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ON THE SPECTRUM OF THE P-LAPLACIAN OPERATOR FOR NEUMANN EIGENVALUE PROBLEMS WITH WEIGHTS

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Abstract. This paper is devoted to study the spectrum for a Neumann eigenvalue problem involving the p-Laplacian operator with weight in a bounded domain.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 1$, $1 < p < +\infty$ and $m \in L^r(\Omega)$, with $r = r(N,p)$ satisfying the conditions

$$\begin{gathered} r > N/p \quad \text{if } \ 1 < p < N \\ r = 1 \quad \text{if } \ p > N \\ r > p \quad \text{if } \ p = N \end{gathered} \tag{1.1}$$

We assume that $\operatorname{meas}(\Omega^+) \neq 0$, where $\Omega^+ = \{x \in \Omega / m(x) > 0\}$. We consider the nonlinear eigenvalue problem

$$\begin{gathered} -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator.

The goal of this work is to prove the existence of a sequence of non trivial eigenvalues for the problem (1.2), and we prove the simplicity, isolation and monotonicity with respect to the weight of the first eigenvalue λ_1 defined by

$$\lambda_1 = \inf\{\|\nabla u\|_p^p : u \in W^{1,p}(\Omega) \text{ and } \int_\Omega m(x)|u|^p dx = 1\}. \tag{1.3}$$

The semilinear elliptic problems has been treated by many authors; see, e.g. [3, 11, 12, 16] and the references therein. In the case of bounded weight: Anane [1] with Dirichlet boundary conditions and Dakkak [8] with Neumann boundary conditions, proved the existence, simplicity and isolation of the first eigenvalue. Cuesta [7] (for the p-Laplacian) and in Touzani [17] (for the A_p-Laplacian: $A_p u = \Sigma_{i,j=1}^N \frac{\partial}{\partial x_i}(|\nabla u|_a^{p-2} a_{ij}(x) \frac{\partial u}{\partial x_j})$, $|\xi|_a^2 = \Sigma_{i,j=1}^N a_{ij}(x)\xi_i\xi_j$) studied the above properties

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in the case of Dirichlet problem with weight in $L^r(\Omega)$, with r satisfying (1.1). Cuesta [7] showed these results for any r satisfying the conditions (1.1), by using Harnack's inequality and Picone's identity. However, in [17], the isolation was establisehd with some appropriate condition on r ($r > Np'$) in order to use the regularity results established by Di Benedetto [9]. We will try to adapt these results to the case of Neumann boundary conditions.

This paper is organized as follows. In section 2, we recall some definitions and results that we will use later. In section 3, we prove that the problem (1.2) has a sequence of eigenvalues, by using a perturbation of the initial problem ([7]), and then by applying the Ljusternik-Schnirelmann theory ([4, 5]) to the perturbed problem. In section 4, we show simplicity, isolation and monotonicity of the first eigenvalue λ_1.

2. Preliminaries

Note by $W^{1,p}(\Omega)$ the Sobolev space with norm $\|.\|_{1,p} = (\|.\|_p^p + \|\nabla(.)\|_p^p)^{1/p}$, where $\|.\|_p$ is the L^p-norm.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.2) if there exists $u \in W^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, dx = \lambda \int_\Omega m(x)|u|^{p-2}u\varphi \, dx \qquad \forall \varphi \in W^{1,p}(\Omega). \tag{2.1}$$

Theorem 2.1 ([1]). *Let $v > 0$ and $u \geq 0$ be two continuous functions in Ω, differentiable a.e, and*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u,$$
$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\nabla v.$$

Then we have

(i) *$L(u,v) = R(u,v)$*
(ii) *$L(u,v) \geq 0$ a.e. in Ω*
(iii) *$L(u,v) = 0$ a.e. in Ω if and only if there exists $k \in \mathbb{R}$ such that $u = kv$.*

Proposition 2.2. *Let $u \in W^{1,p}(\Omega)$ be an eigenfunction associated to λ then*

