ON THE SPECTRUM OF THE LAPLACIAN ON REGULAR METRIC TREES

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Abstract. A metric tree is a tree whose edges are viewed as line segments of positive length. The Laplacian $\mathbf{\Delta}$ on such tree is the operator of second derivative on each edge, complemented by the Kirchhoff matching conditions at the vertices. The spectrum of $\mathbf{\Delta}$ can be quite different, reflecting geometry of the tree.

We discuss a special case of trees, namely the so-called regular trees. They possess a rich group of symmetries. This allows one to construct an orthogonal decomposition of the space $\mathsf{L}^2(\Gamma)$ which reduces the Laplacian. Based upon this decomposition, a detailed spectral analysis of $\mathbf{\Delta}$ on the regular metric trees is possible. In the preprint a survey of the recent results on the subject are presented.

1. introduction

In the classical graph theory a graph is considered as a combinatorial object. A function on such graph is a function defined on the set of its vertices, and the Laplacian is a discrete operator.

Opposite to this, a metric graph $\mathcal{G}$ is a graph whose edges are regarded as non-degenerate line segments. A function on $\mathcal{G}$ is a family of functions defined on its edges, and the Laplacian on $\mathcal{G}$ acts as $\mathbf{\Delta} f = -f''$; we include the sign "$-$" in the definition of the Laplacian. Functions from its domain satisfy certain matching conditions at the vertices. The spectral theory of the Laplacian on metric graphs is much less developed than its counterpart for the discrete Laplacian.

Regular metric trees form an important sub-class of general metric graphs. They possess a very rich group of symmetries. This allows one to construct an orthogonal decomposition (*the basic decomposition*) of the space $\mathsf{L}^2(\Gamma)$ which reduces the Laplacian. Based upon this decomposition, a detailed spectral analysis of the Laplacian on regular metric trees is possible.

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Our goal is to present a survey of the known results on the spectrum of the Laplacian on such trees. The paper can be considered as an expanded version of the survey given in [17]. Many results are presented here with more detail, but we often give references and informal explanations rather than rigorous proofs. Among the few new results we specially mention Theorem 5.3 and Example 6.2. On the other hand, we do not reproduce the results of [17] concerning the Schrödinger operator.

There are several papers, devoted to differential operators on regular metric trees and to related problems. In [4] Hill's equation on the *homogeneous trees* was considered. A tree is called homogeneous if all its edges have equal length and all its vertices are of the same degree. In particular, the band - gap structure of the spectrum of the Laplacian on such trees was revealed in [4].

In [10] the weighted spectral problems of the type $\boldsymbol{\Delta} f = \lambda V f$, where $V \in \mathsf{L}^1(\Gamma)$ is a non-negative weight function, were investigated. In general, the tree Γ was not assumed regular. For the regular trees, in [10] the basic decomposition of $\mathsf{L}^2(\Gamma)$ was constructed. It is reproduced here as the formula (3.2). With its help for such trees much more advanced results were obtained than in the general case. Let us note that for the combinatorial trees a decomposition similar to (3.2) was used before, see e.g. [13] and [1].

In the paper [5] the basic decomposition was rediscovered and applied to the spectral analysis of the Laplacian and the Schrödinger operator on the regular trees Γ of finite height. In particular, it was proved in [5] that for such trees each operator $\mathbf{A}_k$ appearing in the decomposition (3.14) is compact. Our Theorem 4.1 substantially refines this result.

In [11] a new, more detailed exposition of the results on the basic decomposition of $\mathsf{L}^2(\Gamma)$ was given. The scheme was applied to the analysis of the Hardy-type inequalities on regular trees. As a consequence, a necessary and sufficient condition of the positive definiteness of the Laplacian (Theorem 5.2 of the present paper) was established.

It is necessary to mention here also the papers [6], [7] and [8], though formally they do not deal with the Laplacian on trees. In [6] the Hardy-type integral operators on trees were introduced in connection with the spectral analysis of the Neumann Laplacian in certain irregular domains. There is a close relation between the approximation numbers of the Hardy-type integral operators in $\mathsf{L}^2(\Gamma)$ and the eigenvalues of the problem $\mathbf{\Delta} f = \lambda V f$ on the tree. It was the paper [6] which attracted the author's attention to operators on trees.

In the paper [8] a criterion on a function $V \geq 0$ to be a "Hardy weight" in $\mathsf{L}^p(\Gamma)$ was established. For $p = 2$ the term means that the inequality $\int_\Gamma V|f|^2 dx \leq c \int_\Gamma |f'|^2 dx$ holds for any function f for which the integral in the right-hand side is finite. The result of [8] does not apply to the problem of positive definiteness of the Laplacian, since the latter requires a Hardy-type inequality on another (narrower) function space; see [11] for a detailed discussion.

In [7] the behaviour of the approximation numbers of the Hardy-type integral operators on trees is studied in detail, not only in L^2-case but also in the general L^p-case. When applied to the Laplacian, the estimates obtained for $p = 2$ refine some results of [10].

In the paper [15] a new approach to the eigenvalue estimates for the equation $\mathbf{\Delta} f = \lambda V f$ on metric graphs (not necessarily trees) was developed. As one of applications, the Weyl-type asymptotics for this equation was justified. Our Theorem 4.1 (ii) is a special case of this result. Its another proof was given in [17]. The asymptotic results of the paper [7] also can serve as a basis for obtaining this theorem.

The approach of [15] was based upon a special techniques of approximation of functions from the Sobolev space $\mathsf{H}^1 = \mathsf{W}_2^1$ on graphs. In [16] this approach was extended to the spaces W_p^1 with $p \neq 2$.

In [14] a detailed spectral analysis of the Laplacian and the Schrödinger operator on the homogeneous trees was carried out. One of the results of [14] is reproduced below as Example 6.3.

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2. Regular rooted metric trees

2.1. **Geometry of a tree.** Let Γ be a rooted tree with the root o, the set of vertices $\mathcal{V} = \mathcal{V}(\Gamma)$ and the set of edges $\mathcal{E} = \mathcal{E}(\Gamma)$. We suppose that $\#\mathcal{V} = \#\mathcal{E} = \infty$. Each edge e of a metric tree is viewed as a non-degenerate line segment of length $|e|$. The distance $\rho(x, y)$ between any two points $x, y \in \Gamma$, and thus the metric topology on Γ, is introduced in a natural way, and $|x|$ stands for $\rho(x, o)$. A subset $E \subset \Gamma$ is compact if and only if it is closed and has non-empty intersections with only a finite number of edges.

