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ON THE SPECTRUM OF THE N = 2 SU(3) ⊗ SU(2) ⊗ U(1) GAUGE THEORY FROM D = 11 SUPERGRAVITY

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ABSTRACT

We determine the spectrum of the representations contributing to the 0,1,2,3-forms on M^{pqr} spaces: then, we calculate the eigenvalues of the Laplacian on 0-forms and of the *d operator on 3-forms. These latter can be given only implicitly since they are the roots of a 15th order secular equation. In the sector of the 2-forms there is a 0-mode corresponding to the fact that the second Betti number B_2 is equal to one. Hence there is an extra U_1 vector multiplet and the effective theory in four-dimensions is N = 2 supergravity coupled to the N = 2 gauge multiplet of SU(3) ⊗ SU(2) ⊗ U(1). There are also scalar 0-modes in the $(\underline{10},\underline{3})$ and $(\underline{\overline{10}},\underline{3})$ of SU(3)⊗SU(2). We show that they are not elements of a N = 2 hypermultiplet but rather of a bigger multiplet containing also vectors and other spins. Hypermultiplets whose states are all massive are not excluded and using our results for the spectrum can now be systematically searched.

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INTRODUCTION

D = 11 supergravity[1] admits a Freund-Rubin[2] spontaneous compactification with $SU(3) \otimes SU(2) \otimes U(1)$ symmetry and N = 2 supersymmetry[3]. The fermionic spectrum of this "vacuum" (limited to the longitudinal representations) has been computed in Ref. 4), where the formalism for harmonic analysis on any coset space of relevance to supergravity has been laid down. Questions of stability for this solution have been considered in Ref. 5) and a computation of the $SU(3) \otimes SU(2) \otimes U(1)$ gauge coupling constants has been performed in Ref. 6).

Recently general relations among the spectra of bosonic and fermionic fields, holding on any coset space with Killing spinors, have been worked out in Ref. 7): utilizing this tool, one can therefore obtain the entire spectrum of the solution if, to the knowledge of the fermionic longitudinal representations, one adds the knowledge of the spectrum of a few more bosonic operators. In particular, in view of the relation among the spectrum of the three-form $Y_{\alpha\beta\gamma}$ and the transverse and longitudinal fermionic spectra, the most interesting objects are the $Y_{[\alpha\beta\gamma]}$, for their richness of information, and the scalar function Y for the simplicity of calculations involved. In this paper we shall:

(i) give the spectrum of the $SU(3) \otimes SU(2) \otimes U(1)$ representations contributing to the 0, 1, 2 and 3-forms on the manifold M^{111} hence to the Laplacian scalars 0^+, the vector 1^+, the vectors 1^- and the pseudoscalars 0^-.

(ii) Calculate the eigenvalues of $\Box$ and determine the Laplacian scalar 0-modes.

(iii) Calculate the *d operator on 3-forms and give their eigenvalues as roots of the secular equation for a 15×15 or 10×10 matrix. Determine the pseudoscalar 0-modes.

(iv) Show that M^{111} has Betti number $B_2 = 1$ [already observed by Witten[8]] giving the explicit form of the harmonic 2-form and, via the results of Ref. 7) of the entire extra U(1) vector multiplet.

Our results can be summarized in the following way.

a) The solution[3] leads to a four-dimensional theory of gauged N = 2, supergravity coupled to the vector multiplet of SU(3) ⊗ SU(2) ⊗ U(1) (and not SU(3) ⊗ SU(2) as originally thought). This is due to the Betti vector multiplet.

b) The four-dimensional theory contains, among the many other multiplets, one which is in the $\underline{10} \otimes \underline{3}$ representation of SU(3) ⊗ SU(2) and contains states with the following labels with respect to spin, O(2) assignments and mass:

$$\left.\begin{array}{lll} 0^+ : & \square\!\square \text{ of } O(2) & m^2_{0^+} = 0 \\ 0^- : & \square\!\square \text{ of } O(2) & m^2_{0^-} = 480 \\ 1/2 : & \square\!\square\!\square \text{ of } O(2) & m_{1/2} = -24 \end{array}\right\} \tag{1.1}$$

These states match the number of Bose and Fermi degrees of freedom, and can in principle, constitute an N = 2 hypermultiplet. However, as already pointed in Ref. 7), the masses (1.1) do not satisfy the mass sum rule

$$m^2_{0^+} + m^2_{0^-} - 2\, m^2_{1/2} = 0 \tag{1.2}$$

implyed by N = 1 supersymmetry. Hence either the states (1.1) are just part of a bigger multiplet containing also spin 1 and possibly spin 3/2 fields or we are in presence of a hypermultiplet which cannot be described in N = 1 language. By explicit check we find that the spinor of (1.1) transforms also into a massive vector state so that (1.1) is not a hypermultiplet. The question whether hypermultiplets exist at all on this compactification remains therefore open: we see, however, that we do not have to look for massless states but rather for states satisfying Eq. (1.2).

This gives an example of why it is important to calculate the complete spectrum of the various Freund-Rubin solutions: indeed there are infinite multiplets contained in the higher dimensional theory and it is not yet well established which of them are to be retained in the four-dimensional theory and which of them are to be discarded: these multiplets are moreover coupled to supergravity and there is probably something to be learned about the possible couplings in four dimensions. Further motivations for the investigation of the spectra come from the quantum theory: indeed all the massive states are needed in the calculation of the Casimir energy and of the β function[9].

We go now to the derivation of the results.

2. - HARMONIC EXPANSIONS FOR THE SCALAR AND THE 1,2,3-FORMS ON M^{pqr}-SPACES

We use the notations and the formalism developed in Ref. 4). The essential thing is to decompose our SO(7) representations into R-representations where the subgroup R is given by:

$$R = SU_2^{(color)} \otimes U_1' \otimes U_1'' \tag{2.1}$$

The generators of R are:

$$\begin{aligned}
J^c_{\dot m} &= \frac{1}{2}\lambda_{\dot m} \Longrightarrow SU_2^{(color)} \\
Z' &= i\left[rp\,Y^c + 2rq\,J_3^{(weak)} - (3p^2+q^2)\,Y^{(weak)}\right] \Longrightarrow U_1' \\
Z'' &= i\left[-\frac{q}{2}Y^c + 3p\,J_3^{(weak)}\right] \Longrightarrow U_1''
\end{aligned} \tag{2.2}$$

and the embedding of R in SO(7) is described by the following $T_H^{\alpha\beta}$ matrices:

$$T_{\dot{m}}^{\alpha\beta} = \begin{pmatrix} f_{\dot{m}AB} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3a}$$

$$T_{Z'}^{\alpha\beta} = \begin{pmatrix} 2\sqrt{3}\, rp\, f_{8AB} & 0 & 0 \\ 0 & 2rq\,\varepsilon_{mu} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3b}$$

$$T_{Z''}^{\alpha\beta} = \begin{pmatrix} -\sqrt{3}\, q\, f_{8AB} & 0 & 0 \\ 0 & 3p\,\varepsilon_{mu} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.3c}$$

These formulae are just recalled from Section 8 of Ref. 4). Decomposing the SO(7) indices of 0,1,2,3-form, we get:

i) $\Upsilon = \Upsilon$ (2.4a)

ii) $\Upsilon_\alpha = \left\{ \Upsilon_A , \Upsilon_m , \Upsilon_3 \right\}$ (2.4b)

iii) $\Upsilon_{[\alpha\beta]} = \left\{ \Upsilon_{AB} , \Upsilon_{Am} , \Upsilon_{mu} , \Upsilon_{3A} , \Upsilon_{3m} \right\}$ (2.4c)

iv) $\Upsilon_{[\alpha\beta\gamma]} = \left\{ \Upsilon_{ABC} , \Upsilon_{3AB} , \Upsilon_{Amu} , \Upsilon_{Am3} , \Upsilon_{mu3} , \Upsilon_{Amu} \right\}$ (2.4d)

The next step is that of converting the indices A,m,3 into the standard $SU(2)_C \otimes U'(1) \otimes U''(1)$ indices used in Section 8 of Ref. 4), which can then be expanded in harmonics.

This is done by the use of the Gell-Mann lambda matrices and the Pauli σ-matrices, as follows:

<u>0-form:</u>

$$\Upsilon = [0|I] \tag{2.5}$$

<u>1-form:</u>

$$\Upsilon_A = \lambda^A_{\#i} \langle 1|I\rangle_i + \lambda^A_{i\#} \langle 1|I\rangle_i^* \tag{2.6a}$$

$$\Upsilon_m = \sigma^m_{\uparrow\downarrow} \langle 1|I\rangle_\bullet + \sigma^m_{\downarrow\uparrow} \langle 1|I\rangle_\bullet^* \tag{2.6b}$$

$$\Upsilon_3 = [1|I] \tag{2.6c}$$

2-form:

$$\Upsilon_{AB} = \lambda^{A}_{i\#}\lambda^{B}_{j\#}\,\varepsilon_{ij}\langle 2|\mathrm{I}\rangle^{*}_{\cdot} + \lambda^{A}_{\#i}\lambda^{B}_{\#j}\,\varepsilon_{ij}\langle 2|\mathrm{I}\rangle_{\cdot}$$
$$+\, i\,\lambda^{[A}_{m\#}\lambda^{B]}_{\#m}[2|\mathrm{I}]_{\cdot} + \lambda^{[A}_{i\#}\lambda^{B]}_{\#j}\,\varepsilon_{im}[2|\mathrm{I}]_{mj} \qquad (2.7a)$$

$$\Upsilon_{mn} = \varepsilon_{mn}\,[2|\mathrm{II}]_{\cdot} \qquad (2.7b)$$

$$\Upsilon_{3A} = \lambda^{A}_{i\#}\langle 2|\mathrm{I}\rangle^{*}_{i} + \lambda^{A}_{\#i}\langle 2|\mathrm{I}\rangle_{i} \qquad (2.7c)$$

$$\Upsilon_{3m} = \sigma^{m}_{\uparrow\downarrow}\langle 2|\mathrm{II}\rangle_{\cdot} + \sigma^{m}_{\downarrow\uparrow}\langle 2|\mathrm{II}\rangle^{*}_{\cdot} \qquad (2.7d)$$

$$\Upsilon_{Am} = \lambda^{A}_{i\#}\left\{\sigma^{m}_{\uparrow\downarrow}\langle 2|\mathrm{II}\rangle^{*}_{i} + \sigma^{m}_{\downarrow\uparrow}\langle 2|\mathrm{III}\rangle^{*}_{i}\right\}$$
$$+ \lambda^{A}_{\#i}\left\{\sigma^{m}_{\downarrow\uparrow}\langle 2|\mathrm{II}\rangle_{i} + \sigma^{m}_{\uparrow\downarrow}\langle 2|\mathrm{II}\rangle_{i}\right\} \qquad (2.7e)$$