(i) *$u \in L^\infty(\Omega)$*
(ii) *u is locally Hölder continuous; i.e., there exists $\alpha = \alpha(p, N, \|\lambda m\|_r)$ in $]0,1[$ such that for each $\Omega' \subset \Omega$, there exists $C = C(p, N, \|\lambda m\|_s, \operatorname{dist}(\Omega', \partial\Omega))$ such that*

$$|u(x) - u(y)| \leq C\|u\|_\infty |x-y|^\alpha \quad \forall x, y \in \Omega'.$$

The proof of (i) can be found in [10, propositions 1.2 and 1.3], and of (ii) in [14, theorem 8].

Proposition 2.3. *Let $u \in W^{1,p}(\Omega)$ be a nonnegative weak solution of (1.2), then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \Omega$.*

The proof is a direct consequence of Harnack's inequality (see [15, theorem 5, 6, 9]).

3. Existence of solutions

In this section, we establish the existence of solutions by using a perturbation of problem (1.2) in order to use the general theory of nonlinear eigenvalue problems. So, let us consider the perturbed problem

$$\begin{gathered} -\Delta_p u + \varepsilon |u|^{p-2}u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where ε is enough small ($0 < \varepsilon < 1$), $\lambda > 0$. We first show that problem (3.1) has at least one sequence of eigenvalues, and deduce the solutions of problem (1.2) when ε tends to 0.

Let $X = W^{1,p}(\Omega)$, $G_\varepsilon(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{\varepsilon}{p}\|u\|_p^p$, and $F(u) = \frac{1}{p}\int_\Omega m(x)|u|^p dx$. It is well known that F and G_ε are differentiable [8]. The problem (3.1) is equivalent to the problem $G'_\varepsilon(u) = \lambda F'(u)$. Let us consider the functional $\Phi_\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ defined by $\Phi_\varepsilon(v) = (G_\varepsilon(v))^2 - F(v)$.

Lemma 3.1. *The eigenvalues and eigenfunctions associated to the problem* (3.1) *are entirely determined by a non trivial critical values of* Φ_ε.

Proof. Let $u \not\equiv 0$ be a critical point of Φ_ε associated with a critical value c_ε then $\Phi_\varepsilon(u) = c_\varepsilon$ and $\Phi'_\varepsilon(u) = 0$, i.e $c_\varepsilon = -(G_\varepsilon(u))^2 < 0$ and $\langle \Phi'_\varepsilon(u), v\rangle = \frac{1}{2\sqrt{-c_\varepsilon}}\langle F'(u), v\rangle$ for any $v \in C_c^\infty(\Omega)$. Thus we deduce that $\lambda = \frac{1}{2\sqrt{-c_\varepsilon}}$ is a positive eigenvalue of (3.1) and u is its associated eigenfunction.

Conversely, let $(u \not\equiv 0, \lambda)$ be a solution of (3.1), then for every $\beta \in \mathbb{R}^*, \beta u$ is also an eigenfunction associated to λ. In particular for $\beta = (\frac{1}{2\lambda G_\varepsilon(u)})^{\frac{1}{p}}$, $v = (2\lambda G_\varepsilon(u))^{-\frac{1}{p}} u$ is an eigenfunction associated to $\lambda = \frac{1}{2\sqrt{-c_\varepsilon}}$, thus v is a critical point associated to the critical value $c_\varepsilon = -\frac{1}{4\lambda^2}$. □

Let us now consider the sequence

$$c_{n,\varepsilon} = \inf_{K \in A_n} \sup_{v \in K} \Phi_\varepsilon(v), \tag{3.2}$$

where $A_n = \{K \subset W^{1,p}(\Omega) : K \text{ is compact symmetric and } \gamma(K) \geq n\}$, $n \geq 1$.

Theorem 3.2. *The values* $c_{n,\varepsilon}$ *defined by* (3.2) *are the critical values of* Φ_ε, *moreover* $c_{n,\varepsilon} < 0$ *for* $n \geq 1$ *and* $\lim_{n\to\infty} c_{n,\varepsilon} = 0$.