For any two points $x, y \in \Gamma$ there exists a simple polygonal path in Γ which starts at x and terminates at y. This path is unique and we denote it by $\langle x, y\rangle$. We write $x \prec y$ if $x \in \langle o, y\rangle$ and $x \neq y$. The relation $\prec$ defines a partial ordering on Γ.

For any vertex v its *generation* $\operatorname{gen}(v)$ is defined as

$$\operatorname{gen}(v) = \#\{x \in \mathcal{V}(\Gamma) : x \prec v\}.$$

In particular, $v = o$ is the only vertex such that $\operatorname{gen}(v) = 0$. For any edge emanating from the vertex v (which means that $e = \langle v, w\rangle$ and $v \prec w$) we define its generation as $\operatorname{gen}(e) := \operatorname{gen}(v)$.

The *branching number* $b(v)$ of a vertex v is defined as the number of edges emanating from v. We assume that $\operatorname{gen}(v) < \infty$ for any v and $b(v) > 1$ for $v \neq o$. We denote by e_v^- the only edge which terminates at a vertex $v \neq o$, and by $e_v^1, \dots, e_v^{b(v)}$ the edges emanating from $v \in \mathcal{V}$.

Definition 2.1. We call a tree Γ *regular* if all the vertices of the same generation have equal branching numbers, and all the edges of the same generation are of the same length.

Evidently, any regular tree is fully determined by specifying two number sequences (*generating sequences*) $\{b_n\} = \{b_n(\Gamma)\}$ and $\{t_n\} = \{t_n(\Gamma)\}$, $n = 0, 1, \dots$ such that

$$b(v) = b_{\operatorname{gen}(v)}, \ |v| = t_{\operatorname{gen}(v)} \qquad \forall v \in \mathcal{V}(\Gamma).$$

According to our assumptions, $b_n \geq 2$ for any $n > 0$. It is clear that $t_0 = 0$ and the sequence $\{t_n\}$ is strictly increasing, and we denote

$$h(\Gamma) = \lim_{n\to\infty} t_n = \sup_{x\in\Gamma} |x|. \tag{2.1}$$

We call $h(\Gamma)$ *the height of* Γ. Another useful characteristics of the regular tree is its *branching function*

$$g_\Gamma(t) = \#\{x \in \Gamma : |x| = t\}, \ 0 \leq t < h(\Gamma).$$

Clearly,

$$g_\Gamma(0) = 1; \qquad g_\Gamma(t) = b_0 \dots b_n, \ t_n < t \leq t_{n+1}, \ n = 0, 1, \dots \tag{2.2}$$

We also introduce *the reduced height of* Γ

$$L(\Gamma) = \int_0^{h(\Gamma)} \frac{dt}{g_\Gamma(t)}. \tag{2.3}$$

Clearly $h(\Gamma) < \infty$ implies $L(\Gamma) < \infty$. For the trees of infinite height both $L(\Gamma) < \infty$ and $L(\Gamma) = \infty$ is possible.

The natural measure dx on Γ is induced by the Lebesgue measure on the edges. The spaces $\mathsf{L}^p(\Gamma)$ are understood as L^p-spaces with respect to this measure. We denote by $|E|$ the measure of a (measurable) subset $E \subset \Gamma$ and call the number $|\Gamma|$ *the total length* of Γ. It is clear that $\rho(x, y) = |\langle x, y\rangle|$ for any pair of points $x, y \in \Gamma$ and that $|\Gamma| = \int_\Gamma g_\Gamma(t)dt$.

2.2. **Special subtrees of Γ.** Subtrees $T \subset \Gamma$ of the following two types play a special part in the further analysis. For any vertex v and for any edge $e = \langle v, w\rangle,\ v \prec w$ we set

$$T_v = \{x \in \Gamma : x \succeq v\}, \qquad T_e = e \cup T_w.$$

Evidently, $T_o = \Gamma$ and

$$T_v = \bigcup_{1\le j\le b(v)} T_{e_v^j}, \qquad \forall v \in \mathcal{V}(\Gamma).$$

Due to the regularity of Γ, all the subtrees $T_e,\ \mathrm{gen}(e) = k$ can be identified with a single tree Γ_k whose generating sequences are

$$b_0(\Gamma_k) = 1, \qquad b_n(\Gamma_k) = b_{k+n}(\Gamma),\ n \in \mathbb{N};$$
$$t_0(\Gamma_k) = 0; \qquad t_n(\Gamma_k) = t_{k+n}(\Gamma) - t_k(\Gamma),\ n \in \mathbb{N}.$$

It follows from here and from (2.2) that the branching function g_{Γ_k} is given by

$$g_{\Gamma_k}(t) = \frac{g_\Gamma(t_k + t)}{b_0 \dots b_k} = \frac{g_\Gamma(t_k + t)}{g_\Gamma(t_k+)}, \qquad k = 0, 1, \dots \tag{2.4}$$

Note also that any subtree $T_v,\ \mathrm{gen}(v) = k$, can be identified with the union of b_k copies of the tree Γ_k emanating from the common root v.

3. The Laplacian on a regular tree

The notion of differential operator on any metric graph, in particular on a tree, is well known. Still, for the sake of completeness we present here the variational definition of the Dirichlet Laplacian on a tree.

3.1. **The operator $\boldsymbol{\Delta}$.** We say that a scalar-valued function f on Γ belongs to the Sobolev space $\mathsf{H}^1 = \mathsf{H}^1(\Gamma)$ if f is continuous, $f \restriction e \in \mathsf{H}^1(e)$ for each edge e, and

$$\|f\|^2_{\mathsf{H}^1} := \int_\Gamma \big(|f'(x)|^2 + |f(x)|^2\big)dx < \infty.$$

The derivative of a function $f \restriction e$ at an interior point $x \in e$ is always taken in the direction compatible with the partial ordering on Γ. This agreement is indifferent for the definition of H^1 but we shall use it later. By $\mathsf{H}^{1,0} = \mathsf{H}^{1,0}(\Gamma)$ we denote the subspace $\{f \in \mathsf{H}^1 : f(o) = 0\}$.

Let H^1_c stand for the set of all functions from H^1 having compact support and $\mathsf{H}^{1,0}_c = \mathsf{H}^1_c \cap \mathsf{H}^{1,0}$.

Lemma 3.1. *Let* $h(\Gamma)=\infty$. *Then* H^1_c *is dense in* H^1, *and therefore* $\mathsf{H}^{1,0}_c$ *is dense in* $\mathsf{H}^{1,0}$.

