

# ON THE SPECTRUM OF TWO DIFFERENT FRACTIONAL OPERATORS

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ABSTRACT. In this paper we deal with two nonlocal operators, that are both well known and widely studied in the literature in connection with elliptic problems of fractional type. Precisely, for a fixed  $s \in (0, 1)$  we consider the *integral* definition of the fractional Laplacian given by

$$(-\Delta)^s u(x) := \frac{c(n, s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n,$$

where  $c(n, s)$  is a positive normalizing constant, and another fractional operator obtained via a *spectral* definition, that is

$$A_s u = \sum_{i \in \mathbb{N}} a_i \lambda_i^s e_i,$$

where  $e_i, \lambda_i$  are the eigenfunctions and the eigenvalues of the Laplace operator  $-\Delta$  in  $\Omega$  with homogeneous Dirichlet boundary data, while  $a_i$  represents the projection of  $u$  on the direction  $e_i$ .

Aim of this paper is to compare these two operators, with particular reference to their spectrum, in order to emphasize their differences.

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## 1. INTRODUCTION

Recently in the literature a great attention has been devoted to the study of nonlocal problems driven by fractional Laplace type operators, not only for a pure academic interest, but also for the various applications in different fields. Indeed, many different problems driven by the fractional Laplacian were considered in order to get existence, non-existence and regularity results and, also, to obtain qualitative properties of the solutions.

In particular, two notions of fractional operators were considered in the literature, namely the *integral* one (which reduces to the *classical fractional Laplacian*, see, for instance, [7, 8, 9, 10, 14, 15, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and references therein) and the

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*spectral* one (that is sometimes called the *regional, or local, fractional Laplacian*, see, e.g. [2, 4, 5, 6, 35] and references therein).

For any fixed  $s \in (0, 1)$  the fractional Laplace operator  $(-\Delta)^s$  at the point  $x$  is defined by

$$(1.1) \quad (-\Delta)^s u(x) := \frac{c(n, s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy,$$

where  $c(n, s)$  is a positive normalizing constant<sup>1</sup> depending only on  $n$  and  $s$ .

A different operator, which is sometimes denoted by  $A_s$ , is defined as the power of the Laplace operator  $-\Delta$ , obtained by using the spectral decomposition of the Laplacian. Namely, let  $\Omega$  be a smooth bounded domain of  $\mathbb{R}^n$ , and let  $\lambda_k$  and  $e_k$ ,  $k \in \mathbb{N}$ , be the eigenvalues and the corresponding eigenfunctions of the Laplacian operator  $-\Delta$  in  $\Omega$  with zero Dirichlet boundary data on  $\partial\Omega$ , that is

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega \\ e_k = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized in such a way that  $\|e_k\|_{L^2(\Omega)} = 1$ . For any  $s \in (0, 1)$  and any  $u \in H_0^1(\Omega)$  with

$$u(x) = \sum_{i \in \mathbb{N}} a_i e_i(x), \quad x \in \Omega,$$

one considers the operator

$$(1.2) \quad A_s u = \sum_{i \in \mathbb{N}} a_i \lambda_i^s e_i.$$

Aim of this paper is to compare the two previous definitions of fractional Laplace operators. First of all, we would like to note that these two fractional operators (i.e. the ‘integral’ one and the ‘spectral’ one) are *different* (in spite of some confusion that it is possible to find in some of the existent literature in which the two operators are somehow freely interchanged). Indeed, the spectral operator  $A_s$  depends on the domain  $\Omega$  considered (since its eigenfunctions and eigenvalues depend on  $\Omega$ ), while the integral one  $(-\Delta)^s$  evaluated at some point is independent on the domain in which the equation is set.<sup>2</sup>

Of course, by definition of  $A_s$ , it is easily seen that *the eigenvalues and the eigenfunctions of  $A_s$  are respectively  $\lambda_k^s$  and  $e_k$ ,  $k \in \mathbb{N}$* , that is the  $s$ -power of the eigenvalues of the Laplacian and the very same eigenfunctions of the Laplacian, respectively.

On the other hand, the spectrum of  $(-\Delta)^s$  may be less explicit to describe. We refer to [28, Proposition 9 and Appendix A], [23, 24], [25, Proposition 5] and [30, Proposition 4] for the variational characterization of the eigenvalues and for some basic properties.