Proof. The proof of this theorem is based on the fundamental theorem of multiplicity [6] and the approximation of Sobolev imbedding by operators of finite rank. We first show that for all $n \geq 1$, $c_{n,\varepsilon}$ is a critical value of Φ_ε and $c_{n,\varepsilon} < 0$.

Since ϕ_ε is even and is C^1 on $W^{1,p}(\Omega)$, then the result follows from the fundamental theorem of multiplicity if Φ_ε satisfies the following conditions:

- (i) Φ_ε is bounded below
- (ii) Φ_ε verify the Palais Smale condition (PS).
- (iii) for all $n \geq 1$, there exists a compact symmetric subset K such that $\gamma(K) = n$ and $\sup_{v\in K}\{\Phi_\varepsilon(v)\} < 0$.

Let us verify assertion (i): Let us take ε fixed ($0 < \varepsilon < 1$) and let $u \in W^{1,p}(\Omega)$, then by Hölder's inequality and the Sobolev imbeddings; there exist $k_1 > 0$, $k_2 > 0$

and $k_3 > 0$ such that:

$$\int_\Omega m|u|^p dx \leq \begin{cases} k_1\|m\|_r\|u\|_{1,p}^p & \text{if } 1 < p < N \\ k_2\|m\|_r\|u\|_{1,N}^N & \text{if } p = N \\ k_3\|m\|_r\|u\|_{1,p}^p & \text{if } p > N \end{cases}$$

In addition, note that $\|u\|_{1,p,\varepsilon} = (\|\nabla u\|_p^p + \varepsilon\|u\|_p^p)^{\frac{1}{p}}$ defines a norm on $W^{1,p}(\Omega)$ equivalent to the usual norm on $W^{1,p}(\Omega)$. It is easy to see that there exists $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that for Φ_ε, we have the following inequalities:

$$\Phi_\varepsilon(u) \geq \begin{cases} \frac{c_1}{p^2}\|u\|_{1,p}^p(\varepsilon^2\|u\|_{1,p}^p - p\|m\|_r) & \text{if } 1 < p < N \\ \frac{c_2}{N^2}\|u\|_{1,N}^N(\varepsilon^2\|u\|_{1,N}^N - p\|m\|_r) & \text{if } p = N \\ \frac{c_3}{p^2}\|u\|_{1,p}^p(\varepsilon^2\|u\|_{1,p}^p - p\|m\|_r) & \text{if } p > N \end{cases}$$

From where by treating each case one has: Φ_ε is bounded below and $\Phi_\varepsilon(u) \to +\infty$ as $\|u\|_{1,p} \to +\infty$.

(ii) Let us now show that Φ_ε verify the palais Smale condition: Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that $(\Phi(u_n))_n$ is bounded and $\Phi'(u_n) \to 0$ in $(W^{1,p}(\Omega))'$. Since Φ_ε is coercive then $(u_n)_n$ is bounded in $W^{1,p}(\Omega)$, thus there exists a subsequence still denoted (u_n) such that u_n converges to u strongly in $L^p(\Omega)$ and weakly in $W^{1,p}(\Omega)$. Suppose that $\|u_n\|_{1,p}$ converges to $\alpha \geq 0$. We distinguish two cases:

Case 1: $\alpha = 0$**.** Since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\|u_n\|_{1,p} \to 0$ then $u_n \to 0$ in $W^{1,p}(\Omega)$, consequently, the condition (PS) is satisfied.