Proof. For any number $L>0$ let $\varphi_L(t)$ be the continuous function on $\mathbb{R}_+$, which is 1 for $t\le L$, is 0 for $t\ge L+1$ and is linear on $[L,L+1]$. Given a function $f\in\mathsf{H}^1(\Gamma)$, denote $f_L(x)=\varphi_L(|x|)f(x)$. Then $f_L\in\mathsf{H}^1_{\mathrm{c}}$ and an elementary calculation shows that $f_L\to f$ in H^1 as $L\to\infty$. □

We define the (positive) Dirichlet and Neumann Laplacians $\boldsymbol{\Delta}_D$ and $\boldsymbol{\Delta}_N$ as the self-adjoint operators in $\mathsf{L}^2(\Gamma)$, associated with the quadratic form $\int_\Gamma|f'|^2dx$ considered on the form domains

$$\mathrm{Quad}(\boldsymbol{\Delta}_D)=\mathsf{H}^{1,0}(\Gamma),\qquad \mathrm{Quad}(\boldsymbol{\Delta}_N)=\mathsf{H}^1(\Gamma)$$

respectively. The difference between the properties of the operators $\boldsymbol{\Delta}_D$ and $\boldsymbol{\Delta}_N$ is minor, and for definiteness we address the first of them in the most part of the paper. For this reason, we use for it a shortened notation

$$\boldsymbol{\Delta}:=\boldsymbol{\Delta}_D.$$

It is easy to describe the operator domain $\mathrm{Dom}(\boldsymbol{\Delta})$ and the action of $\boldsymbol{\Delta}$. Evidently $f\in\mathrm{Dom}(\boldsymbol{\Delta})\Rightarrow f\restriction e\in\mathsf{H}^2(e)$ for each edge e and the Euler – Lagrange equation reduces on e to $\boldsymbol{\Delta}f=-f''$. At the root we have the boundary condition $f(o)=0$ which appears in the definition of $\mathrm{Quad}(\boldsymbol{\Delta})$. At each vertex $v\neq o$ the functions $f\in\mathrm{Dom}(\boldsymbol{\Delta})$ satisfy certain matching conditions. In order to describe them, denote by f_- the restriction $f\restriction e_v^-$ and by $f_j,\ j=1,\dots,b(v)$ the restrictions $f\restriction e_v^j$. The matching conditions at $v\neq o$ are

$$f_-(v)=f_1(v)=\ldots=f_{b(v)}(v);\quad f_1'(v)+\ldots+f_{b(v)}'(v)=f_-'(v). \tag{3.1}$$

These are nothing but the Kirchhoff laws well known in the theory of electrical networks. The first condition in (3.1) comes from the requirement $f\in\mathsf{H}^1(\Gamma)$ which includes continuity of f, and the second arises as the natural condition in the sense of Calculus of Variations. The conditions listed are also sufficient for $f\in\mathrm{Dom}(\boldsymbol{\Delta})$.

Due to the boundary condition $f(o)=0$, the Dirichlet Laplacian on Γ splits into the orthogonal sum of the Dirichlet Laplacians on the subtrees whose initial edges are $e_o^1,\dots,e_o^{b(o)}$. For this reason, in what follows we assume $b(o)=1$ (except for Subsection 7.2). Under the latter assumption, the only difference for the Neumann Laplacian $\boldsymbol{\Delta}_N$ is that the boundary condition $f(o)=0$ is replaced by $f'(o)=0$.

3.2. **Reduction of the Laplacian.** Our further analysis is based upon an orthogonal decomposition of the space $\mathsf{L}^2(\Gamma)$ which, for the case of regular trees, reduces the Laplacian. Let us describe this decomposition.

Given a subtree $T \subset \Gamma$, we say that a function $f \in \mathsf{L}^2(\Gamma)$ belongs to the set (a closed subspace) $\mathcal{F}_T$ if

$$f(x) = 0 \text{ for } x \notin T; \qquad f(x) = f(y) \qquad \text{if } x, y \in T \text{ and } |x| = |y|.$$

In particular, $\mathcal{F}_\Gamma$ consists of all *symmetric* (i.e. depending only on $|x|$) functions from $\mathsf{L}^2(\Gamma)$.

We need the subspaces $\mathcal{F}_T$ associated with the subtrees T_e and T_v, introduced in Subsection 2.2. To simplify our notations, we shall write $\mathcal{F}_e$, $\mathcal{F}_v$ instead of $\mathcal{F}_{T_e}, \mathcal{F}_{T_v}$. It is clear that for each vertex $v \neq o$ the subspaces $\mathcal{F}_{e_v^j}, j = 1, \dots, b(v)$ are mutually orthogonal and their orthogonal sum $\widetilde{\mathcal{F}_v}$ contains $\mathcal{F}_v$. Denote

$$\mathcal{F}'_v = \widetilde{\mathcal{F}_v} \ominus \mathcal{F}_v.$$

Theorem 3.2. *Let Γ be a regular metric tree and $b(o) = 1$. Then the subspaces $\mathcal{F}'_v$, $v \in \mathcal{V}(\Gamma)$ are mutually orthogonal and orthogonal to $\mathcal{F}_\Gamma$. Moreover,*

$$\mathsf{L}^2(\Gamma) = \mathcal{F}_\Gamma \oplus \sum_{v \in \mathcal{V}(\Gamma)} \oplus \mathcal{F}'_v, \tag{3.2}$$

and this decomposition reduces the Dirichlet Laplacian and the Neumann Laplacian on Γ.

Now we have to describe the parts of the Laplacian in the subspaces $\mathcal{F}_\Gamma$ and $\mathcal{F}'_v$. For this purpose, along with the operator $\mathbf{\Delta}$ on Γ let us consider the Dirichlet Laplacian on each tree Γ_k defined in Subsection 2.2. Below we denote this operator by $\mathbf{\Delta}_k$. Consider also its part $\mathfrak{A}_k = \mathbf{\Delta}_k \restriction \mathcal{F}_{\Gamma_k}$ which is a natural analog of the operator $\mathfrak{A}_0 = \mathbf{\Delta} \restriction \mathcal{F}_\Gamma$.

Theorem 3.3. *Let $v \in \mathcal{V}(\Gamma)$ and $\operatorname{gen}(v) = k > 0$. Then the operator $\mathbf{\Delta} \restriction \mathcal{F}'_v$ is unitarily equivalent to the orthogonal sum of $(b_k - 1)$ copies of the operator $\mathfrak{A}_k$.*

The proof is given in [11]. A naive but justifiable explanation is that $\mathbf{\Delta} \restriction \mathcal{F}_v$ is the orthogonal sum of b_k copies of the operator $\mathfrak{A}_k$ and the passage to $\mathbf{\Delta} \restriction \mathcal{F}'_v$ corresponds to the withdrawal of one of these copies.