3-form:

$$\Upsilon_{ABC} = \varepsilon_{ABCD}\left\{\lambda^{D}_{i\#}\langle 3|\mathrm{I}\rangle^{*}_{i} + \lambda^{D}_{\#i}\langle 3|\mathrm{I}\rangle_{i}\right\} \qquad (2.8a)$$

$$\Upsilon_{3AB} = \lambda^{A}_{i\#}\lambda^{B}_{j\#}\,\varepsilon_{ij}\langle 3|\mathrm{I}\rangle^{*}_{\cdot} + \lambda^{A}_{\#i}\lambda^{B}_{\#j}\,\varepsilon_{ij}\langle 3|\mathrm{I}\rangle_{\cdot}$$
$$+\, i\,\lambda^{[A}_{i\#}\lambda^{B]}_{\#i}[3|\mathrm{I}]_{\cdot} + \lambda^{[A}_{i\#}\lambda^{B]}_{\#j}\,\varepsilon_{im}[3|\mathrm{I}]_{mj} \qquad (2.8b)$$

$$Y_{Amn} = \varepsilon_{mn} \left\{ \lambda^{A}_{i\#} \langle 3|\mathrm{II}\rangle^{*}_{i} + \lambda^{A}_{\# i} \langle 3|\mathrm{II}\rangle_{i} \right\} \tag{2.8c}$$

$$\begin{aligned} Y_{Am3} = \; & \lambda^{A}_{i\#} \left\{ \sigma^{m}_{\uparrow\downarrow} \langle 3|\mathrm{IV}\rangle^{*}_{i} + \sigma^{m}_{\downarrow\uparrow} \langle 3|\mathrm{III}\rangle^{*}_{i} \right\} \\ & + \lambda^{A}_{\# i} \left\{ \sigma^{m}_{\downarrow\uparrow} \langle 3|\mathrm{IV}\rangle_{i} + \sigma^{m}_{\uparrow\downarrow} \langle 3|\mathrm{III}\rangle_{i} \right\} \end{aligned} \tag{2.8d}$$

$$Y_{mn3} = \varepsilon_{mn} \left[3|\mathrm{II} \right] . \tag{2.8e}$$

$$\begin{aligned} Y_{ABm} = \; & \lambda^{A}_{i\#} \lambda^{B}_{j\#} \varepsilon^{ij} \left\{ \sigma^{m}_{\uparrow\downarrow} \langle 3|\mathrm{II}\rangle_{\cdot} + \sigma^{m}_{\downarrow\uparrow} \langle 3|\mathrm{III}\rangle_{\cdot} \right\} \\ & + \lambda^{A}_{\# i} \lambda^{B}_{\# j} \varepsilon^{ij} \left\{ \sigma^{m}_{\downarrow\uparrow} \langle 3|\mathrm{II}\rangle^{*}_{\cdot} + \sigma^{m}_{\uparrow\downarrow} \langle 3|\mathrm{III}\rangle^{*}_{\cdot} \right\} \\ & + i \lambda^{[A}_{i\#} \lambda^{B]}_{\# i} \left\{ \sigma^{m}_{\uparrow\downarrow} \langle 3|\mathrm{IV}\rangle_{\cdot} + \sigma^{m}_{\downarrow\uparrow} \langle 3|\mathrm{IV}\rangle^{*}_{\cdot} \right\} \\ & + \lambda^{[A}_{i\#} \lambda^{B]}_{\# j} \left\{ \sigma^{m}_{\uparrow\downarrow} \varepsilon_{ik} \langle 3|\mathrm{I}\rangle_{kj} - \sigma^{m}_{\downarrow\uparrow} \varepsilon_{jk} \langle 3|\mathrm{I}\rangle^{*}_{ik} \right\} \end{aligned} \tag{2.8f}$$

where $\langle \; | \quad \rangle_{ijk\ldots}$ denotes a complex field which transforms as an $SU(2)^{C}$ symmetric tensor, each latin index i,j,k taking the two values $\uparrow$ and $\downarrow$. The digit in the brackets remembers the type of Y the object we are dealing with came from, while the roman number is simply a label which distinguishes among the fields with the same $SU(2)^C$ structure. The fields $[\ |\ ]_{ij\ldots}$ are J^c = integer representations on which the self-conjugation constraint has been imposed.

$$[\ \ |\ \ ]_{\dot{i}_1\ldots\dot{i}_{2m}} = \varepsilon_{\dot{i}_1\dot{j}_1}\ldots\varepsilon_{\dot{i}_{2m}\dot{j}_{2m}}[\ \ |\ \ ]^*_{\dot{j}_1\ldots\dot{j}_{2n}} \qquad (2.9)$$

We conclude therefore that there is a one-to-one correspondence between the 0, 1, 2, 3-forms and the following sets of irreducible $SU(2)^C$ representations:

$$\overset{o}{Y} \Longleftrightarrow [0|0] \Big\} \ J^c = 0 \ \text{self-conjugate} \qquad (2.10a)$$

$$\overset{(1)}{Y} \Longleftrightarrow \left(\begin{array}{c} \langle 1|\mathrm{I}\rangle_i \\ \hline \langle 1|\mathrm{I}\rangle_{\cdot} \\ \hline [1|\mathrm{I}]_{\cdot} \end{array}\right) \begin{array}{l} J_c = 1/2 \\ \\ J^c = 0 \ \text{complex} \\ \\ J^c = 0 \ \text{self-conjugate} \end{array} \qquad (2.10b)$$

$$\overset{(2)}{Y} \Longleftrightarrow \left(\begin{array}{c} [2\,|\,\mathrm{I}]_{ij} \\ \hline \langle 2\,|\,\mathrm{I}\rangle_{i} \\ \langle 2\,|\,\mathrm{II}\rangle_{i} \\ \langle 2\,|\,\mathrm{III}\rangle_{i} \\ \hline \langle 2\,|\,\mathrm{I}\rangle_{\cdot} \\ \langle 2\,|\,\mathrm{II}\rangle_{\cdot} \\ \hline [2\,|\,\mathrm{I}]_{\cdot} \\ [2\,|\,\mathrm{II}]_{\cdot} \end{array} \right) \begin{array}{l} J^{c}=1 \text{ self-conjugate} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} J^{c}=1/2 \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} J^{c}=0 \text{ complex} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} J^{c}=0 \text{ self-conjugate} \end{array} \qquad (2.10c)$$

$$\overset{(3)}{Y} \Longleftrightarrow \left(\begin{array}{c} \langle 3|\mathrm{I}\rangle_{ij} \\ \hdashline [3|\mathrm{I}]_{ij} \\ \hdashline \langle 3|\mathrm{I}\rangle_{i} \\ \langle 3|\mathrm{II}\rangle_{i} \\ \langle 3|\mathrm{III}\rangle_{i} \\ \langle 3|\mathrm{IV}\rangle_{i} \\ \hdashline \langle 3|\mathrm{I}\rangle_{\cdot} \\ \langle 3|\mathrm{II}\rangle_{\cdot} \\ \langle 3|\mathrm{III}\rangle_{\cdot} \\ \langle 3|\mathrm{IV}\rangle_{\cdot} \\ \hdashline [3|\mathrm{I}]_{\cdot} \\ [3|\mathrm{II}]_{\cdot} \end{array}\right) \begin{array}{l} J^c = 1 \quad \text{complex} \\ J^c = 1 \quad \text{self-conjugate} \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} J^c = 1/2 \\ \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} J^c = 0 \quad \text{complex} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} J^c = 0 \quad \text{self-conjugate.} \end{array} \tag{2.10d}$$

Given the $SU(2)^C$ irreducible fields, we must determine their eigenvalues with respect to the two U(1) subgroups. This will determine the constraints on the $SU(3) \otimes SU(2) \otimes U(1)$ representations contributing to each field.

<u>0-form:</u>

Given these $SU(2)^C$ irreducible fields, we must calculate their eigenvalues with respect to the two U(1) subgroups. This will determine the constraints on the $SU(3) \otimes SU(2) \otimes U(1)$ quantum numbers and therefore the spectrum of the $SU(3) \otimes SU(2) \otimes U(1)$ representations contributing to each field. The result of this computation is given below:

0-form:

$$[0|0] : \begin{Bmatrix} Z' = 0 \\ Z'' = 0 \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} \right| \\ 3p\,Y^{w} = Y^{c} = 2\left(M_2 - M_1\right) \end{cases} \tag{2.11}$$

1-form:

$$\langle 1|I\rangle_i : \begin{Bmatrix} Z' = 3irp \\ Z'' = -\frac{3i}{2} q \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} \right| \\ 3\left(1 + p\,Y^{w}\right) = Y^{c} \equiv \begin{matrix} \nearrow 2\left(M_2 - M_1\right) - 3 \\ \searrow 2\left(M_2 - M_1\right) + 3 \end{matrix} \end{cases} \tag{2.12a}$$

$$\langle 1|I\rangle_0 : \begin{Bmatrix} Z' = -2irq \\ Z'' = -3ip \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} - 1 \right| \\ 3p\,Y^{w} = Y^{c} \equiv 2\left(M_2 - M_1\right) \end{cases} \tag{2.12b}$$

$$[1|I] : \begin{Bmatrix} Z' = 0 \\ Z'' = 0 \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} \right| \\ 3p\,Y^{w} = Y^{c} \equiv 2\left(M_2 - M_1\right) \end{cases} \tag{2.12c}$$

<u>2-form</u>:

$$[2|\mathrm{I}]_{ij} : \begin{Bmatrix} Z' = 0 \\ Z'' = 0 \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} \right| \\ 3pY^{w} = Y^{c} \equiv \begin{cases} 2(M_2 - M_1) - 6 \\ 2(M_2 - M_1) \\ 2(M_2 - M_1) + 6 \end{cases} \end{cases} \tag{2.13a}$$

$$\langle 2|\mathrm{I}\rangle_{i} : \begin{Bmatrix} Z' = 3irp \\ Z'' = -\frac{3}{2} iq \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left| \frac{q}{2} Y^{w} \right| \\ 3(1 + pY^{w}) = Y^{c} \equiv \begin{cases} 2(M_2 - M_1) - 3 \\ 2(M_2 - M_1) + 3 \end{cases} \end{cases} \tag{2.13b}$$

$$\langle 2|\mathrm{II}\rangle_i : \begin{Bmatrix} Z' = 3irp - 2irq \\ \\ Z'' = -\frac{3}{2}iq - 3ip \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left|\frac{q}{2}Y^W - 1\right| \\ \\ 3\left(1 + pY^W\right) = Y^c \equiv \begin{cases} 2(M_2 - M_1) - 3 \\ \\ 2(M_2 - M_1) + 3 \end{cases} \end{cases} \tag{2.13c}$$

$$\langle 2|\mathrm{III}\rangle_i : \begin{Bmatrix} Z' = 3irp + 2irq \\ \\ Z'' = -\frac{3}{2}iq + 3ip \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left|\frac{q}{2}Y^W + 1\right| \\ \\ 3\left(1 + pY^W\right) = Y^c \equiv \begin{cases} 2(M_2 - M_1) - 3 \\ \\ 2(M_2 - M_1) + 3 \end{cases} \end{cases} \tag{2.13d}$$

$$\langle 2|\mathrm{I}\rangle_{\bullet} : \begin{Bmatrix} Z' = 6irp \\ \\ Z'' = -3iq \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left|\frac{q}{2}Y^W\right| \\ \\ 3\left(2 + pY^W\right) = Y^c \equiv 2(M_2 - M_1) \end{cases} \tag{2.13e}$$

$$\langle 2|\mathrm{II}\rangle_{\bullet} : \left\{\begin{array}{l} Z_1' = -2irq \\ \\ Z_1'' = -3ip \end{array}\right\} \Rightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w} - 1\right| \\ \\ 3pY^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.13f}$$

$$[2|\mathrm{I}] : \left\{\begin{array}{l} Z_1' = 0 \\ \\ Z_1'' = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w}\right| \\ \\ 3pY^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.13g}$$

$$[2|\mathrm{II}] : \left\{\begin{array}{l} Z_1' = 0 \\ \\ Z_1'' = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w}\right| \\ \\ 3pY^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.13h}$$

