf{C}$-analytic functions in Ω. Suppose that $f \not\equiv$ const. on V and suppose that for all $x \in V$ it holds $\lambda(x) \nabla f(x) = \nabla g(x)$. Then for every $x_0 \in \Omega$ there exists a neighborhood $\mathcal{U}$ of x_0 such that for every $r \in \mathbf{R}^1$, $r \neq f(x_0)$ the set*

$$\lambda(V \cap \mathcal{U} \cap \{x \mid f(x) = r\})$$

is at most finite.

Proof. If $x_0 \notin V$, then the above assertion holds trivially, hence we can suppose that $x_0 \in V$.

Let us denote

$$W = \{x \in \Omega \mid \nabla f(x) = 0\} .$$

From Theorem 2.6 (where we set $V = \Omega$) we obtain that the function f is locally constant on W, hence we can choose a neighborhood $\mathcal{U}$ of x_0 in such a way that $f(x) = f(x_0)$ for all $x \in W \cap \mathcal{U}$.

If $W \supset V$, then the assertion of Lemma holds with the above $\mathcal{U}$, hence we can suppose that $V \not\subset W$.

First part. First we prove that for every $r \neq f(x_0)$, the function λ is locally constant on the set $V \cap \mathcal{U} \cap \{x \mid f(x) = r\}$. As in the proof of Theorem 3.8 we can choose a neighborhood $\Omega^* \subset \mathbf{C}^n$ of the set Ω such that it holds

(i) $\Omega^* \cap \mathbf{R}^n = \Omega$,

(ii) there exist functions f^*, g^*, λ^* holomorphic in Ω^* such that

$$f^*|_\Omega = f\,, \quad g^*|_\Omega = g\,, \quad \lambda^*|_\Omega = \lambda\,,$$

(iii) there exists a (complex) analytic set $V^* \subset \Omega^*$ such that $V^* \cap \Omega = V$ and we can suppose that V^* is the least set with this property, thus (see Theorem C2) V^* is irreducible,

(iv) the set

$$B^* = \{z \in \Omega^* \mid \lambda^*(z)\, \nabla f^*(z) = \nabla g^*(z)\}$$

contains V and hence also V^*.

Let us denote

$$W^* = \left\{z \in \Omega^* \mid \sum_1^n \left(\frac{\partial f^*}{\partial z_i}\right)(z) = 0\right\}.$$

If $W^* \supset B^*$, then also $W^* \supset V^*$, hence $W \supset V$ which is a contradiction.

Now, let us consider the analytic set $B^* \setminus W^*$ in the open set $\Omega^* \setminus W^*$. From Theorem B5 (iv) it follows that $B^* \setminus W^*$ is an irreducible analytic set in $\Omega^* \setminus W^*$. The function f^* is not constant on $B^* \setminus W^*$, since $B^* \setminus W^*$ is dense in B^* (see Theorem B5 (iv)) and by assumption f^* is not constant on $V \subset B^*$.

Now it is sufficient to use Lemma 3.6. Let $r \in \mathbf{R}^1$, $r \neq f(x_0)$. Then the set $V \cap \mathscr{U} \cap \{x \mid f(x) = r\}$ is contained in the set $[V \setminus W] \cap \{x \mid f(x) = r\}$ and hence also in the set $(B^* \setminus W^*) \cap \{z \mid f^*(z) = r\}$ but it follows from Lemma 3.6 that λ^* is locally constant on this set.

Second part. Let us choose a neighborhood $\mathscr{U}'$ of x_0 such that $\overline{\mathscr{U}'}$ (closure in $\mathbf{R}^n$) is compact and $\overline{\mathscr{U}'} \subset \mathscr{U}$.

Let r, $r \neq f(x_0)$ be fixed. We can suppose that there exists a point $y_0 \in \mathscr{U}'$ such that $f(y_0) = r$ (in the opposite case $\lambda(V \cap \mathscr{U}' \cap \{x \mid f(x) = r\}) = \emptyset$).

Then it holds obviously

$$(*) \qquad [(V \cap \mathscr{U}) \setminus W] \cap \{x \in \mathscr{U} \setminus W \mid f(x) = f(y_0)\} = V \cap \mathscr{U} \cap \{x \in \Omega \mid f(x) = r\}$$

and this set is clearly a **C**-analytic set in $\mathscr{U}$. Since the set $\overline{U}'$ is compact, there exists at most finite number of **C**-irreducible branches of this set which intersect $\mathscr{U}'$. Each such **C**-irreducible branch is connected and the function λ is locally constant on the set $(*)$. Hence

$$\lambda(V \cap \mathscr{U}' \cap \{x \in \Omega \mid f(x) = r\})$$

is at most finite set.

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