Our next step is to understand the nature of each operator $\mathfrak{A}_k$. Below we introduce a family $\{\mathbf{A}_k\}$, $k = 0, 1, \dots$ of operators acting in $\mathsf{L}^2(t_k, h(\Gamma))$ and then show that $\mathfrak{A}_k \sim \mathbf{A}_k$ for each k; here and in the sequel the symbol "$\sim$" stands for the unitary equivalence.

Denote $I_j = (t_{j-1}, t_j),\ j \in \mathbb{N}$. The domain $\mathrm{Dom}(\mathbf{A}_k)$ consists of all functions u on $[t_j, h(\Gamma))$, such that $u\restriction I_j \in \mathsf{H}^2(I_j)$ for any $j > k$,

$$\sum_{j>k} \int_{I_j} \big(|u''|^2 + |u|^2\big) dt < \infty, \tag{3.3}$$

and the following boundary condition at t_k and the matching conditions at the points $t_j,\ j > k$ are satisfied:

$$u(t_k) = 0; \qquad u(t_j+) = b_j^{1/2} u(t_j-), \qquad j > k; \tag{3.4}$$

$$u'(t_j+) = b_j^{-1/2} u'(t_j-), \qquad j > k. \tag{3.5}$$

The operator $\mathbf{A}_k$ acts on this domain as

$$(\mathbf{A}_k u)(t) = -u''(t), \qquad t \neq t_j,\ j \geq k. \tag{3.6}$$

It is self-adjoint and non-negative. Its quadratic domain is

$$\mathrm{Quad}(\mathbf{A}_k) = \big\{u \in \mathsf{L}^2(t_k, h(\Gamma)) : u\restriction I_j \in \mathsf{H}^1(I_j)\ \forall j > k, \\ \sum_{j>k} \int_{I_j} |u'|^2 dt < \infty,\ \text{and the conditions (3.4) are satisfied}\big\}. \tag{3.7}$$

The quadratic form of $\mathbf{A}_k$ is

$$\mathbf{a}_k[u] = \sum_{j>k} \int_{I_j} |u'|^2 dt, \qquad u \in \mathrm{Quad}(\mathbf{A}_k). \tag{3.8}$$

Lemma 3.4. *For any $k = 0, 1, \dots$*

$$\mathfrak{A}_k \sim \mathbf{A}_k. \tag{3.9}$$

Proof. We prove (3.9) for $k = 0$, the rest is similar. It is natural to identify a function $f \in \mathfrak{F}_\Gamma$ with the function φ_f on $[0, h(\Gamma))$ satisfying $f(x) = \varphi_f(t)$ for any $x \in \Gamma$ such that $|x| = t$. Then

$$\int_\Gamma |f(x)|^2 dx = \int_0^{h(\Gamma)} |\varphi_f(t)|^2 g_\Gamma(t) dt,\ \forall f \in \mathfrak{F}_\Gamma. \tag{3.10}$$

If $f \in \mathsf{H}^{1,0}(\Gamma)$, then φ_f is continuous on $[0, h(\Gamma))$, has the distributional derivative $\varphi_f' \in \mathsf{L}^2_{\mathrm{loc}}(0, h(\Gamma))$, satisfies $\varphi_f(0) = 0$, and

$$\int_\Gamma |f'(x)|^2 dx = \int_0^{h(\Gamma)} |\varphi_f'(t)|^2 g_\Gamma(t) dt,\ \forall f \in \mathsf{H}^{1,0}(\Gamma) \cap \mathfrak{F}_\Gamma. \tag{3.11}$$

We also see that

$$f \in \mathrm{Quad}(\boldsymbol{\Delta}\restriction \mathfrak{F}_\Gamma) \Leftrightarrow \varphi_f \in \mathsf{H}^{1,\bullet}\big((0, h(\Gamma)); g_\Gamma\big) \tag{3.12}$$

where $\mathsf{H}^{1,\bullet}\big((0,h(\Gamma));g_\Gamma\big)$ stands for the weighted Sobolev space which is defined by the conditions

$$\int_0^{h(\Gamma)}(|\varphi'(t)|^2+|\varphi(t)|^2)g_\Gamma(t)dt<\infty;\ \varphi(0)=0. \tag{3.13}$$

Correspondingly, $\mathfrak{A}_0$ turns into the operator in the weighted space $\mathsf{L}^2\big((0,h(\Gamma));g_\Gamma\big)$, associated with the quadratic form in the right-hand side of the equality in (3.11).

Now we substitute $u(t)=\varphi(t)\sqrt{g_\Gamma(t)}$. Then $u'(t)=\varphi'(t)\sqrt{g_\Gamma(t)}$ for $t\neq t_1,t_2,\dots$, since $g_\Gamma(t)$ is constant on each interval I_j. Clearly $u(0)=0$; at any point $t_j,\ j\in\mathbb{N}$ the function u meets the matching condition $u(t_j+)=b_j^{1/2}u(t_j-)$ which comes from the continuity of φ. All this shows that $\|f\|^2_{\mathsf{L}^2(\Gamma)}=\|u\|^2_{\mathsf{L}^2(0,h(\Gamma))}$,

$$f\in\mathrm{Quad}(\mathfrak{A}_0)\Leftrightarrow u\in\mathrm{Quad}(\mathbf{A}_0)$$

and $\|f'\|^2_{\mathsf{L}^2(\Gamma)}=\mathbf{a}_0[u]$. It follows that $\mathfrak{A}_0\sim\mathbf{A}_0$. □

The outcome of this analysis is the following result. It was proved in [11] for the general case of Schrödinger operators. Below $\mathbf{A}^{[r]}$ stands for the orthogonal sum of r copies of a self-adjoint operator $\mathbf{A}$.