3-form:

$$
\langle 3|I\rangle_{ij} : \left\{ \begin{array}{l} Z' = -2irq \\[2em] Z'' = -3ip \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \dfrac{q}{2} Y^{w} - 1 \right| \\[2em] 3p\, Y^{w} = Y^{c} \equiv \left\{ \begin{array}{l} 2(M_2 - M_1) - 6 \\ 2(M_2 - M_1) \\ 2(M_2 - M_1) + 6 \end{array} \right. \end{array} \right. \qquad (2.14a)
$$

$$
[3|I]_{ij} : \left\{ \begin{array}{l} Z' = 0 \\[2em] Z'' = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \dfrac{q}{2} Y^{w} \right| \\[2em] 3p\, Y^{w} = Y^{c} \equiv \left\{ \begin{array}{l} 2(M_2 - M_1) - 6 \\ 2(M_2 - M_1) \\ 2(M_2 - M_1) + 6 \end{array} \right. \end{array} \right. \qquad (2.14b)
$$

$$\langle 3|\mathrm{I}\rangle_i : \begin{Bmatrix} Z' = 3irp \\[2ex] Z_1'' = -\frac{3}{2}iq \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left|\frac{q}{2}Y^w\right| \\[4ex] 3\left(1+pY^w\right) = Y^c \equiv \begin{cases} \nearrow 2(M_2-M_1)-3 \\[2ex] \searrow 2(M_2-M_1)+3 \end{cases} \end{cases} \tag{2.14c}$$

$$\langle 3|\mathrm{II}\rangle_i : \begin{Bmatrix} Z_1' = 3irp \\[2ex] Z'' = -\frac{3}{2}iq \end{Bmatrix} \Longrightarrow \begin{cases} J \geq \left|\frac{q}{2}Y^w\right| \\[4ex] 3\left(1+pY^w\right) = Y^c \equiv \begin{cases} \nearrow 2(M_2-M_1)-3 \\[2ex] \searrow 2(M_2-M_1)+3 \end{cases} \end{cases} \tag{2.14d}$$

$$\langle 3|\mathrm{III}\rangle_i : \left\{ \begin{array}{l} Z' = 3irp - 2irq \\ \\ Z'' = -3ip - \frac{3}{2} iq \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \frac{q}{2} Y^w - 1 \right| \\ \\ 3(1 + pY^w) = Y^c \equiv \begin{cases} \rightarrow 2(M_2 - M_1) - 3 \\ \rightarrow 2(M_2 - M_1) + 3 \end{cases} \end{array} \right. \tag{2.14e}$$

$$\langle 3|\mathrm{IV}\rangle_i : \left\{ \begin{array}{l} Z' = 3irp + 2irq \\ \\ Z'' = 3ip - \frac{3}{2} iq \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \frac{q}{2} Y^w + 1 \right| \\ \\ 3(1 + pY^w) = Y^c \equiv \begin{cases} \rightarrow [2(M_2 - M_1) - 3] \\ \rightarrow 2(M_2 - M_1) + 3 \end{cases} \end{array} \right. \tag{2.14f}$$

$$\langle 3|\mathrm{I}\rangle : \left\{ \begin{array}{l} Z' = 6irp \\ \\ Z'' = -3iq \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \frac{q}{2} Y^w \right| \\ \\ 3(2 + pY^w) = Y^c = 2(M_2 - M_1) \end{array} \right. \tag{2.14g}$$

$$\langle 3|\mathrm{II}\rangle_{\bullet} : \left\{ \begin{array}{l} Z_1' = -6irp - 2irq \\ \\ Z_1'' = 3qi - 3ip \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| \frac{q}{2} Y^w - 1 \right| \\ \\ 3\left(pY^w - 2\right) = Y^c \equiv 2\left(M_2 - M_1\right) \end{array} \right. \tag{2.14h}$$

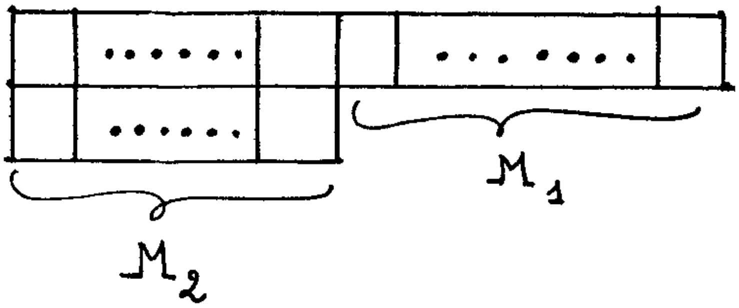
$$\langle 3|\mathrm{III}\rangle_{\bullet} : \left\{ \begin{array}{l} Z_1' = -6irp + 2irq \\ \\ Z_1'' = 3iq + 3ip \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} J \geq \left| 1 + \frac{q}{2} Y^w \right| \\ \\ 3\left(pY^w - 2\right) = Y^c \equiv 2\left(M_2 - M_1\right) \end{array} \right. \tag{2.14i}$$

$$\langle 3|\mathrm{IV}\rangle_{\bullet} : \left\{\begin{array}{l} Z_1' = -2irq \\ \\ Z_1'' = -3ip \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w} - 1\right| \\ \\ 3p Y^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.14j}$$

$$[3|\mathrm{I}]_{\bullet} : \left\{\begin{array}{l} Z_1' = 0 \\ \\ Z_1'' = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w}\right| \\ \\ 3p Y^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.14k}$$

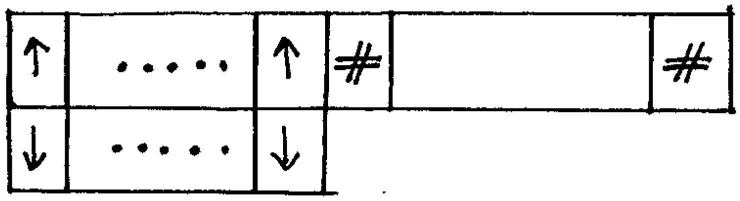
$$[3|\mathrm{II}]_{\bullet} : \left\{\begin{array}{l} Z_1' = 0 \\ \\ Z_1'' = 0 \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} J \geq \left|\frac{q}{2} Y^{w}\right| \\ \\ 3p Y^{w} = Y^{c} \equiv 2(M_2 - M_1) \end{array}\right. \tag{2.14ℓ}$$

In all the above formulae Y^W is the weak hypercharge, namely the eigenvalue of the O(2) generator sitting in Osp(4/2), while J and M_1, M_2 are the SU(2) isospin and the SU(3) quantum numbers respectively. These latter are defined by the number of boxes in the two rows of the SU(3) Young tableau:



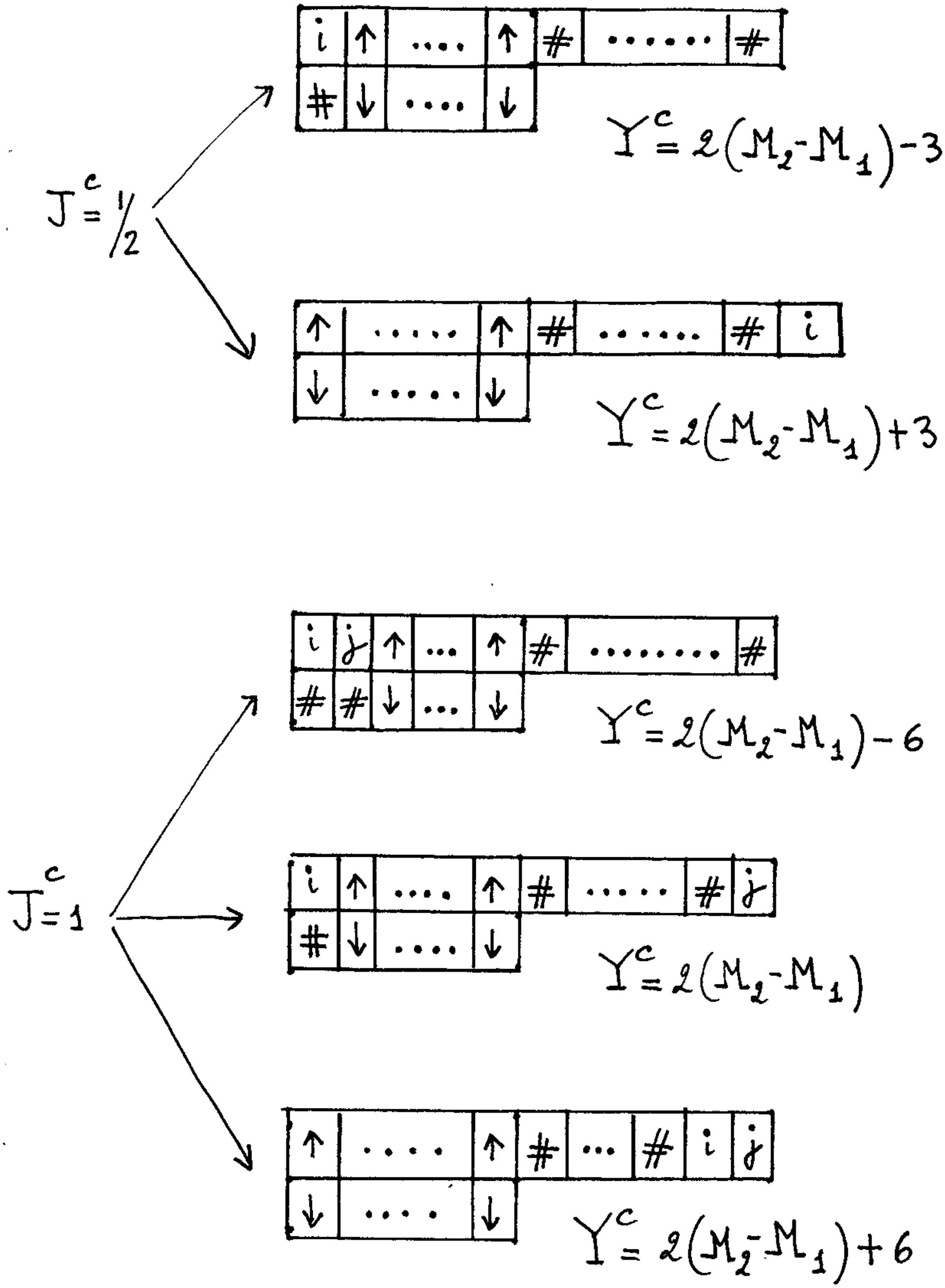
Y^C, on the other hand, is the eigenvalue of the colour hypercharge, namely the U(1) generator sitting in SU(3). Depending on the value of the colour isospin J^C the value of Y^C is linked in a different way to the labels M_1, M_2 of the SU(3) irreducible representation. As explained in Ref. 4), the possible values of Y^C for a given $\{M_1,M_2\}$ pair and a given $SU(2)^C \subset SU(3)$ irreducible representation are in one-to-one correspondence with the different ways of filling the Young tableau. Explicitly we find:

$J^c = 0$



(2.15a)

$$Y^c = 2(M_2 - M_1)$$



(2.15b)

(2.15c)

(2.15d)

(2.15e)

(2.15f)

From Eqs. (2.14), it is evident that for all bosonic representations the value of Y^W is an even number. In Ref. 4) it was seen that for all fermionic representations Y^W was odd. This is very reasonable because bosons transform into fermions and vice versa under the action of the Killing spinor which carries one unit of Y^W.

Given the decompositions into R-representations we can now calculate the eigenvalues of the invariant operators.