Case 2: $\alpha > 0$**.** For $n \geq 1$ we have:

$$\Phi'_\varepsilon(u_n) = 2G_\varepsilon(u_n)G'_\varepsilon(u_n) - F'(u_n)$$

then

$$G'_\varepsilon(u_n) = \frac{1}{2G_\varepsilon(u_n)}(\Phi'_\varepsilon(u_n) + F'(u_n))$$

i.e.,

$$[\frac{p}{2}(\Phi'_\varepsilon(u_n) + m|u_n|^{p-2}u_n)]/(\|\nabla u_n\|_p^p + \varepsilon\|u_n\|_p^p) = G'_\varepsilon(u_n).$$

Since $u \mapsto m|u_n|^{p-2}u$ is strongly continuous, $\|u_n\|_{1,p} \to \alpha > 0$ and $\Phi'_\varepsilon(u_n) \to 0$, then the expression

$$\widehat{u_n} = [\frac{2}{p}(\Phi'_\varepsilon(u_n) + m|u_n|^{p-2}u_n)]/(\|\nabla u_n\|_p^p + \varepsilon\|u_n\|_p^p)$$

converges strongly in $(W^{1,p}(\Omega))'$. However, G'_ε is continuous, thus $u_n = (G'_\varepsilon)^{-1}\widehat{u_n}$ converges strongly in $W^{1,p}(\Omega)$, from where the (PS) condition holds.

(iii) For all $n \geq 1$, there exists a compact symmetric subset K such that

$$\gamma(K) = n \quad \text{and} \quad \sup_{u \in K} \Phi_\varepsilon(u) < 0.$$

Indeed, since $\text{meas}(\Omega^+) \neq 0$, then there exists a family of balls $(B_i)_{1 \leq i \leq n}$ in Ω such that $B_i \cap B_j = \emptyset$ if $i \neq j$ and $\text{meas}(\Omega^+ \cap B_i) \neq 0$. By approximating the characteristic function $\chi_{\Omega^+ \cap B_i}$ by $C_c^\infty(\Omega)$ functions in L^p, there exists a sequence $(u_i)_{1 \leq i \leq n} \subset C_c^\infty(B_i)$ such that $\int_\Omega m|u_i|^p dx > 0$ for all $i = 1, \dots, n$. We normalize u_i in order to have $F(u_i) = 1$. Let X_n be the subset generated by $(u_i)_i$. For all $u \in X_n$, we have $u = \Sigma_{i=1}^n \alpha_i u_i$, and $F(u) = \Sigma_{i=1}^n |\alpha_i|^p$. Then $u \mapsto (F(u))^{\frac{1}{p}}$ define one norm on X_n; moreover, since ε is fixed it follows that $(\|\nabla u\|_p^p + \varepsilon\|u\|_p^p)^{1/p}$ is also a norm on $W^{1,p}(\Omega)$. However, the dimension of X_n is finite, then there exists $c > 0$ such that

$$cF(u) \leq G_\varepsilon(u) \leq \frac{1}{c}F(u) \quad \forall\, u \in X_n.$$

Let K be defined as

$$K = \{u \in W^{1,p}(\Omega) \text{ such that } \frac{c^2}{3} \leq F(u) \leq \frac{c^2}{2}\}$$

It is clear that $K_1 = K \cap X_n \neq \emptyset$ and $\sup_{u\in K_1} \Phi_\varepsilon(u) \leq -\frac{c^2}{12} < 0$. Since X_n is isomorphous to $\mathbb{R}^n$, one can identify K_1 to a crown K_1' of $\mathbb{R}^n$ such that $S^{n-1} \subset K_1' \subset \mathbb{R}^n \setminus \{0\}$ where S^{n-1} is the unit sphere of $\mathbb{R}^n$. then $\gamma(K_1) = n$ and the result follows.

For the proof of $\lim_{n\to\infty} c_{n,\varepsilon} = 0$, we use an approximation of Sobolev imbeddings by operators of finite rank; see for example [17]. $\square$

Remark 3.3. It is clear that the sequence $(\lambda_{n,\varepsilon})_n$ defined by the formula $\lambda_{n,\varepsilon} = \frac{1}{2\sqrt{-c_{n,\varepsilon}}}$ for all $n \geq 1$ is a sequence of positive eigenvalues of (3.1) which satisfy $\lim_{n\to+\infty} \lambda_{n,\varepsilon} = +\infty$.