Theorem 3.5. *Let Γ be the regular tree with the generating sequences $\{b_n\}$ (with $b_0=1$) and $\{t_n\}$. Then*

$$\mathbf{\Delta}\sim\mathbf{A}_0\oplus\sum_{k=1}^{\infty}\oplus{\mathbf{A}_k}^{[b_0\dots b_{k-1}(b_k-1)]}. \tag{3.14}$$

3.3. Spectrum of $\mathbf{A}_0$ and spectrum of $\mathbf{\Delta}$.

We conclude from (3.7) and (3.8) that

$$\mathrm{Quad}(\mathbf{A}_0)\supset\mathrm{Quad}(\mathbf{A}_1)\supset\mathrm{Quad}(\mathbf{A}_2)\supset\dots$$

and

$$\mathbf{a}_k=\mathbf{a}_0\restriction\mathrm{Quad}(\mathbf{A}_k),\qquad\forall k\in\mathbb{N}.$$

By the variational principle, this implies that the spectral properties of all the operators $\mathbf{A}_k$ and of the whole operator $\mathbf{\Delta}$ are basically determined by the properties of the single operator $\mathbf{A}_0$. In particular, the following statement holds. As usual, we denote by $\sigma(\mathbf{A})$ and $\sigma_p(\mathbf{A})$ the spectrum and the point spectrum of a self-adjoint operator $\mathbf{A}$.

Theorem 3.6. *Let $\mathbf{A}_k, k=0,1,\dots$ be the operators in $\mathsf{L}^2(t_k,h(\Gamma))$ whose domain and action are defined by the equations* (3.3) – (3.6). *Then* *(i) If* $\mathbf{A}_0$ *is positive definite, then the same is true for any operator* $\mathbf{A}_k,\ k\in\mathbb{N}$*, and*

$$\min\sigma(\mathbf{A}_0)\le\min\sigma(\mathbf{A}_1)\le\ldots\le\min\sigma(\mathbf{A}_k)\le\ldots$$

(ii) If the spectrum of $\mathbf{A}_0$ *is discrete, then the same is true for any operator* $\mathbf{A}_k,\ k\in\mathbb{N}$.

(iii) If the spectrum of $\mathbf{A}_0$ *is discrete, then*

$$\min\sigma(\mathbf{A}_k)\to\infty \text{ as } k\to\infty. \tag{3.15}$$

Note that the statement Theorem 3.6 (iii) does not follow from the above construction and needs a separate proof. It will be given in Section 4 (see (4.4)) for the trees of finite height and in the end of Subsection 5.3 for the trees of infinite height.

It follows from Theorem 3.2 that

$$\sigma_p(\mathbf{\Delta})=\bigcup_{k=0}^{\infty}\sigma_p(\mathbf{A}_k);\qquad \sigma(\mathbf{\Delta})=\overline{\bigcup_{k=0}^{\infty}\sigma(\mathbf{A}_k)}. \tag{3.16}$$

Together with Theorem 3.6, this leads to the following result.

Corollary 3.7. *(i) The Dirichlet Laplacian* $\mathbf{\Delta}$ *on a regular tree is positive definite if and only if the operator* $\mathbf{A}_0$ *is positive definite. Moreover,*

$$\inf\sigma(\mathbf{\Delta})=\inf\sigma(\mathbf{A}_0).$$

(ii) The spectrum of $\mathbf{\Delta}$ *is discrete if and only if the spectrum of* $\mathbf{A}_0$ *is discrete.*

4. Laplacian on regular trees of finite height

Here we assume $h(\Gamma)<\infty$, cf. (2.1).

Theorem 4.1. *(i) Let* Γ *be a regular tree and* $h(\Gamma)<\infty$*. Then the spectrum* $\sigma(\mathbf{\Delta})$ *is discrete.*

(ii) Suppose in addition that $|\Gamma|<\infty$*. Then the Weyl asymptotic formula for the eigenvalue counting function of* $\mathbf{\Delta}$ *is satisfied:*

$$N(\lambda;\mathbf{\Delta})=\pi^{-1}|\Gamma|\lambda^{1/2}(1+o(1)),\qquad \lambda\to\infty. \tag{4.1}$$

Proof. (i) First of all, we prove that the spectrum of $\mathbf{A}_0$ is discrete. Taking (3.10) and (3.11) into account and using the variational description of the spectrum, we find that it is sufficient to check that the Rayleigh quotient

$$\frac{\int_0^{h(\Gamma)}|\varphi(t)|^2g_\Gamma(t)dt}{\int_0^{h(\Gamma)}|\varphi'(t)|^2g_\Gamma(t)dt},\qquad \varphi\in \mathsf{H}^1_{\mathrm{loc}}\big((0,h(\Gamma)),g_\Gamma\big),\ \varphi(0)=0$$

generates a compact operator. After the change of variables

$$s = s(t) = \int_0^t \frac{d\tau}{g_\Gamma(\tau)}$$

we come to another Rayleigh quotient:

$$\text{(4.2)} \qquad \frac{\int_0^{L(\Gamma)} |\psi(s)|^2 W(s) ds}{\int_0^{L(\Gamma)} |\psi'(s)|^2 ds}, \qquad \psi \in \mathsf{H}^1(0, h(\Gamma)),\ \psi(0) = 0$$

where $L(\Gamma)$ is the reduced height of Γ and $W(s) = g_\Gamma^2(t(s))$. The function W is monotone and

$$\int_0^{L(\Gamma)} \sqrt{W(s)} ds = \int_0^{L(\Gamma)} g_\Gamma(t(s)) ds = \int_0^{L(\Gamma)} t'(s) ds = h(\Gamma) < \infty.$$

It is well known ([2], Theorem 3.1; see also an exposition in [3]) that under these conditions the Rayleigh quotient (4.2) generates a compact operator, say $\mathbf{K}$, whose eigenvalues satisfy the estimate

$$\sqrt{\lambda_n(\mathbf{K})} \le C n^{-1} \int_0^{L(\Gamma)} \sqrt{W(s)} ds = C h(\Gamma) n^{-1}$$

where C is an absolute constant. It follows in particular that

$$\|\mathbf{K}\| = \lambda_1(\mathbf{K}) \le \big(C h(\Gamma)\big)^2.$$

Equivalently, this means that

$$\text{(4.3)} \qquad \min \sigma(\mathbf{A}_0) \ge (C h(\Gamma))^{-2}.$$

Applying the same argument to the operators $\mathbf{A}_k$, we find that

$$\text{(4.4)} \qquad \min \sigma(\mathbf{A}_k) \ge (C(h(\Gamma) - t_k))^{-2}, \qquad k \in \mathbb{N}.$$

This implies (3.15) and therefore, discreteness of $\sigma(\mathbf{\Delta})$.

(ii) Two different proofs of (4.1) were given by the author in [15] and [17]. Here we outline the first of them since it is more straightforward.