3. CALCULATION OF THE LAPLACIAN ($\boxtimes_{(0)^3}$) EIGENVALUES

According to Ref. 7) the relevant mass operator for the 0-forms from which one sector of the scalar fields of the four-dimensional theory originates, is the Laplacian operator:

$$\boxtimes_{(0)^3} Y = \Box Y = \mathcal{D}^{\mu}\mathcal{D}_{\mu} Y = M_{(0)^3} Y \tag{3.1}$$

To calculate its eigenvalues $M_{(0)^3}$, we begin by writing the expression of the field $[0|0]$ according to the constraints given in Eq. (2.11). We obtain

$$[0|0] = \sum_{\substack{\{M_1, M_2, J, Y\} \\ \text{satisfying (2.11)}}} \left\langle \begin{matrix} \uparrow \ldots \uparrow \, \# \ldots \# \\ \downarrow \ldots \downarrow \end{matrix} \middle| \begin{matrix} I_1 \ldots I_{M_2} \, K_1 \ldots K_{M_1} \\ J_1 \ldots J_{M_2} \end{matrix} \right\rangle \otimes$$

$$\otimes \langle \underbrace{\uparrow \ldots \uparrow}_{m_\uparrow} \underbrace{\downarrow \ldots \downarrow}_{m_\downarrow} | i_1 \ldots i_{2J} \rangle \otimes E^{Y^W} \otimes \tag{3.2}$$

$$S(x)^{I_1 \ldots I_{M_2} K_1 \ldots K_{M_1} \otimes i_1 \ldots i_{2J}}_{J_1 \ldots J_{M_2}}$$

where the harmonics used in (3.2) are explicitly defined in Eqs. (8.40) and (8.43) of Ref. 4). The symbol S(x) denotes the x-space scalar field which inherits from its associated harmonic a set of SU(3) and SU(2) indices, transforming in the irreducible $\{M_1,M_2,J\}$ representation. S(x) is complex. Using the abbreviations introduced in Eqs. (9.11) of Ref. 4), Eq. (3.2) can be rewritten as follows:

$$Y = [0|0] = \sum_{\{M_1,M_2,J,Y\}} \{0|M_1,M_2\} \otimes \{m_\uparrow, m_\downarrow|2J,Y\} \cdot S(M_1,M_2,2J,Y) \quad (3.3)$$

and the numbers $m_\uparrow$ and $m_\downarrow$ of spin-ups and spin-downs of the SU(2) harmonic are given below:

$$m_\uparrow = J + \frac{q}{2} Y^W \quad (3.4a)$$

$$m_\downarrow = J - \frac{q}{2} Y^W \quad (3.4b)$$

The self-conjugation condition of the field $[0|0]$ yields the following self-conjugation condition on the space-time field S:

$$\left\{ \begin{array}{l} \left[\tilde{S}(M_1,M_2,2J,Y) \right]^* = S(M_1^*,M_2^*,2J^*,Y^*) \\ \\ M_1^* = M_2 \\ M_2^* = M_1 \\ J^* = J \\ Y^* = -Y \end{array} \right. \quad (3.5)$$

Written explicitly, Eq. (3.5) reads:

$$
S\;\begin{array}{|c|c|c|c|c|c|}\hline I_1 & \quad\quad & I_{M_2^*} & K_1 & \quad\quad & K_{M_1^*} \\ \hline J_1 & & J_{M_2^*} & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|c|}\hline i_1 & \quad\quad\quad & i_{2J} \\ \hline \end{array} =
$$

$$
= \left(\frac{1}{2}\right)^{M_1} \varepsilon^{I_1 J_1 H_1} \ldots \varepsilon^{I_{M_2^*} J_{M_2^*} H_{M_2^*}} \varepsilon^{K_1 F_1 G_1} \ldots \varepsilon^{K_{M_1^*} F_{M_1^*} G_{M_1^*}} \times
\tag{3.6}
$$

$$
\times\, \varepsilon^{\dot{i}_1 \dot{j}_1} \ldots \varepsilon^{\dot{i}_{2J} \dot{j}_{2J}}
$$

$$
\left\{ S\;\begin{array}{|c|c|c|c|c|c|}\hline F_1 & \ldots & F_{M_2} & H_1 & \ldots\ldots & H_{M_1} \\ \hline G_1 & \ldots & G_{M_2} & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array} \otimes \begin{array}{|c|c|c|}\hline \dot{j}_1 & \ldots\ldots\ldots & \dot{j}_{2J} \\ \hline \end{array} \right\}^*
$$

where the relation between $\{M_1, M_2, J, Y\}$ and the labels of the conjugate representation are the following ones:

$$
\begin{cases} M_1^* = M_2 \quad ; \quad J^* = J \quad ; \quad Y^* = -Y \\ M_2^* = M_1 \end{cases}
\tag{3.7}
$$

To calculate the explicit form of the eigenvalues, we now use the splitting of the covariant derivative D_α into an R-derivative and the extra term due to the non-symmetricity of the coset S/R. Recalling Eqs. (10.8) and (10.13) of Ref. 4), we can write:

$$
\mathcal{D}_\alpha = \mathcal{D}_\alpha^R + M_\alpha^{\beta\gamma} T_{\beta\gamma}
\tag{3.8}
$$

where the explicit form of the term $M_{\alpha}^{\beta\gamma}$ is recalled here also for later use:

$$M_{C}^{AB} = 0 \; ; \; M_{m}^{AB} = 0 \; ; \; M_{3}^{AB} = f^{8AB}\left(\frac{\sqrt{3}\,pc}{3p^2+q^2+2r^2} - \frac{\sqrt{3}}{2}\,p\,\frac{a^2}{c}\right) \tag{3.9a}$$

$$M_{A}^{mn} = 0 \; ; \; M_{\ell}^{mn} = 0 \; ; \; M_{3}^{mn} = \varepsilon^{mn}\left(\frac{qc}{3p^2+q^2+2r^2} - \frac{qb^2}{2c}\right) \tag{3.9b}$$

$$M_{B}^{Am} = 0 \; ; \; M_{n}^{Am} = 0 \; ; \; M_{3}^{Am} = 0 \tag{3.9c}$$

$$M_{B}^{3A} = \frac{\sqrt{3}}{2}\,p\,\frac{a^2}{c}\,f^{8AB} \; ; \; M_{m}^{3A} = 0 \; ; \; M_{3}^{3A} = 0 \tag{3.9d}$$

$$M_{n}^{3m} = \frac{1}{2}\,\frac{qb^2}{c}\,\varepsilon^{mn} \; ; \; M_{A}^{3m} = 0 \; ; \; M_{3}^{3m} = 0 \tag{3.9e}$$

In Eqs. (3.9) p, q, r are the integer numbers defining the M^{pqr} coset and a,b, c are the vielbein rescalings needed to obtain an Einstein space. These latter are defined in Ref. 3) and for the case $p \neq 0$; $q \neq 0$ can be replaced by the more convenient parameters α, β, γ defined in Eq. (4.11) of Ref. 3).

In complete generality we obtain

$$\Box = \mathcal{D}_{\alpha}^{R}\mathcal{D}_{\alpha}^{R} + M_{\alpha}^{\alpha\beta}\mathcal{D}_{\beta}^{R} \tag{3.10}$$

and since $M_{\alpha}^{\alpha\beta} = 0$, as it is visible from (3.9), we get

$$\Box = \mathcal{D}^{R}_{\alpha}\mathcal{D}^{R}_{\alpha} = \frac{\alpha}{6}\frac{q^2}{p^2}\gamma^2\lambda_A\lambda_A + \frac{1}{2}\gamma^2\beta\,\sigma_m\sigma_m + \frac{1}{4}q^2\gamma^2\hat{Y}^2 \tag{3.11}$$

where λ_A are the 4,5,6,7 Gell-Mann lambda matrices, σ_m the 1,2 Pauli sigma matrices and $\hat{Y}$ is the weak-hypercharge operator. Applying Eq. (3.11) to the harmonic expansion (3.3) we easily obtain the final result:

$$M_{(0)^3} = \gamma^2\left\{\frac{2}{3}\alpha\frac{q^2}{p^2}\left(M_1+M_2+M_1M_2\right)+2\beta\left(J+J^2-\frac{q^2}{4}Y^2\right)+\frac{q^2}{4}Y^2\right\} \tag{3.12}$$

which holds for all M^{pqr} spaces. In particular on the supersymmetric space M^{111} we have

$$p = q = 1\,;\quad \alpha = \frac{1}{2}\,;\quad \beta = \frac{1}{4}\,;\quad \gamma = 8 \tag{3.13}$$

which yields:

$$M_{(0)^3} = 64\left\{\frac{1}{3}\left(M_1+M_2+M_1M_2\right)+\frac{1}{2}J(J+1)+\frac{1}{8}Y^2\right\} \tag{3.14}$$

Hence the condition for 0-modes, discussed in Table II of Ref. 7) gives:

$$M_{(0)^3} = 64 \Longrightarrow \begin{cases} M_1 = M_2 = 1\,;\ J = 0\,;\ Y = 0 \\ M_1 = M_2 = 0\,;\ J = 1\,;\ Y = 0 \end{cases} \tag{3.15a}$$

$$\mathcal{M}_{(0)^3} = 160 \Longrightarrow \begin{cases} \mathcal{M}_1 = 0 \,;\, \mathcal{M}_2 = 3 \,;\, J = 1 \,;\, Y = 2 \\ \\ \mathcal{M}_1 = 3 \,;\, \mathcal{M}_2 = 0 \,;\, J = 1 \,;\, Y = -2 \end{cases} \tag{3.15b}$$

The 0-modes of Eq. (3.15a) are the scalar fields sitting in the SU(3) and SU(2) gauge multiplets. The 0-modes of Eq. (3.15b), on the other hand, are the states quoted in the introduction. They might be elements of a hypermultiplet but they are not since, as we shall show in the last section, they also mix with spin 1, spin 3/2 and spin 2 fields.

4. CALCULATION OF THE *d ($\boxtimes_{(1)^3}$) EIGENVALUES

The relevant mass operator for 3^-forms, as discussed in Ref. 7) is:

$$\boxtimes_{(1)^3} Y_{\alpha\beta\gamma} = \frac{1}{24} \varepsilon_{\alpha\beta\gamma\mu\nu\rho\sigma} \mathcal{D}_\mu Y_{\nu\rho\sigma} = \mathcal{M}_{(1)^3} Y_{\alpha\beta\gamma} \tag{4.1}$$

Decomposing the vector indices $\alpha = (A,m,3)$ and the derivative D_α according to Eq. (3.8) we obtain

$$\boxtimes_{(1)^3} Y_{ABC} = \varepsilon_{ABCD} \varepsilon_{mn} \Big\{ \frac{ia}{16} \lambda_D Y_{mn3} - \frac{ib}{8} \sigma_m Y_{Dm3} - \frac{ic}{16} \hat{Y} Y_{Dmn} \Big\} \tag{4.2a}$$

$$\boxtimes_{(1)^3}\Upsilon_{ABm} = \varepsilon_{ABCD}\,\varepsilon_{mn}\left\{\frac{ia}{8}\lambda_C\Upsilon_{Dm3} + \frac{ib}{16}\sigma_m\Upsilon_{3CD} - \frac{ic}{16}\hat{\Upsilon}\,\Upsilon_{CDn}\right\} \tag{4.2b}$$

$$\boxtimes_{(1)^3}\Upsilon_{Amn} = \varepsilon_{ABCD}\,\varepsilon_{mn}\left\{\frac{ia}{16}\lambda_B\Upsilon_{CD3} - \frac{ic}{48}\hat{\Upsilon}\,\Upsilon_{BCD}\right\} \tag{4.2c}$$

$$\boxtimes_{(1)^3}\Upsilon_{AB3} = \varepsilon_{ABCD}\,\varepsilon_{mn}\left\{\frac{ia}{16}\lambda_C\Upsilon_{Dmn} + \frac{ib}{16}\sigma_m\Upsilon_{CDn}\right\} + \tag{4.2d}$$

$$+\frac{i}{8}\frac{g}{p}\alpha\gamma\,\lambda^{[A}_{i\#}\lambda^{B]}_{\#i}\,\varepsilon^{mn}\Upsilon_{3mn} - \frac{\gamma\beta}{4}\varepsilon_{ABCD}\Upsilon_{CD3}$$

$$\boxtimes_{(1)^3}\Upsilon_{Am3} = \varepsilon_{ABCD}\,\varepsilon_{mn}\left\{-\frac{ia}{16}\lambda_B\Upsilon_{CDm} + \frac{ib}{48}\sigma_m\Upsilon_{BCD}\right\} \tag{4.2e}$$

$$+\frac{1}{2\sqrt{3}}\frac{g}{p}\alpha\gamma\, f^{8AB}\varepsilon^{mn}\Upsilon_{Bm3}$$

$$\boxtimes_{(1)^3}\Upsilon_{mn3} = \varepsilon_{ABCD}\,\varepsilon_{mn}\left\{\frac{ia}{48}\lambda_A\Upsilon_{BCD}\right\} \tag{4.2f}$$

$$-\frac{1}{4\sqrt{3}}\frac{g}{p}\alpha\gamma\,\varepsilon^{mn} f^{8AB}\Upsilon_{AB3}$$