Next, we show that problem (1.2) has a sequence of eigenvalues $(\lambda_n)_n$ which is the limit of the sequence $(\lambda_{n,\varepsilon})_n$, as $\varepsilon \to 0$.

Remark 3.4. We write $\lambda_{n,\varepsilon}$ as $\lambda_{n,\varepsilon} = \inf_{K\in\Gamma_n} \sup_{u\in K} G_\varepsilon(u)$, where

$$\Gamma_n = \{K \subset W^{1,p}(\Omega) \setminus \{0\} : K \text{ is compact, symmetric } \gamma(K) \geq n,\ \int m|u|^p = 1\}.$$

For more details see [8]. Put

$$G(u) = \frac{1}{p}\|\nabla u\|_p^p \quad \text{and} \quad \lambda_n = \inf_{K\in A_n} \sup_{u\in K} G(u).$$

Lemma 3.5. *The following assertions hold:*

(i) $\lim_{\varepsilon\to0} \lambda_{n,\varepsilon} = \lambda_n$
(ii) $\lambda_n \to +\infty$ *as* $n \to +\infty$.

Proof. (i) Let $\varepsilon > 0$, from Remark 3.4, we have $\lambda_{n,\varepsilon} \geq \lambda_n$. Let $\alpha > 0$ such that $\lambda_n < \alpha$. Then there exists $K = K(\alpha) \in A_n$ such that $\lambda_n \leq \sup_{u\in K} G(u) < \alpha$. Set $\delta = \sup_{u\in K} \|u\|_p$. Then

$$\lambda_n \leq \lambda_{n,\varepsilon} \leq \sup_{u\in K} G_\varepsilon(u) \leq \sup_{u\in K} G(u) + \frac{\varepsilon\delta}{p}$$

For $\varepsilon = \frac{1}{k} \to 0$, there exists $k(\alpha)$ such that for every $k \geq k(\alpha)$, we obtain $\sup_{u\in K} G(u) + \frac{\delta}{kp} \leq \alpha$. Thus $\lambda_n \leq \lambda_{n,\varepsilon} \leq \alpha$ for all $k \geq k(\alpha)$. From where we obtain the desired result.

(ii) From (3.2), we have for all $m, m' \in L^r(\Omega)$ such that $m' \geq m$; $c_{n,\varepsilon}(m) \leq c_{n,\varepsilon}(m')$, thus $\lambda_{n,\varepsilon}(m) \geq \lambda_{n,\varepsilon}(m')$. When $\varepsilon \to 0$, we obtain $\lambda_n(m') \leq \lambda_n(m)$. Let $\delta > 0$, and set

$$m'(x) = \begin{cases} m(x) & \text{if } m(x) \geq \delta \\ \delta & \text{if } m(x) < \delta \end{cases}$$

It is clear that $\lambda_{n,\varepsilon}(m') \leq \lambda_n(m') + \varepsilon/\delta$. Since $(\lambda_{n,\varepsilon}(m'))_n \nearrow \infty$, $\lim_{n\to\infty} \lambda_n(m') = +\infty$. The result follows. $\square$

Corollary 3.6. *$(\lambda_n)_n$ is a sequence of positives eigenvalues associated to the Neumann problem* (1.2).

Proof. Let $n \in \mathbb{N}^*$ fixed and $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}^*$. There exists a sequence $(u_k)_{k\in\mathbb{N}^*}$ of eigenfunctions associated with $(\lambda_{n,k})_{k\in\mathbb{N}^*}$ which verify $G_k(u_k) + \|u_k\|_p^p = 1$ then $(u_k)_k$ is bounded in $W^{1,p}(\Omega)$. Hence for a subsequence $(u_k)_k$, $(u_k)_k$ converges strongly in $L^p(\Omega)$, and weakly in $W^{1,p}(\Omega)$ towards a limit u. $u_k \to u$. Indeed, set $J(u) = |u|^{p-2}u$. The operator $G' + J$ is of S^+ type and $G' + J : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is an homeomorphism (see [13]), then $(u_k)_k$ converges strongly to u. However, $G'(u_k) + \frac{1}{k}|u_k|^{p-2}u_k = \lambda_{n,k}F'(u_k)$ and F' is strongly continuous on $W^{1,p}(\Omega)$, thus $G'(u)= \lambda_n F'$ and $G(u) + \|u\|_p^p = 1$. Consequently, (u, λ_n) is a solution of (1.2). □