If $f \in \mathsf{H}^{1,0}(\Gamma)$, then

$$|f(x)|^2 = \Big| \int_{\langle o,x \rangle} f'(y) dy \Big|^2 \le h(\Gamma) \|f'\|^2_{\mathsf{L}^2(\Gamma)}.$$

Hence, for any non-negative function $V \in \mathsf{L}^1(\Gamma)$ we have

$$\int_\Gamma V|f|^2 dx \le h(\Gamma) \|f'\|^2_{\mathsf{L}^2(\Gamma)} \int_\Gamma V dx.$$

This shows that in the case $h(\Gamma) < \infty$ the quadratic form in the left-hand side generates a bounded self-adjoint operator, say $\mathbf{T}_{\Gamma,V}$, in the space $\mathsf{H}^{1,0}(\Gamma)$ equipped with the norm $\|f'\|_{\mathsf{L}^2(\Gamma)}$. It is easy to check that this operator is compact. If $|\Gamma|<\infty$, its eigenvalues satisfy the estimate

$$\lambda_n(\mathbf{T}_{\Gamma,V})\le|\Gamma|n^{-2}\int_\Gamma V\,dx,\ \forall n\in\mathbb{N} \tag{4.5}$$

which was established in [15] for the general case of operators on metric graphs. The Rayleigh quotient for the operator $\mathbf{T}_{\Gamma,V}$ is

$$\frac{\int_\Gamma V|f|^2dx}{\int_\Gamma|f'|^2dx},\ f(o)=0.$$

This shows in particular that the operator $\mathbf{T}_{\Gamma,\mathbf{1}}$, i.e. $\mathbf{T}_{\Gamma,V}$ for $V\equiv 1$, can be identified with $\mathbf{\Delta}^{-1}$.

Given a number $\varepsilon>0$, find a compact subtree $T_\varepsilon\subset\Gamma$ such that $|\Gamma\setminus T_\varepsilon|<\varepsilon$. Let V_ε be the characteristic function of T_ε. Evidently,

$$\mathbf{T}_{\Gamma,\mathbf{1}}=\mathbf{T}_{\Gamma,V_\varepsilon}+\mathbf{T}_{\Gamma,\mathbf{1}-V_\varepsilon}.$$

For the first term the Weyl asymptotics holds,

$$\pi\lambda^{1/2}\#\{n:\lambda_n(\mathbf{T}_{\Gamma,V_\varepsilon})>\lambda\}=|T_\varepsilon|+o(1),\qquad\lambda\to 0. \tag{4.6}$$

This follows from the classical result for a single interval. Indeed, insertion of the additional condition $f(v_k)=0$ at a finite number of vertices allows one to withdraw the non-compact part $\Gamma\setminus T_\varepsilon$ and does not affect the asymptotic behaviour of the eigenvalues. Further, we derive from (4.5) that

$$\lambda_n(\mathbf{T}_{\Gamma,\mathbf{1}-V_\varepsilon})\le\varepsilon|\Gamma|n^{-2},\ \forall n\in\mathbb{N}.$$

This estimate allows us to pass in (4.6) to the limit as $\varepsilon\to 0$, which leads to (4.1). □

5. Laplacian on regular trees of infinite height

5.1. **Trees with arbitrarily long edges.** Our next result is quite elementary and its proof is standard. The result applies to arbitrary metric graphs rather than to trees only, see [17]. Still, below we formulate only the particular case we are interested in in this paper.

Theorem 5.1. *Let $\mathcal{G}$ be a regular metric tree and* $\sup_{e\in\mathcal{E}(\Gamma)}|e|=\infty$. *Then* $\sigma(\mathbf{\Delta})=[0,\infty)$.

Proof. It is enough to show that for any $r>0$ the point $\lambda=r^2$ belongs to the spectrum. For this purpose we fix a non-negative function $\zeta\in C_0^\infty(-1,1)$ such that $\zeta(t)=1$ on $(-1/2,1/2)$. Further, choose an edge $e\in\mathcal{E}(\Gamma)$. In an appropriate coordinate system, e can be identified with the interval $(-l,l)$ where $l=|e|/2$. The function f on Γ,

$$f(t)=\zeta(t/l)\sin rt \text{ on } e,\qquad f(t)=0 \text{ otherwise},$$

belongs to $\mathrm{Dom}(\mathbf{\Delta})$. An elementary calculation shows that

$$\|\mathbf{\Delta} f - r^2 f\| \le \varepsilon(l)\|f\|, \qquad \varepsilon(l) \to 0 \text{ as } l \to \infty.$$

Choosing a sequence of edges e such that $|e| \to \infty$, we obtain a Weyl sequence for the operator $\mathbf{\Delta}$ and the point $\lambda = r^2$. This implies that $\lambda \in \sigma(\mathbf{\Delta})$. □

The assumption of Theorem 5.1 does not exclude the embedded eigenvalues. This will be shown in the next section, see Example 6.2.

5.2. **Criterion of positive definiteness of the Laplacian.**

Theorem 5.2. *Let* Γ *be a regular tree and* $h(\Gamma) = \infty$*. Then the Laplacian on* Γ *is positive definite if and only if* $L(\Gamma) < \infty$ *and*

$$B(\Gamma) := \sup_{t>0}\left(\int_0^t g_\Gamma(s)ds \cdot \int_t^\infty \frac{ds}{g_\Gamma(s)}\right) < \infty. \tag{5.1}$$

Moreover,

$$(4B(\Gamma))^{-1} \le \min \sigma(\mathbf{\Delta}) \le B(\Gamma)^{-1}. \tag{5.2}$$

Recall that the "reduced height" $L(\Gamma)$ is defined by (2.3).

Proof. According to Corollary 3.7, we have to study positive definiteness of the operator $\mathbf{A}_0$ or, equivalently, of $\mathfrak{A}_0$. Let $c_0 := \min \sigma(\mathfrak{A}_0)$. Taking into account the relations (3.10) – (3.12), we come to the inequality

$$c_0 \int_0^\infty |\varphi(t)|^2 g_\Gamma(t)dt \le \int_0^\infty |\varphi'(t)|^2 g_\Gamma(t)dt, \ \forall \varphi \in \mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma). \tag{5.3}$$

Due to the density of the class $\mathsf{H}^{1,0}_c(\Gamma)$ in $\mathsf{H}^{1,0}(\Gamma)$, it is sufficient to have this inequality for functions with compact support.