Now recalling the definition of the creation and annihilation operators acting on Young tableaux, which was given in Eqs. (10.10) of Ref. 4) and inserting the expansions (2.8) into Eq. (4.2), we find:

$$\boxtimes_{(1)^3} \langle 3|\mathrm{I}\rangle_{\ddot{ij}} = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{\varepsilon_{jk}\left(A^{\dagger}_{\#}A_{k}\right)\langle 3|\mathrm{III}\rangle_{i} - \varepsilon_{jk}\left(A^{\dagger}_{\dot{i}}A_{\#}\right)\langle 3|\mathrm{IV}\rangle_{k} + (i\leftrightarrow j)\right\} - \frac{\gamma}{8}\sqrt{2\beta}\left\{B^{\dagger}_{\downarrow}B_{\uparrow}[3|\mathrm{I}]_{\dot{i}\dot{j}}\right\} + \frac{\gamma}{8}q\left\{\hat{Y}\langle 3|\mathrm{I}\rangle_{\ddot{ij}}\right\} \tag{4.3a}$$

$$\boxtimes_{(1)^3}[3|\mathrm{I}]_{\dot{i}\dot{j}} = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{i\left[\varepsilon_{jl}\left(A^{\dagger}_{\#}A_{l}\right)\langle 3|\mathrm{II}\rangle_{i} - \varepsilon_{jl}\left(A^{\dagger}_{\dot{i}}A_{\#}\right)\langle 3|\mathrm{II}\rangle^{*}_{l} + (i\leftrightarrow j)\right\} + \frac{\gamma\beta}{2}[3|\mathrm{I}]_{\dot{i}\dot{j}} - \frac{\gamma}{8}\sqrt{2\beta}\left\{2\left(B^{\dagger}_{\uparrow}B_{\downarrow}\right)\langle 3|\mathrm{I}\rangle_{\ddot{ij}} - 2\varepsilon_{il}\varepsilon_{jk}\left(B^{\dagger}_{\downarrow}B_{\uparrow}\right)\langle 3|\mathrm{I}\rangle^{*}_{lk}\right\} \tag{4.3b}$$

$$\boxtimes_{(1)^3}\langle 3|\mathrm{I}\rangle_{i} = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{iA^{\dagger}_{\dot{i}}A_{\#}[3|\mathrm{II}]_{\bullet}\right\} + + \frac{\gamma}{8}\sqrt{2\beta}\left\{2B^{\dagger}_{\uparrow}B_{\downarrow}\langle 3|\mathrm{III}\rangle_{i} - 2B^{\dagger}_{\downarrow}B_{\uparrow}\langle 3|\mathrm{IV}\rangle_{i}\right\} - \frac{\gamma}{8}q\left\{i\hat{Y}\langle 3|\mathrm{II}\rangle_{i}\right\} \tag{4.3c}$$

$$\boxtimes_{(1)^3} \langle 3|\mathrm{II}\rangle_i = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2i\varepsilon_{ij} A^{+}_{\#} A_j \langle 3|\mathrm{I}\rangle_{\cdot} +$$

$$+ A^{+}_{i} A_{\#} [3|\mathrm{I}]_{\cdot} + i A^{+}_{k} A_{\#} \varepsilon_{km} [3|\mathrm{I}]_{mi}\Big\} \tag{4.3d}$$

$$+ \frac{\gamma}{8} q \Big\{ i \hat{\Upsilon} \langle 3|\mathrm{I}\rangle_i \Big\}$$

$$\boxtimes_{(1)^3} \langle 3|\mathrm{III}\rangle_i = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2\varepsilon_{ij}\left(A^{+}_{\#} A_j\right)\langle 3|\mathrm{III}\rangle^{*}$$

$$-i\left(A^{+}_{i} A_{\#}\right)\langle 3|\mathrm{IV}\rangle + \left(A^{+}_{j} A_{\#}\right)\varepsilon_{jk}\langle 3|\mathrm{I}\rangle_{ki}\Big\} \tag{4.3e}$$

$$-\frac{\gamma}{8}\sqrt{2\beta}\Big\{\left(B^{+}_{\downarrow} B_{\uparrow}\right)\langle 3|\mathrm{I}\rangle_i\Big\} - \frac{1}{4}\frac{q}{p}\alpha\gamma\langle 3|\mathrm{III}\rangle_i$$

$$\boxtimes_{(1)^3} \langle 3|\mathrm{IV}\rangle_i = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2\varepsilon_{ij}\left(A^{+}_{\#} A_j\right)\langle 3|\mathrm{II}\rangle^{*}$$

$$-i\left(A^{+}_{i} A_{\#}\right)\langle 3|\mathrm{IV}\rangle^{*} + \left(A^{+}_{j} A_{\#}\right)\varepsilon_{ik}\langle 3|\mathrm{I}\rangle^{*}_{jk}\Big\} \tag{4.3f}$$

$$+\frac{\gamma}{8}\sqrt{2\beta}\Big\{\left(B^{+}_{\uparrow} B_{\downarrow}\right)\langle 3|\mathrm{I}\rangle_i\Big\} + \frac{1}{4}\frac{q}{p}\alpha\gamma\langle 3|\mathrm{IV}\rangle_i$$

$$\boxtimes_{(1)^3}\langle 3|\mathrm{I}\rangle_\bullet = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{ i\,\varepsilon_{ij}\left(A^\dagger_i A_\#\right)\langle 3|\mathrm{II}\rangle_j \right. \tag{4.3g}$$

$$+\frac{\gamma}{8}\sqrt{2\beta}\left\{ 2\left(B^\dagger_\uparrow B_\downarrow\right)\langle 3|\mathrm{III}\rangle^*_\bullet - 2\left(B^\dagger_\downarrow B_\uparrow\right)\langle 3|\mathrm{II}\rangle^*_\bullet \right\}$$

$$-\frac{\gamma}{2}\beta\langle 3|\mathrm{I}\rangle_\bullet$$

$$\boxtimes_{(1)^3}\langle 3|\mathrm{II}\rangle_\bullet = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{ \varepsilon_{ij}\left(A^\dagger_\# A_i\right)\langle 3|\mathrm{IV}\rangle_j \right\} \tag{4.3h}$$

$$+\frac{\gamma}{8}\sqrt{2\beta}\left\{\left(B^\dagger_\downarrow B_\uparrow\right)\langle 3|\mathrm{I}\rangle^*_\bullet\right\} - \frac{\gamma}{8}q\left\{\hat{Y}\langle 3|\mathrm{II}\rangle_\bullet\right\}$$

$$\boxtimes_{(1)^3}\langle 3|\mathrm{III}\rangle_\bullet = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{ \varepsilon_{ij}\,A^\dagger_\# A_i\,\langle 3|\mathrm{III}\rangle^*_j \right\}$$

$$-\frac{\gamma}{8}\sqrt{2\beta}\left\{B^\dagger_\uparrow B_\downarrow\langle 3|\mathrm{I}\rangle^*_\bullet\right\} + \frac{\gamma}{8}q\left\{\hat{Y}\langle 3|\mathrm{III}\rangle_\bullet\right\} \tag{4.3i}$$

$$\boxtimes_{(1)^3}\langle 3|\mathrm{IV}\rangle_\bullet = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\left\{ i\,A^\dagger_i A_\#\langle 3|\mathrm{IV}\rangle^*_i - i\,A^\dagger_\# A_i\langle 3|\mathrm{III}\rangle_i \right\}$$

$$+\frac{\gamma}{8}\sqrt{2\beta}\left\{B^\dagger_\downarrow B_\uparrow[3|\mathrm{I}]_\bullet\right\} - \frac{\gamma}{8}q\left\{\hat{Y}\langle 3|\mathrm{IV}\rangle_\bullet\right\} \tag{4.3j}$$

$$\boxtimes_{(1)^3}[3|\mathrm{I}]_{\bullet} = \frac{\gamma}{8}\sqrt{\frac{2\alpha}{3}}\,\frac{q}{p}\Big\{\big(\overset{+}{A}_{\#}A_i\big)\langle 3|\mathrm{II}\rangle_i - \big(\overset{+}{A}_i A_{\#}\big)\langle 3|\mathrm{II}\rangle_i^{*}\Big\} - \frac{\gamma}{8}\sqrt{2\beta}\Big\{2\big(\overset{+}{B}_{\uparrow}B_{\downarrow}\big)\langle 3|\mathrm{IV}\rangle_{\bullet} - 2\big(\overset{+}{B}_{\downarrow}B_{\uparrow}\big)\langle 3|\mathrm{IV}\rangle_{\bullet}^{*}\Big\} + \frac{1}{4}\frac{q}{p}\alpha\gamma\,[3|\mathrm{II}]_{\bullet} - \frac{\gamma\beta}{2}\,[3|\mathrm{I}]. \tag{4.3k}$$

$$\boxtimes_{(1)^3}[3|\mathrm{II}]_{\bullet} = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2i\,\overset{+}{A}_{\#}A_i\,\langle 3|\mathrm{I}\rangle_i + \overset{+}{A}_i A_{\#}\langle 3|\mathrm{I}\rangle_i^{*}\Big\} + \frac{1}{2}\frac{q}{p}\alpha\gamma\,[3|\mathrm{I}]. \tag{4.3ℓ}$$

Now in order to obtain the numerical matrix whose eigenvalues are the desired eigenvalues of $\boxtimes_{(1)^3}$ on a given irreducible representation, we have to insert the harmonic expansion of each $\langle\,|\,\rangle$ field into (4.3) and let the creation-annihilation operators act on the tableaux. To this purpose we have to consider the spectra (32.14) and, for each possible $SU(3)\otimes SU(2)\otimes U(1)$ irreducible representation, see how many fragments $\langle\,|\,\rangle$ have that irreducible representation in their expansion. Let us begin by noting that, calling $m_{\uparrow}$ the number of spin-ups and $m_{\downarrow}$ the number of spin-downs of the SU(2) harmonic, the result (3.4), holding for the $[0|0]$ case, generalizes to all the fragments in the following way:

$$m_{\uparrow} = J + \frac{q}{2}Y^{W} + \Delta_{\uparrow} \tag{4.4a}$$

$$m_{\downarrow} = J - \frac{q}{2}Y^{W} - \Delta_{\uparrow} \tag{4.4b}$$

where

$$\Delta_{\uparrow} = \begin{cases} \nearrow 1 \\ \rightarrow 0 \\ \searrow -1 \end{cases} \tag{4.5}$$

Hence the complete SU(3) $\otimes$ U(2) $\otimes$ U(1) harmonic is specified, in each irreducible representation, by giving the SU(3) tableau filling pattern and the number $\Delta_{\uparrow}$. With this proviso we conclude that there are in total three types of irreducible representations contributing to the 3-form expansion and the allowed fragments in each type are exhibited in Table 1.