4. On the first eigenvalue

In this section, we are going to prove some properties of the first eigenvalue λ_1 of problem (1.2) defined by (1.3). We refer in this section to the work of Cuesta [7]; so we prove that λ_1 is simple, isolated and strictly monotone with respect to the weight.

Proposition 4.1. *λ_1 is an eigenvalue of problem* (1.2)*. Moreover $\lambda_1 > 0$ if and only if m changes its sign on Ω and $\int_\Omega m(x)dx < 0$.*

The proof of the above proposition is a straight application of [15, Theorem 1.2].

Proposition 4.2. *The eigenfunctions associated with $\lambda_1 > 0$ are either positive or negative in Ω.*

Proof. Let u be an eigenfunction associated to λ_1. Without loss of generality, we can suppose that $u \in M = \{u \in W^{1,p}(\Omega) : \int_\Omega m|u|^p = 1\}$. Then, the infimum in (1.3) is achieved at one $u \in W^{1,p}(\Omega)$. It's easy to prove that $\|\nabla|u|\|_p = \|\nabla u\|_p$ and $|u| \in M$, it follows then that $|u|$ is an eigenfunction associated to λ_1, and therefore, from Proposition 2.3, the result holds. □

Proposition 4.3. *Any eigenfunction associated with a positive eigenvalue $\lambda \neq \lambda_1$ changes its sign.*

Proof. For the proof, we use the Picone's identity (see Theorem 2.1). Indeed, let (v, λ), (u, λ_1) be two positive solutions of (1.2) and suppose that $u > 0$. Suppose also that $v \geq 0$ (the case $v \leq 0$ is proved similarly). By Theorem 2.1, we have

$$\int_\Omega L(u, v + \varepsilon)dx = \int_\Omega R(u, v + \varepsilon)dx \geq 0 \quad \forall\varepsilon > 0.$$

However,

$$\int_\Omega R(u, v + \varepsilon)dx = \int_\Omega [|\nabla u|^p - |\nabla v|^{p-2}\nabla(\frac{u^p}{(v + \varepsilon)^{p-1}})\nabla v]dx,$$

By taking $\frac{u^p}{(v+\varepsilon)^{p-1}}$ as a test function in the formula $\langle -\Delta_p v, \varphi\rangle = \lambda\langle m|v|^{p-2}v, \varphi\rangle$, we obtain

$$\int_\Omega [\lambda_1 m u^p - \lambda m v^{p-1}\frac{u^p}{(v + \varepsilon)^{p-1}}]dx \geq 0$$

i.e.,

$$\int_\Omega m u^p(\lambda_1 - \lambda\frac{v^{p-1}}{(v + \varepsilon)^{p-1}})dx \geq 0.$$

Let ε goes to 0, we obtain $\int_\Omega mu^p(\lambda_1-\lambda)dx \geq 0$ which is impossible because $\lambda > \lambda_1$ and $\int_\Omega mu^p dx > 0$. Hence, v changes its sign on Ω. $\square$

Proposition 4.4 (Simplicity)**.** *λ_1 is simple in the sense that if (u, λ_1) and (v, λ_1) are two solutions associated to the problem* (1.2)*, then there exists $\alpha \in \mathbb{R}$ such that $u = \alpha v$.*

The proof of the above proposition is the same as in the preceding proposition. We use the assertion (iii) of Theorem 2.1 (see preliminaries section).

