

# ON THE SPECTRUM OF A ONE-PARAMETER STRONGLY CONTINUOUS REPRESENTATION

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Let  $\alpha_t$  be a bounded strongly continuous one-parameter group on a Banach space  $X$ , with infinitesimal generator  $iZ$ . If  $\text{sp } \alpha$  denotes the spectrum introduced by Arveson [1] of the representation  $\alpha$ , we show that  $\text{sp } \alpha = \sigma Z$ . This generalises Olesen's result [3, 2.5] for norm continuous one-parameter groups of isometries.

We recall [1] that if  $f \in L^1(\mathbb{R})$ ,

$$Ax = \int \alpha_t(x) f(t) dt \quad x \in X,$$

defines a bounded linear operator  $A$  on  $X$ . Thus we can lift  $\alpha$  to a representation  $\pi_\alpha$  of  $L^1(\mathbb{R})$ . The spectrum of  $\alpha$ , written  $\text{sp } \alpha$ , is defined as the hull of the ideal kernel  $(\pi_\alpha)$ .

The following proposition and the usual characterisation of  $\text{sp } \alpha$  (e.g. [4, 2.4.1]) prove our claim that  $\text{sp } \alpha = \sigma Z$ . If  $I$  is a subset of  $\mathbb{R}$ , we will denote by  $L_I^1$ , the set of  $f$  in  $L^1(\mathbb{R})$  such that  $\hat{f}$  has support in  $I$ .

**PROPOSITION.** *Let  $I$  be a compact interval in  $\mathbb{R}$ . Then*

- i)  $I \subseteq \varrho(Z)$  implies that  $\pi_\alpha(f) = 0$  for all  $f \in L_I^1$ .
- ii)  $\pi_\alpha(f) = 0$  for all  $f \in L_I^1$  implies that  $I^0 \subseteq \varrho(Z)$ .

**PROOF.** The proof is inspired by that of [2, Lemma 5.5].

- i) Take  $f \in L_I^1$ . Then

$$f(t) = \int_{-\infty}^{+\infty} e^{-ita} \hat{f}(a) da \quad \text{a.e. in } \mathbb{R}.$$

For  $\delta > 0$ , define

$$\begin{aligned} f_\delta^+(t) &= \int_{-\infty}^{+\infty} \hat{f}(a) \exp(-iat - \delta t) da & \text{for } t > 0 \\ f_\delta^-(t) &= \int_{-\infty}^{+\infty} \hat{f}(a) \exp(-iat + \delta t) da & \text{for } t < 0. \end{aligned}$$

Then

$$f_\delta^+(t) - f(t) = f(t)(e^{-\delta t} - 1) \quad \text{for } t > 0,$$

and

$$f_{\delta}^{-}(t) - f(t) = f(t)(e^{+\delta t} - 1) \quad \text{for } t < 0.$$

Hence by dominated convergence, there exists  $\delta_0 > 0$  such that

$$\int_0^{\infty} |f_{\delta}^{+}(t) - f(t)| dt, \int_{-\infty}^0 |f_{\delta}^{-}(t) - f(t)| dt < \varepsilon/2 \quad \text{for } 0 < \delta < \delta_0.$$

Since  $\|\alpha_t x\| \leq M\|x\|$  for some  $M < \infty$ , all  $t \in \mathbb{R}$ , and

$$\pi_{\alpha}(f)x = \int_0^{\infty} f(t)\alpha_t(x) dt + \int_{-\infty}^0 f(t)\alpha_t(x) dt \quad \forall x \in X$$

we get for  $0 < \delta < \delta_0$  that

$$\begin{aligned} \|\pi_{\alpha}(f)x\| &\leq M\|x\|\varepsilon + \|\int_0^{\infty} \int_{-\infty}^{\infty} \alpha_t(x) \hat{f}(a) \exp(-iat - \delta t) da dt + \\ &\quad + \int_{-\infty}^0 \int_{-\infty}^{\infty} \alpha_t(x) \hat{f}(a) \exp(-iat + \delta t) da dt\| \\ &= M\|x\|\varepsilon + \|\int_{-\infty}^{\infty} \hat{f}(a) \int_0^{\infty} \exp(-iat - \delta t) \alpha_t(x) dt da + \\ &\quad + \int_{-\infty}^{\infty} \hat{f}(a) \int_{-\infty}^0 \exp(-iat + \delta t) \alpha_t(x) dt da\| \\ &= M\|x\|\varepsilon + \|\int_{-\infty}^{\infty} \hat{f}(a)[R(a - i\delta, Z)x - R(a + i\delta, Z)x] da\|. \end{aligned}$$

The integral tends to zero as  $\delta \downarrow 0$ , since  $\hat{f}$  has compact support in  $I$  and  $I \subseteq \varrho(Z)$ . Hence  $\pi_{\alpha}(f) = 0$ .

(ii) Suppose  $\pi_{\alpha}(f) = 0 \quad \forall f \in L_I^1$ .

We restrict ourselves to  $f \in \mathcal{S}(\mathbb{R})$ , such that  $\hat{f}$  has support in  $I$ . Then with the same notation, for  $\delta > 0$

$$\begin{aligned} \int_0^{\infty} |f_{\delta}^{+}(t) - f(t)| dt &= \int_0^{\infty} |f(t)| |e^{-\delta t} - 1| dt \\ &= \int_0^{\infty} (1 + t^2) |f(t)| |e^{-\delta t} - 1| / (t^2 + 1) dt \\ &\leq \varepsilon \int_0^{\infty} |f(t)|^2 (1 + t^2)^2 dt. \end{aligned}$$

for suitably small  $\delta$ , independently of  $f$ .

Then looking at the previous manipulations we see

$$\|\int_I [R(a - i\delta, Z) - R(a + i\delta, Z)] \times \hat{f}(a) da\| \leq |\hat{f}| \varepsilon \|x\| M,$$

where  $|\cdot|$  is a continuous seminorm on  $\mathcal{S}(\mathbb{R})^{\wedge}$ .

We now apply an edge of the wedge theorem [5, Theorem 2.16] and deduce that  $R(\lambda, Z)$  is analytic for  $\lambda \in I^0$ . Hence  $I^0 \subseteq \varrho Z$ .

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## REFERENCES

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