If we collectively denote by ρ the set of quantum numbers which identify an irreducible representation

$$\rho = \{M_1, M_2, J, Y\} \tag{4.6}$$

We see that set $\{\rho\}^{I}$ is closed under conjugation:

$$* : \{\rho\}^{I} \longrightarrow \{\rho\}^{I} \tag{4.7}$$

while the sets $\{\rho\}^{II}$ and $\{\rho\}^{III}$ are the conjugate of each other. This implies that from Eqs. (4.3), we actually obtain two numerical matrices: one corresponding to the representation group $\{\rho\}^{I}$, the other to the representation group: $\{\rho\}^{I} \oplus \{\rho\}^{II}$. Indeed since the operator $\boxtimes_{(1)^3}$ involves $*$-operations on the fields, one representation and its conjugate contribute to the same matrix.

We introduce the following notation for the x-space fields which are coefficients of the harmonics in each fragment:

$$\pi\langle 1, I | \rho\rangle \longleftrightarrow \langle 3 | I\rangle_{ij}$$

$$\begin{aligned}
&\pi[1,\mathrm{I}|\rho] \longleftrightarrow [3|\mathrm{I}]_{ij}\\
&\pi\langle \tfrac{1}{2},\mathrm{I}|\rho\rangle \longleftrightarrow \langle 3|\mathrm{I}\rangle_i\\
&\qquad\vdots\\
&\pi\langle 0,\mathrm{I}|\rho\rangle \longleftrightarrow \langle 3|\mathrm{I}\rangle .
\end{aligned} \tag{4.8}$$

$\langle\pi\ ,\ |\rangle$ are pseudoscalar fields in x-space: their first label (1,½ or 0) recalls the colour isospin of the fragment they are coming from, the second roman lable tells you which particular fragment and ρ is the set of $SU(3) \otimes SU(2) \otimes U(1)$ representation labels. It is understood that π has also a set of $SU(3) \otimes SU(2) \otimes U(1)$ indices running in the ρ-representation. On these indices the conjugation operation $\sim$ defined in Eq. (9.5) of Ref. 4) can be implemented. Utilizing these conventions from Eqs. (4.3) we obtain the following results:

A) <u>Action of</u> $\boxtimes_{(1)^3}$ <u>in the first representation group $\{\rho\}^{\mathrm{I}}$</u>:

$$\begin{aligned}
\boxtimes_{(1)^3}\, \pi\langle 1,\mathrm{I}|\rho\rangle = 2\,\frac{\gamma}{8}\,\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{ M_2\, \pi\langle \tfrac{1}{2},\mathrm{III}|\rho\rangle \\
- M_1\, \widetilde{\pi\langle 1/2\,,\mathrm{IV}|\rho^*\rangle}\Big\} - \frac{\gamma}{8}\sqrt{2\beta}\,\Big\{ J + \frac{q}{2}Y\Big\}\,\pi[1,\mathrm{I}|\rho] \\
+ \frac{\gamma}{8}\, q\, Y\, \pi\langle 1,\mathrm{I}|\rho\rangle
\end{aligned} \tag{4.9a}$$

$$\begin{aligned}\boxtimes_{(1)^3}\pi[1,\mathrm{I}|\varrho] &= 2i\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{M_2\pi\langle\tfrac{1}{2},\mathrm{II}|\varrho\rangle \\ &+ M_1\overgroup{\pi\langle\tfrac{1}{2},\mathrm{II}|\varrho^*\rangle}\Big\} - \frac{\gamma}{8}\sqrt{2\beta}\Big\{2\left(1+J-\frac{q}{2}Y\right)\times \\ &\times\pi\langle 1,\mathrm{I}|\varrho\rangle - 2\left(1+J+\frac{q}{2}Y\right)\overgroup{\pi\langle 1,\mathrm{I}|\varrho^*\rangle}\Big\}\end{aligned} \tag{4.9b}$$

$$\begin{aligned}\boxtimes_{(1)^3}\pi\langle\tfrac{1}{2},\mathrm{I}|\varrho\rangle &= i\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{M_1\pi[0,\mathrm{II}|\varrho]\Big\} \\ &+ \frac{\gamma}{8}\sqrt{2\beta}\Big\{2\left(1+J-\frac{q}{2}Y\right)\pi\langle\tfrac{1}{2},\mathrm{III}|\varrho\rangle \\ &- 2\left(1+J+\frac{q}{2}Y\right)\pi\langle\tfrac{1}{2},\mathrm{IV}|\varrho\rangle\Big\} \\ &- \frac{i\gamma}{8}qY\pi\langle\tfrac{1}{2},\mathrm{II}|\varrho\rangle\end{aligned} \tag{4.9c}$$

$$\begin{aligned}\boxtimes_{(1)^3}\pi\langle\tfrac{1}{2},\mathrm{II}|\varrho\rangle &= \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{M_1\pi[0,\mathrm{I}|\varrho] \\ &- \frac{i}{2}(M_1+2)\pi[1,\mathrm{I}|\varrho]\Big\}+ \\ &+ \frac{\gamma}{8}qiY\pi\langle 1/2,\mathrm{I}|\varrho\rangle\end{aligned} \tag{4.9d}$$

$$\boxtimes_{(1)^3} \pi \langle \tfrac{1}{2}, \mathrm{III} | \varrho \rangle = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{ -iM_1 \pi \langle 0, \mathrm{IV} | \varrho \rangle$$

$$-\frac{1}{2}(M_1+2)\pi\langle 1, \mathrm{I} | \varrho \rangle \Big\} - \frac{\gamma}{8}\sqrt{2\beta}\Big\{ J + \frac{q}{2} Y \Big\} \pi \langle \tfrac{1}{2}, \mathrm{I} | \varrho \rangle$$

$$-\frac{1}{4}\frac{q}{p}\alpha\gamma\, \pi \langle \tfrac{1}{2}, \mathrm{III} | \varrho \rangle \tag{4.9e}$$

$$\boxtimes_{(1)^3} \pi \langle \tfrac{1}{2}, \mathrm{IV} | \varrho \rangle = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{ iM_1 \widetilde{\pi \langle 0, \mathrm{IV} | \varrho^* \rangle}$$

$$-\frac{1}{2}(M_1+2)\widetilde{\pi\langle 1, \mathrm{I} | \varrho^* \rangle} \Big\} + \frac{\gamma}{8}\sqrt{2\beta}\Big\{ J - \frac{q}{2} Y \Big\} \pi \langle \tfrac{1}{2}, \mathrm{I} | \varrho \rangle$$

$$+\frac{1}{4}\frac{q}{p}\alpha\gamma\, \pi \langle \tfrac{1}{2}, \mathrm{IV} | \varrho \rangle \tag{4.9f}$$

$$\boxtimes_{(1)^3} \pi \langle 0, \mathrm{IV} | \varrho \rangle = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{ i(M_1+2) \widetilde{\pi \langle \tfrac{1}{2}, \mathrm{IV} | \varrho^* \rangle}$$

$$-i(M_2+2)\pi\langle \tfrac{1}{2}, \mathrm{III} | \varrho \rangle \Big\} + \frac{\gamma}{8}\sqrt{2\beta}\Big\{ J + \frac{q}{2} Y \Big\} \pi [0, \mathrm{I} | \varrho]$$

$$-\frac{\gamma}{8} q Y\, \pi \langle 0, \mathrm{IV} | \varrho \rangle \tag{4.9g}$$

$$\boxtimes_{(1)^3}\pi[0,\mathrm{I}|\rho] = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{(M_2+2)\pi\langle\tfrac{1}{2},\mathrm{II}|\rho\rangle + (M_1+2)\widetilde{\pi\langle\tfrac{1}{2},\mathrm{II}|\rho^*\rangle}\Big\}$$

$$+\frac{\gamma}{8}\sqrt{2\beta}\Big\{2\big(J-\tfrac{q}{2}Y+1\big)\pi\langle 0,\mathrm{IV}|\rho\rangle + 2\big(1+J+\tfrac{q}{2}Y\big)\widetilde{\pi\langle 0,\mathrm{IV}|\rho^*\rangle}\Big\}$$

$$+\frac{1}{4}\frac{q}{p}\alpha\gamma\,\pi[0,\mathrm{II}|\rho] - \frac{\gamma\beta}{2}\pi[0,\mathrm{I}|\rho] \tag{4.9h}$$

$$\boxtimes_{(1)^3}\pi[0,\mathrm{II}|\rho] = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big[2i(M_2+2)\pi\langle\tfrac{1}{2},\mathrm{I}|\rho\rangle - 2i(M_1+2)\widetilde{\pi\langle\tfrac{1}{2},\mathrm{I}|\rho^*\rangle}\Big] + \frac{1}{2}\frac{q}{p}\alpha\gamma\,\pi[0,\mathrm{I}|\rho] \tag{4.9i}$$

B) Action of $\boxtimes_{(1)^3}$ in the second representation group $\{\rho\}^{\mathrm{II}}$

$$\boxtimes_{(1)^3}\pi\langle 1,\mathrm{I}|\rho\rangle = \frac{\gamma}{4}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{(M_2-1)\pi\langle\tfrac{1}{2},\mathrm{III}|\rho\rangle\Big\}$$

$$-\frac{\gamma}{8}\sqrt{2\beta}\big(J+\tfrac{q}{2}Y\big)\pi[1,\mathrm{I}] + \frac{\gamma}{8}qY\pi\langle 1,\mathrm{I}|\rho\rangle \tag{4.10a}$$

$$\boxtimes_{(1)^3}\,\pi[1,\mathrm{I}|\varrho] = \frac{\gamma}{4}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{ i\left(M_2-1\right)\pi\langle \tfrac{1}{2},\mathrm{II}|\varrho\rangle \Big\}$$

$$-\frac{\gamma}{8}\sqrt{2\beta}\Big\{2\left(1+J-\frac{q}{2}Y\right)\pi\langle 1,\mathrm{I}|\varrho\rangle +$$

$$+2\left(1+J+\frac{q}{2}Y\right)\widetilde{\pi\langle 1,\mathrm{I}|\varrho^{*}\rangle}\Big\} + \frac{\gamma\beta}{2}\pi[1,\mathrm{I}|\varrho] \qquad (4.10b)$$

$$\boxtimes_{(1)^3}\,\pi\langle \tfrac{1}{2},\mathrm{I}|\varrho\rangle = \frac{\gamma}{8}\sqrt{2\beta}\Big\{2\left(1+J-\frac{q}{2}Y\right)\pi\langle \tfrac{1}{2},\mathrm{III}|\varrho\rangle$$

$$-2\left(1+J+\frac{q}{2}Y\right)\pi\langle \tfrac{1}{2},\mathrm{IV}|\varrho\rangle\Big\} - \frac{\gamma}{8}\,i q Y\,\pi\langle \tfrac{1}{2},\mathrm{II}|\varrho\rangle$$