The inequality (5.3) with $c_0 > 0$ is a special case of the Hardy inequality with two weights. Necessary and sufficient conditions for such inequalities to be satisfied (Muckenhoupt conditions) are well known, see e.g. [9], Section 1.3.1. Since $g_\Gamma(t) \ge 1$, they are never satisfied if $L(\Gamma) = \infty$, so that the condition $L(\Gamma) < \infty$ is necessary for positive definiteness of the Laplacian.

Let $\varphi \in \mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma)$ be a function with compact support. Then its derivative $\omega = \varphi'$ lies in $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$ and also has compact support. Besides, $\int_{\mathbb{R}_+} \omega dt = 0$. Denote by Ω the class of all such functions ω. For any $\omega \in \Omega$ the function $\varphi(t) = -\int_t^\infty \omega(s)ds$ lies in $\mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma)$ and has compact support. For this reason, the inequality (5.3) is equivalent to

$$c_0 \int_0^\infty \Big|\int_t^\infty \omega(s)ds\Big|^2 g_\Gamma(t)dt \le \int_0^\infty |\omega(t)|^2 g_\Gamma(t)dt, \qquad \forall \omega \in \Omega. \tag{5.4}$$

Since $\mathbf{1} \notin \mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$, the set Ω is dense in the whole of $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$. Hence, the validity of (5.4) on Ω is equivalent to its validity on $\mathsf{L}^2(\mathbb{R}_+; g_\Gamma)$. The condition (5.1) is exactly the Muckenhoupt condition for (5.4) to be satisfied with some constant $c_0 > 0$, see [9], Theorem 1.3.1/3. The inequality (5.2) is also a part of this Theorem. □

It follows from Theorem 5.2 that in the case $L(\Gamma) = \infty$ the point $\lambda = 0$ lies in $\sigma(\mathbf{\Delta})$. A straightforward calculation shows that $\lambda = 0$ is not an eigenvalue an hence, $0 \in \sigma_{\mathrm{ess}}(\mathbf{\Delta})$.

5.3. Discreteness of $\sigma(\mathbf{\Delta})$.

Theorem 5.3. *Let Γ be a regular tree and $h(\Gamma) = \infty$. Then the Laplacian on Γ has discrete spectrum if and only if $L(\Gamma) < \infty$, $B(\Gamma) < \infty$, and*

$$\lim_{t\to\infty}\left(\int_0^t g_\Gamma(s)ds \cdot \int_t^\infty \frac{ds}{g_\Gamma(s)}\right) = 0. \tag{5.5}$$

Proof. The necessity of the assumption $L(\Gamma) < \infty$ is clear from Theorem 5.2. Under this assumption, the condition $B(\Gamma) < \infty$ is necessary and sufficient for the boundedness of the operator $\mathbf{Q}_0$, generated in the space $\mathsf{H}^{1,\bullet}(\mathbb{R}_+; g_\Gamma)$ by the Rayleigh quotient

$$\frac{\int_0^\infty |\varphi(t)|^2 g_\Gamma(t)dt}{\int_0^\infty |\varphi'(t)|^2 g_\Gamma(t)dt}.$$

It follows in a standard way that the condition (5.5) is necessary and sufficient for the compactness of $\mathbf{Q}_0$ or, equivalently, that of $\mathbf{A}_0^{-1}$. By Theorem 3.6 (ii), each operator $\mathbf{A}_k^{-1}$ is also compact, and it remains for us to show that $\|\mathbf{A}_k^{-1}\| \to 0$ as $k \to \infty$. For this purpose we apply Theorem 5.2 to the tree Γ_k. It follows from (2.4) that

$$\int_0^t g_{\Gamma_k}(s)ds \cdot \int_t^\infty \frac{ds}{g_{\Gamma_k}(s)} = \int_0^t g_\Gamma(t_k+s)ds \cdot \int_t^\infty \frac{ds}{g_\Gamma(t_k+s)}$$
$$\le \int_0^{t_k+t} g_\Gamma(s)ds \cdot \int_{t_k+t}^\infty \frac{ds}{g_\Gamma(s)}.$$

In view of (5.5), $B(\Gamma_k) \to 0$ as $k \to \infty$. By (5.2), $\min\sigma(\Gamma_k) \to \infty$ and we are done. □

6. Examples

In our first example we show that for a tree Γ of finite height but infinite volume the eigenvalues of the Laplacian may have quite an unusual behaviour; see [17] for the proof.

Example 6.1. Fix the numbers $q \in (0,1)$ and $b \in \mathbb{N}$. Consider the tree Γ defined by the sequences $t_n = 1 - q^n$, $n = 0, 1, \dots$ and $b_n = b = \text{const}$, $n = 1, 2, \dots$ Then $h(\Gamma) = 1$, so that the spectrum of the Laplacian on Γ is always discrete. Further, $g_\Gamma(t) = b^n$ for $t_n < t \le t_{n+1}$. The total length of Γ is

$$|\Gamma| = 1 - q + \sum_{n=1}^{\infty} b^n (q^n - q^{n+1}) = (1-q) \sum_{n=0}^{\infty} (bq)^n.$$

Hence, $|\Gamma| = \frac{1-q}{1-bq} < \infty$ if $bq < 1$ and $|\Gamma| = \infty$ otherwise. In the first case, Theorem 4.1 shows that for the eigenvalues of $\mathbf{\Delta}$ the Weyl law (4.1) holds.

If $bq = 1$, then

$$N(\lambda; \mathbf{\Delta}) = \frac{1-q}{2\pi \ln b} \sqrt{\lambda} \big(\ln \lambda + O(1) \big), \qquad \lambda \to \infty,$$

and if $bq > 1$, then there exists a bounded and bounded away from zero periodic function ψ with the period $\ln(q^{-2})$ such that

$$N(\lambda; \mathbf{\Delta}) = \lambda^{\beta/2} \big(\psi(\ln \lambda) + o(1) \big), \qquad \lambda \to \infty$$

where $\beta = -\log_q b > 1$.

In our next example we show that under the assumptions of Theorem 5.1 the spectrum $\sigma(\mathbf{\Delta})$ may have a dense set of embedded eigenvalues.

Example 6.2. Consider the tree Γ with $t_n = 2^{n-1}\pi$, $n \in \mathbb{N}$. The sequence b_n such that $b_0 = 1$ and $b_n > 1$ for $n \ge 1$ can be arbitrary. A direct inspection shows that for any integer l the function $u_l(t) = (g_\Gamma(t))^{-1/2} \sin lt$ is an eigenfunction of the operator $\mathbf{A}_0$, with the eigenvalue $\lambda_l = l^2$. In the same way, the function $u_{l,k}(t) = (g_\Gamma(t))^{-1/2} \sin(2^{-k} lt)$, $t \ge t_n$ is an eigenfunction of any operator $\mathbf{A}_n$ with $n > k$. The corresponding eigenvalue is $\lambda_{l,k} = 2^{-2k} l^2$, and the result follows from (3.16).