Now, we establish the isolation of the first eigenvalue. We use for this the same method as in [2, 7, 17]. We will show first the following estimates:

Proposition 4.5. *For $m \in L^r(\Omega)$ with r satisfying the conditions* (1.1)*, the following estimates hold:*

$$\min(\text{meas}(\Omega^+), \text{meas}(\Omega^-)) \geq (\lambda C^p \|m\|_r)^\sigma$$

with

$$\sigma = \begin{cases} \frac{rN}{N-rp} & \text{if } 1 < p < N \\ \frac{r}{1-N} & \text{if } p = N \\ \frac{N}{N-p} & \text{if } p > N \end{cases} \tag{4.1}$$

Proof. Let u be an eigenfunction associated with λ, then

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \, dx = \lambda \int_\Omega m(x)|u|^{p-2}uv \, dx \quad \forall v \in W^{1,p}(\Omega). \tag{4.2}$$

For $\lambda \neq \lambda_1$, u changes its sign i.e. $u^+ \neq 0$ and $u^- \neq 0$, so, since $u^+ \in W^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u^+|^p dx = \lambda \int_\Omega m(x)|u^+|^p. \tag{4.3}$$

Case 1: $1 < p < N$**.** By Hölder inequality, we have

$$\lambda \int_\Omega m|u^+|^p dx \leq \|m\|_r \|u^+\|^p_{p^*} (\text{meas}(\Omega^+))^{1-\frac{1}{r}-\frac{p}{p^*}}$$

where

$$p^* = \begin{cases} \frac{pN}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$$

On the other hand, using Sobolev imbeddings, there exists a constant C depending on p and N such that $\|u^+\|_{p^*} \leq C(\int_\Omega |\nabla u^+|^p)^{\frac{1}{p}} dx$. Hence from (4.3) it follows that

$$\|\nabla u^+\|^p_p \leq \lambda C^p \|m\|_r \|u^+\|^p_{1,p} (\text{meas}(\Omega^+))^{\frac{rp-N}{rN}}$$

and $\text{meas}(\Omega^+) \geq (\lambda C^p \|m\|_r)^{\frac{rN}{N-rp}}$.

Case 2: $p = N$**.** In this case, we proceed in the same way as previously. On the one hand, we have

$$\lambda \int_\Omega m|u^+|^N dx \leq \lambda \|m\|_r \|u^+\|^{N-1}_N \|u^+\|_s$$

where $s = \frac{Nr}{r-N}$. From Hölder inequality, we have

$$\|u^+\|^{N-1}_N \leq \|u^+\|^{N-1}_s (\text{meas}(\Omega^+))^{(1-\frac{N}{s})(\frac{N-1}{N})}$$

On the other hand, while $s>1$, by the Sobolev imbedding $W^{1,N}\hookrightarrow L^s(\Omega)$, there exists $C>0$ such that $\|u^+\|_s\le C\|\nabla u^+\|_N$. However, it results

$$\|\nabla u^+\|_N^N\le \lambda C^N\|m\|_r\|\nabla u^+\|_N^N(\operatorname{meas}(\Omega^+))^{(1-\frac{N}{s})(\frac{N-1}{N})}.$$

Finally, we obtain $\operatorname{meas}(\Omega^+)\ge(\lambda C^N\|m\|_r)^{\frac{r}{1-N}}$.

Case 3: $p>N$**.** In this case $r=1$. From (4.3), we have

$$\int_\Omega |\nabla u^+|^p dx\le \lambda\|m\|_1\|u^+\|_\infty^p,$$

and by Morrey's lemma, there exists $C>0$ depending on p and N such that

$$\|u^+\|_\infty\le C(\operatorname{meas}(\Omega^+))^{\frac{1}{N}-\frac{1}{p}}\|\nabla u^+\|_p.$$

Thus, $\operatorname{meas}(\Omega^+)\ge(\lambda C^p\|m\|_1)^{\frac{N}{N-p}}$.

In conclusion , in the three cases we obtain the desired estimate for u^+. By proceeding in the same way for u^-, from (4.3) (for u^-), we deduce the same estimates for u^-. From where the proposition 4.5 holds. ☐

The following proposition is a consequence of the previous proposition.