(4.10c)

$$\boxtimes_{(1)^3}\,\pi\langle \tfrac{1}{2},\mathrm{II}|\varrho\rangle = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2 i M_2\,\pi\langle 0,\mathrm{I}|\varrho\rangle$$

$$- i\left(M_1+3\right)\pi[1,\mathrm{I}|\varrho]\Big\} +$$

$$+\frac{\gamma}{8}\,i q Y\,\pi\langle 1/2,\mathrm{I}|\varrho\rangle \qquad (4.10d)$$

$$\boxtimes_{(1)^3}\pi\langle\tfrac{1}{2},\mathrm{III}|\rho\rangle = -\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2M_2\,\widetilde{\pi\langle 0,\mathrm{III}|\rho^*\rangle}$$
$$-(M_1+3)\,\pi\langle 1,\mathrm{I}|\rho\rangle\Big\} - \frac{\gamma}{8}\sqrt{2\beta}\left(J+\frac{q}{2}Y\right)\pi\langle\tfrac{1}{2},\mathrm{I}|\rho\rangle \tag{4.10e}$$
$$-\frac{1}{4}\frac{q}{p}\alpha\gamma\,\pi\langle\tfrac{1}{2},\mathrm{III}|\rho\rangle$$

$$\boxtimes_{(1)^3}\pi\langle\tfrac{1}{2},\mathrm{IV}|\rho\rangle = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{2M_2\,\widetilde{\pi\langle 0,\mathrm{II}|\rho^*\rangle}$$
$$+(M_1+3)\,\widetilde{\pi\langle 1,\mathrm{I}|\rho^*\rangle}\Big\} + \tag{4.10f}$$
$$+\frac{\gamma}{8}\sqrt{2\beta}\left(J-\frac{q}{2}Y\right)\pi\langle\tfrac{1}{2},\mathrm{I}|\rho\rangle$$
$$+\frac{1}{4}\frac{q}{p}\alpha\gamma\,\pi\langle\tfrac{1}{2},\mathrm{IV}|\rho\rangle$$

$$\boxtimes_{(1)^3}\pi\langle 0,\mathrm{I}|\rho\rangle = \frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{-i(M_1+2)\,\pi\langle\tfrac{1}{2},\mathrm{II}|\rho\rangle\Big\}$$
$$-2\frac{\gamma}{8}\sqrt{2\beta}\Big\{\left(J-\frac{q}{2}Y+1\right)\widetilde{\pi\langle 0,\mathrm{III}|\rho^*\rangle} \tag{4.10g}$$
$$-\left(J+1+\frac{q}{2}Y\right)\widetilde{\pi\langle 0,\mathrm{II}|\rho^*\rangle}\Big\} - \frac{\gamma}{2}\beta\,\pi\langle 0,\mathrm{I}|\rho\rangle$$

C) Action of $\boxtimes_{(1)^3}$ in the third representation group $\{\rho\}^{III}$

$$\boxtimes_{(1)^3} \pi\langle 1, I|\rho\rangle = -\frac{\gamma}{4}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{(M_1-1)\widetilde{\pi\langle \tfrac{1}{2}, IV|\rho^*\rangle}$$
$$-\frac{\gamma}{8}\sqrt{2\beta}\Big\{J+\frac{q}{2}Y\Big\}\pi[1,I|\rho]$$
$$+\frac{\gamma}{8}qY\pi\langle 1,I|\rho\rangle \tag{4.11a}$$

$$\boxtimes_{(1)^3} \pi[1,I|\rho] = \frac{i}{2}\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\Big\{-(M_1-1)\widetilde{\pi\langle \tfrac{1}{2}, II|\rho^*\rangle}\Big\}$$
$$+\frac{\gamma}{8}\sqrt{2\beta}\Big\{2\Big(1+J-\frac{q}{2}Y\Big)\pi\langle 1,I|\rho\rangle$$
$$+2\Big(1+J+\frac{q}{2}Y\Big)\widetilde{\pi\langle 1,I|\rho^*\rangle}\Big\}+\frac{\gamma}{2}\beta\,\pi[1,I] \tag{4.11b}$$

The final answer is now obtained by calculating the eigenvalues of the two matrices given in Table II and III respectively. The second one, corresponding to the representation group $\{\rho\}^{II} + \{\rho\}^{III}$ is simply the recollection of formulae (4.10) and (4.11). The first one, on the other hand, which corresponds to the representation group $\{\rho\}^{I}$ is obtained from Eqs. (4.9) splitting every complex field $\pi\langle\ |\rho\rangle$ in the following way:

$$\pi\langle\ ,\ |\rho\rangle = \varphi_+\langle\ ,\ |\rho\rangle + i\varphi_-\langle\ ,\ |\rho\rangle \tag{4.12}$$

where both ϕ_+ and ϕ_- are self-conjugate:

$$\widetilde{\varphi_\pm\langle\ ,\ |\rho^*\rangle} = \varphi_\pm\langle\ ,\ |\rho\rangle \tag{4.13}$$

When the field is already either self-conjugate or antiself-conjugate we set either ϕ_+ or ϕ_- equal to zero. It is obvious that the eigenvalues cannot be given in closed form due to the order of the secular equations (15th and 10th respectively). They can, however, be computed numerically for each specific representation one is interested in.

In each representation group, there are exceptional cases where some fragments are absent. This is seen by inspection of Table 1. (For instance, if $M_1 = M_2 = 0$ all $J^C = 1$ and $J^C = 0$ fragments are absent or if $J = 0$ all $\Delta_\uparrow = 1$ and $\Delta_\uparrow = -1$ fragments are ruled out.) In these cases, it suffices to scratch the corresponding rows and columns and calculate the eigenvalues of the reduced matrix obtained in this way.

Relying on this it is easily seen that one has the following 0-modes on the space M^{111}:

$M_{(1)^3} = 2$ $\hookrightarrow m_\pi^2 = 0$

- $M_1 = M_2 = 1;\ J = 0;\ Y = 0 \Longrightarrow$ pseudoscalars in the $N = 2$ SU_3 gauge multiplet
- $M_1 = M_2 = 0;\ J = 1;\ Y = 0 \Longrightarrow$ pseudoscalars in the $N = 2$ SU_2 gauge multiplet

$M_{(1)^3} = 1$ $\hookrightarrow m_\pi^2 = 0$

- $M_1 = M_2 = J = Y = 0 \Longrightarrow$ pseudoscalars in the $N = 2$ U_1-gauge multiplet due to the Betti number $B_2 \neq 0$

With a computer programme we have checked that there are no other 0-modes. This means that in the N = 2 theory, there are no massless hypermultiplets. As we have already emphasized this does not exclude the existence of massive hypermultiplets which are equally important. Our present results coupled to those of Ref. 7) on the algebra of harmonics and of Ref. 10) on the Osp(4/N) multiplet structure can be used for their systematic search.

5. THE BETTI-FORM AND THE BETTI-MULTIPLET

In Ref. 8) it has been shown that if a harmonic 2-form exists on the manifold, it has to be in the singlet representation of the isometry group S and moreover from Ref. 7) we know that it must be a linear combination of the generators of the holonomy algebra. Let us then look at the constraints (2.13) for the fragments contributing to the 2-forms. Since we want a singlet of the isometry group we must take only the singlets of the isotropy group. This leaves us with [2,I] and [2,II] hence with the following 2-form:

$$\left.\begin{aligned} Y_{AB} &= ia\,\lambda^{[A}_{m\#}\lambda^{B]}_{\#m} \\ Y_{mu} &= \varepsilon_{mu}\,b \\ Y_{3A} &= Y_{3m} = Y_{Am} = 0 \end{aligned}\right\} \tag{5.1}$$

where a, b are two numerical real constants. Utilizing the identity

$$\lambda^{[A}_{i\#}\lambda^{B]}_{\#i} = i\frac{2}{\sqrt{3}}f^{8AB} \tag{5.2}$$

we see that

$$Y_{AB} = -\frac{2}{\sqrt{3}}a\,f^{8AB}\,;\; Y_{mu} = b\,\varepsilon_{mu}\,;\; Y_{3A} = Y_{3m} = Y_{Am} = 0 \tag{5.3}$$

Hence, comparing with Eq. (6.12) of Ref. 3), we can conclude that

$$\tau_{\alpha\beta}Y_{\alpha\beta} = -2a\tau_{45} - 2a\tau_{67} + 2b\tau_{12} \tag{5.4}$$

is in the holonomy algebra if and only if

$$a = -\frac{1}{2}b \tag{5.5}$$

Hence if a harmonic form exists on M^{111} it can be only the following one

$$\Omega = \Omega_{\alpha\beta} B^\alpha \wedge B^\beta = B^4 \wedge B^5 + B^6 \wedge B^7 - 2 B^1 \wedge B^2 \tag{5.6}$$

Explicit computation with the spin connection given in (4.1) of Ref. 3) yields

$$d\Omega = 0 \quad ; \quad \mathcal{D}^\alpha \Omega_{\alpha\beta} = 0 \tag{5.7}$$

so that Ω is indeed harmonic and the Betti number B_2 is equal to 1 for M^{111}. Actually, one knows that it is 1 for all M^{pqr} spaces. Use of the general formulae given in Table VIII of Ref. 7) enables us to write the harmonics of the entire Betti multiplet.

The important thing is that we know at this point that there is an extra massless N = 2 U(1) gauge multiplet besides those gauging SU(3) and SU(2).

6. A LOOK AT THE MULTIPLET OF THE EXTRA SCALAR 0-MODES

We promised in the introduction to show that the 0-modes (3.15b) of the Laplacian scalar sector are members of a multiplet containing also spin 1 fields. The argument is simple. From the mass-formula

$$\begin{aligned} m_S^2 &= m_{\lambda_L}(m_{\lambda_L} - 4) \qquad \text{if } m_{\lambda_L} > -10 \\ m_S^2 &= (m_{\lambda_L} + 24)(m_{\lambda_L} + 20) \qquad \text{if } m_{\lambda_L} < -10 \end{aligned} \tag{6.1}$$

given in Ref. 7) we know that the multiplet, of which our scalars are members of, contains a spinor field in the $M_1 = 0$; $M_2 = 3$; $J = 1$ $Y = \genfrac{<}{}{0pt}{}{1}{3}$ representation of the isometry group having for its mass one of the following values 4, 0, -20, -24. From Ref. 4), we see that there is a unique solution to these constraints provided by a state belonging to exceptional series "4" namely the state with $M_1 = 0$; $M_2 = 3$; $J = 1$, $Y = 3$ and $m_{\frac{1}{2}} = -24$. Its harmonic is of the form

$$\boxed{4}_{(m=-24)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \chi \\ 0 \\ 0 \\ \chi^* \\ 0 \end{pmatrix} \qquad (6.2)$$

where χ is proportional to the proper $SU(3) \otimes SU(2) \otimes U(1)$ harmonic. Inserting (6.2) in the formula [see Ref. 7)] which gives the harmonic of the vector in which this spin ½ can transform, we get

$$\Upsilon_\alpha = \bar{\eta}\, \tau_\alpha \boxed{4} - \frac{1}{21}\, \bar{\eta}\, \mathcal{D}_\alpha \boxed{4} \neq 0 \qquad (6.3)$$

Hence there is at least a vector field in the multiplet of these ($\underline{10}$,$\underline{3}$) scalars. This suffices to conclude that it is not a hypermultiplet.