Note that in this example $\sigma(\mathbf{\Delta})$ is not pure point, since for each k we have $\sigma(\mathbf{A}_k) \ne \sigma_p(\mathbf{A}_k)$.

Now we present an example (borrowed from [14]) of a tree for which the Laplacian is positive definite.

Example 6.3. Consider the tree $\Gamma = \Gamma_b$ with $b_n = b = \text{const}$, $n \in \mathbb{N}$ and $t_n = n$; so, all the edges of Γ_b are of the same length 1. We have $g_{\Gamma_b}(t) \sim \exp(\beta t)$, $\beta = \ln b$, so that the conditions of Theorem 5.2 are satisfied which yields positive definiteness of the Laplacian.

For the tree Γ_b the spectrum of $\mathbf{\Delta}$ can be calculated explicitly. Define

$$\theta = \arccos \frac{2}{b^{1/2} + b^{-1/2}}.$$

It turns out that $\sigma(\mathbf{\Delta})$ is of infinite multiplicity and consists of the bands
$\big[\big(\pi(l-1)+\theta\big)^2, \big(\pi l - \theta\big)^2\big]$ and the eigenvalues $\lambda_l = \big(\pi l\big)^2$, $l = 1, 2, \dots$

So, in this case the spectrum has the band-gap structure typical for periodic problems.

For comparison, consider the discrete Laplacian $\mathbf{\Delta}_d$ on the combinatorial rooted tree, with the branching numbers as for our tree Γ_b. It is well known (and can be easily calculated, see e.g. [1] where this was done for $b = 2$) that $\sigma(\mathbf{\Delta}_d) = [(b^{1/2}-1)^2, (b^{1/2}+1)^2]$. This shows that in general there is no direct connections between the spectra of the Laplacian on a metric tree and the discrete Laplacian on the combinatorial tree of the same structure.

We conclude this section with an example of a tree for which the Laplacian has discrete spectrum.

Example 6.4. Consider the tree Γ with $b_n = b = \text{const}$, $n \in \mathbb{N}$ and $t_n = n^{1/\alpha}$, $\alpha > 1$. Then $g_\Gamma(t) = b^n$ for $n^{1/\alpha} < t \le (n+1)^{1/\alpha}$, which implies $g_\Gamma(t) \sim \exp(\beta t^\alpha)$ as $t \to \infty$. It is easy to check that the condition (5.5) is satisfied. Hence, $\sigma(\mathbf{\Delta})$ is discrete.

An alternative way to construct a similar example is to take $t_n = n$ and b_n growing fast enough.

7. Concluding remarks

7.1. **The Neumann boundary condition at the root.** Let us discuss the changes in the above scheme appearing if we consider the Neumann Laplacian $\mathbf{\Delta}_N$. The codimension of the subspace $\mathsf{H}^{1,0}(\Gamma)$ in $\mathsf{H}^1(\Gamma)$ is one, which implies that the qualitative properties of the operators $\mathbf{\Delta}_N$ and $\mathbf{\Delta}_D$ are the same. But as a matter of fact, much more can be said about the relations between these two operators.

Let us return to the orthogonal decomposition (3.2) of the space $\mathsf{L}^2(\Gamma)$. If $f \in \mathcal{F}'_v \cap \mathsf{H}^1(\Gamma)$ and $v \neq o$, then $f(v) = 0$. It follows that

$$\mathbf{\Delta}_N \restriction \mathcal{F}'_v = \mathbf{\Delta}_D \restriction \mathcal{F}'_v, \qquad \forall v \neq o.$$

Therefore, the analog of the decomposition (3.14) for the operator $\mathbf{\Delta}_N$ takes the form

$$\mathbf{\Delta}_N \sim \mathbf{A}'_0 \oplus \sum_{k=1}^{\infty} \oplus \mathbf{A}_k^{[b_0 \dots b_{k-1}(b_k - 1)]} \tag{7.1}$$

where only the first term differs from the one in (3.14): namely, in (3.4) for $k = 0$ the condition $u(0) = 0$ should be replaced by $u'(0) = 0$.

The only place where this difference might be important, is Theorem 5.2 where a bound for $\min \sigma(\mathbf{\Delta})$ was found. However, even here the inequality (5.2) remains valid for the Neumann Laplacian. Indeed, the condition $\varphi(t) = 0$ for large t was used when justifying (5.4), rather than the condition $\varphi(0) = 0$.

7.2. **Regular trees without boundary.** Let Γ be a general metric tree. Choose a vertex $o \in \mathcal{E}(\Gamma)$ and suppose that there are $b_0 > 1$ edges of Γ adjacent to o. Then Γ can be split into b_0 rooted subtrees $\Gamma_1, \dots \Gamma_{b_0}$ having the common root o. We say that the tree Γ is regular if and only if all the subtrees Γ_j are regular in the sense of Definition 2.1 and the corresponding sequences $\{t_n\}$ and $\{b_n\}$ are the same for all $j = 1, \dots, b_0$. Note that this definition is not invariant with respect to the choice of the vertex o.

The definition of the Laplacian $\mathbf{\Delta}$ extends to the trees without boundary in a natural way. The only difference is that now we have no boundary condition at o. Instead, the functions from $\mathrm{Quad}(\mathbf{\Delta})$ are required to be continuous at o; the functions $f \in \mathrm{Dom}(\mathbf{\Delta})$ satisfy (3.1) also for $v = o$.

Theorems 3.2 and 3.5 extend to the new situation, with small changes appearing due to the fact that now $b_0 > 1$. As a result, the subspace $\mathcal{F}'_o$ is no more trivial, and the operator $\mathbf{\Delta} \restriction \mathcal{F}'_o$ is unitary equivalent to the orthogonal sum of $(b_0 - 1)$ copies of the operator $\mathbf{A}_0$ described by the equations (3.3) – (3.6) (for $k = 0$). The analog of the decomposition (3.14) now takes the form

$$\mathbf{\Delta} \sim \mathbf{A}'_0 \oplus \mathbf{A}_0^{b_0-1} \oplus \mathbf{A}_1^{b_0(b_1-1)} \oplus \mathbf{A}_2^{b_0 b_1 (b_2-1)} \dots$$

where $\mathbf{A}'_0$ is the same operator as in (7.1).

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