Proposition 4.6 (Isolation)**.** *$\lambda_1(m)$ is isolated, that is, there exists $\beta>\lambda_1$ such that if $\lambda\in]0,\beta[$ then $\lambda=\lambda_1$ or λ is not an eigenvalue associated to* (1.2)*.*

Proof. Let u be an eigenfunction associated with $\lambda\in]0,\beta[$. Then $\int_\Omega|\nabla u|^p dx=\lambda\int_\Omega m(x)|u|^p$. Since $\int_\Omega|\nabla u|^p dx\ge\lambda_1\int_\Omega m(x)|u|^p$, it follows that $\lambda_1\le\lambda$; thus λ_1 is isolated on the left. Assume now by contradiction that there exists a sequence of eigenvalues of (1.2) $(\lambda_n)_n$ decreasing to λ_1, thus one has a sequence of eigenfunctions u_n associated with λ_n. We can suppose that $\|\nabla u_n\|=1$, for all $n\in\mathbb{N}$. (u_n) is bounded in $W^{1,p}(\Omega)$, so there exists a subsequence (still denoted u_n) such that u_n converges weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to a limit $u\in W^{1,p}(\Omega)$. On the other hand, we have

$$\int_\Omega|\nabla u|^p dx\ge\liminf_{n\to\infty}\int_\Omega|\nabla u_n|^p dx=\lambda_1$$

then by the definition of λ_1, we conclude that $\int_\Omega|\nabla u_n|^p dx=\lambda_1$ and u is an eigenfunction associated to λ_1. It follows then by Proposition 4.4 that either $u>0$ or $u<0$. Suppose that $u>0$ (the other case is analogous). By Egorov and Lusin theorem, we obtain for every $\varepsilon>0$, there exists $N_\varepsilon>0$ such that

$$\operatorname{meas}(\{x\in\Omega,u_n(x)>0\})\ge\operatorname{meas}(\Omega)-\varepsilon,\quad\forall n\ge N_\varepsilon.$$

Then for ε suitably chosen by the estimates (4.1) (for more details see [2, p. 24]), we obtain a contradiction with the estimates (4.1) related to $\operatorname{meas}(\Omega^-)$. ☐

The last proposition in this paper deals with the monotonicity of λ_1 with respect to the weight.

Proposition 4.7. *λ_1 verify the monotonicity and the monotonicity strict with respect to the weight, i.e. if $m'\le m$ then $\lambda_1(m)\le\lambda_1(m')$, moreover if $m'\le m$ and* $\operatorname{meas}(\{x\in\Omega:m'(x)<m(x)\})\neq 0$ *then $\lambda_1(m)<\lambda_1(m')$.*

Proof. Let $u>0$ be an eigenfunction associated to $\lambda_1(m')$. By $\lambda_1(m')$ definition we have

$$0<\frac{1}{(\lambda_1(m'))}\int_\Omega|\nabla u|^p dx=\int_\Omega m'u^p dx\le\int_\Omega mu^p dx.$$

Since

$$\lambda_1(m) = \inf\{\|\nabla u\|_p^p / u \in W^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega m u^p dx = 1\}$$

and

$$\frac{u}{(\int_\Omega m u^p dx)^{1/p}} \in W^{1,p}(\Omega) \quad \text{verifies} \quad \int_\Omega m(\frac{u^p}{\int_\Omega m u^p dx}) dx = 1$$

it follows that

$$\lambda_1(m) \leq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega m u^p dx} \leq \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega m' u^p dx} = \lambda_1(m')$$

and from where it follows that $\lambda_1(m) \leq \lambda_1(m')$. The equality holds if and only if

$$\int_\Omega m u^p dx = \int_\Omega m' u^p dx.$$

However, $u > 0$ thus $m \equiv m'$, which is a contradiction with meas$(\{x \in \Omega / m'(x) < m(x)\}) \neq 0$. □

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