Table 1: Table of SU(3) ⊗ SU(2) ⊗ U(1) irreducible representations in the 3-form

	IRREP	$\langle 3\|I\rangle_{ij}$	$[3\|I]_{ij}$	$\langle 3\|I\rangle_{i}$	$\langle 3\|II\rangle_{i}$	$\langle 3\|III\rangle_{i}$	$\langle 3\|IV\rangle_{i}$	$\langle 3\|I\rangle_{\cdot}$	$\langle 3\|II\rangle_{\cdot}$	$\langle 3\|III\rangle_{\cdot}$	$\langle 3\|IV\rangle_{\cdot}$	$[3\|I]_{\cdot}$	$[3\|II]_{\cdot}$
$\{S\}^{I}$	$Y=2k$ $(M_2-M_1)=$ $3pk$	i↑..↑#..#j #↓..↓ $\Delta_\uparrow=-1$	i↑..↑#..#j #↓..↓ $\Delta_\uparrow=0$	↑..↑#..#i ↓..↓ $\Delta_\uparrow=0$	↑..↑#..#i ↓..↓ $\Delta_\uparrow=0$	↑..↑#..#i ↓..↓ $\Delta_\uparrow=-1$	↑..↑#..#i ↓..↓ $\Delta_\uparrow=1$				↑..↑#..# ↓..↓ $\Delta_\uparrow=-1$	↑..↑#..# ↓..↓ $\Delta_\uparrow=0$	↑..↑#..# ↓..↓ $\Delta_\uparrow=0$
$\{S\}^{II}$	$Y=2k$ $(M_2-M_1)=$ $=3pk+3$	ij↑..↑#..# ##↓..↓ $\Delta_\uparrow=-1$	ij↑..↑#..# ##↓..↓ $\Delta_\uparrow=0$	i↑..↑#..# #↓..↓ $\Delta_\uparrow=0$	i↑..↑#..# #↓..↓ $\Delta_\uparrow=0$	i↑..↑#..# #↓..↓ $\Delta_\uparrow=-1$	i↑..↑#..# #↓..↓ $\Delta_\uparrow=1$	↑..↑#..# ↓..↓ $\Delta_\uparrow=0$					
$\{S\}^{III}$	$Y=2k$ $(M_2-M_1)=$ $=3pk-3$	↑..↑#..#ij ↓..↓ $\Delta_\uparrow=-1$	↑..↑#..#ij ↓..↓ $\Delta_\uparrow=0$						↑..↑#..# ↓..↓ $\Delta_\uparrow=-1$	↑..↑#..# ↓..↓ $\Delta_\uparrow=1$			

Table 2: Matrix of $\boxtimes_{(1)^3}$ in the first representation group

	$\varphi_+\langle 1,\mathrm{I}\rangle$	$\varphi_-\langle 1,\mathrm{I}\rangle$	$\varphi[1,\mathrm{I}]$	$\varphi_+\langle\frac{1}{2},\mathrm{I}\rangle$	$\varphi_-\langle\frac{1}{2},\mathrm{I}\rangle$	$\varphi_+\langle\frac{1}{2},\mathrm{II}\rangle$	$\varphi_-\langle\frac{1}{2},\mathrm{II}\rangle$	$\varphi_+\langle\frac{1}{2},\mathrm{III}\rangle$	$\varphi_-\langle\frac{1}{2},\mathrm{III}\rangle$	$\varphi_+\langle\frac{1}{2},\mathrm{IV}\rangle$	$\varphi_-\langle\frac{1}{2},\mathrm{IV}\rangle$	$\varphi_+\langle 0,\mathrm{IV}\rangle$	$\varphi_-\langle 0,\mathrm{IV}\rangle$	$\varphi[0,\mathrm{I}]$	$\varphi[0,\mathrm{II}]$
$\varphi_+\langle 1,\mathrm{I}\rangle$	0	$i\varsigma Y$	$-iB\frac{q}{2}Y$	0	0	0	0	$A\cdot(M_1+M_2)$	$-iA\cdot(M_1-M_2)$	$-A\cdot(M_1+M_2)$	$iA\cdot(M_1-M_2)$	0	0	0	0
$\varphi_-\langle 1,\mathrm{I}\rangle$	$-i\varsigma Y$	0	$-BJ$	0	0	0	0	$iA\cdot(M_1-M_2)$	$A\cdot(M_1+M_2)$	$iA\cdot(M_1-M_2)$	$A\cdot(M_1+M_2)$	0	0	0	0
$\varphi[1,\mathrm{I}]$	$-2iBqY$	$-4B\cdot(1+J)$	Ω	0	0	$2A\cdot(M_1+M_2)$	$-2iA\cdot(M_1-M_2)$	0	0	0	0	0	0	0	0
$\varphi_+\langle\frac{1}{2},\mathrm{I}\rangle$	0	0	0	0	0	$-i\varsigma Y$	0	$2B\cdot(1+J)$	$-iBqY$	$-2B\cdot(1+J)$	$iBqY$	0	0	0	$i/2\cdot A\cdot(M_1-M_2)$
$\varphi_-\langle\frac{1}{2},\mathrm{I}\rangle$	0	0	0	0	0	0	$-i\varsigma Y$	$iBqY$	$2B(1+J)$	$-iBqY$	$-2B(1+J)$	0	0	0	$1/2\cdot A\cdot(M_1+M_2)$
$\varphi_+\langle\frac{1}{2},\mathrm{II}\rangle$	0	0	$A/4\cdot(M_1+M_2+4)$	$i\varsigma Y$	0	0	0	0	0	0	0	0	0	$A/2\cdot(M_1+M_2)$	0
$\varphi_-\langle\frac{1}{2},\mathrm{II}\rangle$	0	0	$-iA/2\cdot(M_1-M_2)$	0	$i\varsigma Y$	0	0	0	0	0	0	0	0	$-iA/4\cdot(M_1-M_2)$	0
$\varphi_+\langle\frac{1}{2},\mathrm{III}\rangle$	$A/4\cdot(M_1+M_2+4)$	$iA/4\cdot(M_1-M_2)$	0	$-B\cdot J$	$-iB\frac{q}{2}Y$	0	0	$-\Delta$	0	0	0	$iA/2\cdot(M_1-M_2)$	$-A/2\,(M_1+M_2)$	0	0
$\varphi_-\langle\frac{1}{2},\mathrm{III}\rangle$	$-iA/4\cdot(M_1-M_2)$	$A/4\cdot(M_1+M_2+4)$	0	$iB\frac{q}{2}Y$	$-B\cdot J$	0	0	0	$-\Delta$	0	0	$A/2\cdot(M_1+M_2)$	$iA/2\cdot(M_1-M_2)$	0	0
$\varphi_+\langle\frac{1}{2},\mathrm{IV}\rangle$	$-A/4\,(M_1+M_2+4)$	$iA/4\,(M_1-M_2)$	0	$B\cdot J$	$-iB\frac{q}{2}Y$	0	0	0	0	Δ	0	$iA/2\cdot(M_1-M_2)$	$A/2\cdot(M_1+M_2)$	0	0
$\varphi_-\langle\frac{1}{2},\mathrm{IV}\rangle$	$iA/4\,(M_1-M_2)$	$A/4\,(M_1+M_2+4)$	0	$iB\frac{q}{2}Y$	$B\cdot J$	0	0	0	0	0	Δ	$A/2\,(M_1+M_2)$	$-iA/2\cdot(M_1-M_2)$	0	0
$\varphi_+\langle 0,\mathrm{IV}\rangle$	0	0	0	0	0	0	0	$i/2\cdot A\cdot(M_1-M_2)$	$A/2\cdot(M_1+M_2+4)$	$iA/2\cdot(M_1-M_2)$	$A/2\cdot(M_1+M_2+4)$	0	$-i\varsigma Y$	$B\cdot J$	0
$\varphi_-\langle 0,\mathrm{IV}\rangle$	0	0	0	0	0	0	0	$-A/2\,(M_1+M_2+4)$	$iA/2\cdot(M_1-M_2)$	$A/2\cdot(M_1+M_2+4)$	$-iA/2\cdot(M_1-M_2)$	$i\varsigma Y$	0	$-\frac{i}{2}BqY$	0
$\varphi[0,\mathrm{I}]$	0	0	0	0	0	$A\cdot(M_1+M_2+4)$	$-iA\cdot(M_1-M_2)$	0	0	0	0	$4B\cdot(1+J)$	$-2iBqY$	$-\Omega$	Δ
$\varphi[0,\mathrm{II}]$	0	0	0	$2iA\cdot(M_1-M_2)$	$2A\cdot(M_1+M_2+4)$	0	0	0	0	0	0	0	0	2Δ	0

$$A = \frac{\gamma}{8}\,\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\;;\quad B = \frac{\gamma}{8}\sqrt{2\beta}\;;\quad \varsigma = \frac{\gamma}{8}q\;;\quad \Delta = \frac{1}{4}\,\frac{q}{p}\,\alpha\gamma\;;\quad \Omega = \frac{\gamma\beta}{2}$$

Table 3: Matrix of $\boxtimes_{(1)}{}^3$ in the second $\oplus$ third representation group

	$\pi\langle 1,I\rangle$	$\widetilde{\pi\langle 1,I\rangle}$	$\pi[1,I]$	$\pi\langle \frac{1}{2},I\rangle$	$\pi\langle \frac{1}{2},II\rangle$	$\pi\langle \frac{1}{2},III\rangle$	$\pi\langle \frac{1}{2},IV\rangle$	$\pi\langle 0,I\rangle$	$\widetilde{\pi\langle 0,II\rangle}$	$\widetilde{\pi\langle 0,III\rangle}$
$\pi\langle 1,I\rangle$	$\zeta\cdot Y$	0	$-B(J+\frac{q}{2}Y)$	0	0	$2A\cdot(M_2-1)$	0	0	0	0
$\widetilde{\pi\langle 1,I\rangle}$	0	$-\zeta\cdot Y$	$-B(J-\frac{q}{2}Y)$	0	0	0	$-2A\cdot(M_2-1)$	0	0	0
$\pi[1,I]$	$-2B\cdot(1+J-\frac{q}{2}Y)$	$-2B\cdot(1+J+\frac{q}{2}Y)$	Ω	0	$2iA\cdot(M_2-1)$	0	0	0	0	0
$\pi\langle \frac{1}{2},I\rangle$	0	0	0	0	$-i\zeta Y$	$2B\cdot(1+J-\frac{q}{2}Y)$	$-2B\cdot(1+J+\frac{q}{2}Y)$	0	0	0
$\pi\langle \frac{1}{2},II\rangle$	0	0	$-iA\cdot(M_1+3)$	$i\zeta Y$	0	0	0	$2iA\cdot M_2$	0	0
$\pi\langle \frac{1}{2},III\rangle$	$A\cdot(M_1+3)$	0	0	$-B\cdot(J+\frac{q}{2}Y)$	0	$-\Delta$	0	0	0	$-2A\cdot M_2$
$\pi\langle \frac{1}{2},IV\rangle$	0	$A\cdot(M_1+3)$	0	$B\cdot(J-\frac{q}{2}Y)$	0	0	Δ	0	$2A\cdot M_2$	0
$\pi\langle 0,I\rangle$	0	0	0	0	$-iA\cdot(M_1+2)$	0	0	$-\Omega$	$2B\cdot(J+1+\frac{q}{2}Y)$	$-2B\cdot(1+J-\frac{q}{2}Y)$
$\widetilde{\pi\langle 0,II\rangle}$	0	0	0	0	0	0	$A\cdot(M_1+2)$	$B\cdot(J-\frac{q}{2}Y)$	$\zeta\cdot Y$	0
$\widetilde{\pi\langle 0,III\rangle}$	0	0	0	0	0	$-A\cdot(M_1+2)$	0	$-B\cdot(J+{}^{q}/_{2}Y)$	0	$-\zeta\cdot Y$

$$A=\frac{\gamma}{8}\frac{q}{p}\sqrt{\frac{2\alpha}{3}}\ ;\ B=\frac{\gamma}{8}\sqrt{2\beta}\ ;\ \zeta=\frac{\gamma}{8}q\ ;\ \Delta=\frac{1}{4}\frac{q}{p}\alpha\gamma\ ;\ \Omega=\frac{\gamma\beta}{2}$$

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