A natural question is whether or not there is a relation between the spectrum of  $A_s$  and  $(-\Delta)^s$  and, of course, between the respective eigenfunctions. In the present paper, by using the classical regularity theory for the eigenfunctions of the Laplace operator  $-\Delta$  and some recent regularity results for the fractional Laplace equation (see [22, 23, 24, 32]), we will show that the eigenfunctions of  $A_s$  and  $(-\Delta)^s$  are different (for more details see Section 2). In particular, we will show that *the eigenfunctions of  $(-\Delta)^s$  are, in general, no better than Hölder continuous up to the boundary*, differently from the eigenfunctions of  $A_s$  (i.e. of the classical Laplacian) that are smooth up to the boundary (if so is the domain).

<sup>1</sup>Different definitions of the fractional Laplacian consider different normalizing constants. The constant  $c(n, s)$  chosen here is the one coming from the equivalence of the integral definition of  $(-\Delta)^s$  and the one by Fourier transform (see, e.g., [7] and [10, (3.1)–(3.3) and (3.8)]) and it has the additional properties that  $\lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$  and  $\lim_{s \rightarrow 0^+} (-\Delta)^s u = u$  (see [10, Proposition 4.4]).

<sup>2</sup>Also, the natural functional domains for the operators  $(-\Delta)^s$  and  $A_s$  are different, but this is a minor distinction, since one could consider both the operators as acting on a very restricted class of functions for which they both make sense - e.g.,  $C_0^\infty(\Omega)$ .

Furthermore, with respect to the eigenvalues of  $A_s$  and  $(-\Delta)^s$ , we will prove that *the first eigenvalue of  $(-\Delta)^s$  is strictly less than the first one of  $A_s$* . To this purpose we will use some extension results for the fractional operators  $A_s$  and  $(-\Delta)^s$  (see [7, 34]).

Summarizing, the results given in this paper are the following:

**Theorem 1.** *The operators  $(-\Delta)^s$  and  $A_s$  are not the same, since they have different eigenvalues and eigenfunctions. In particular:*

- *the first eigenvalues of  $(-\Delta)^s$  is strictly less than the one of  $A_s$ ;*
- *the eigenfunctions of  $(-\Delta)^s$  are only Hölder continuous up to the boundary, differently from the ones of  $A_s$  that are as smooth up the boundary as the boundary allows.*

For further comments on similarities and differences between the operators  $A_s$  and  $(-\Delta)^s$  for  $s = 1/2$  see [13, Remark 0.4].

The paper is organized as follows. Section 2 is devoted to a comparison between the eigenfunctions of  $A_s$  and  $(-\Delta)^s$ . In Section 3 we deal with the spectrum of the two fractional operators we are considering. Section 4 is devoted to the extension of the operator  $A_s$ , while in Section 5 we discuss the relation between the first eigenvalues of  $A_s$  and  $(-\Delta)^s$ .

## 2. A COMPARISON BETWEEN THE EIGENFUNCTIONS OF $A_s$ AND $(-\Delta)^s$

This section is devoted to some remarks about the eigenfunctions of the operators  $A_s$  and  $(-\Delta)^s$ . Precisely, we will consider the following eigenvalue problems in a smooth bounded domain  $\Omega \subset \mathbb{R}^n$ , with Dirichlet homogeneous boundary data, driven, respectively, by  $A_s$  and  $(-\Delta)^s$ ,

$$(2.1) \quad \begin{cases} A_s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$(2.2) \quad \begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Note that in (2.2) the boundary condition is given in  $\mathbb{R}^n \setminus \Omega$  and not simply on  $\partial\Omega$ , due to the nonlocal character of the operator  $(-\Delta)^s$ .

In what follows we will denote by  $e_{k, A_s}$  and  $e_{k, s}$ ,  $k \in \mathbb{N}$ , the  $k$ -th eigenfunction of  $A_s$  and  $(-\Delta)^s$ , respectively.

Taking into account the definition of  $A_s$ , it is easily seen that its eigenfunctions  $e_{k, A_s}$ ,  $k \in \mathbb{N}$ , are exactly the eigenfunctions of the Laplace operator  $-\Delta$ , i.e.

$$e_{k, A_s} = e_k.$$

Also, since  $e_k \in C^\infty(\Omega) \cap C^m(\overline{\Omega})$  for any  $m \in \mathbb{N}$  (see, for instance, [11]), then

$$(2.3) \quad e_{k, A_s} \in C^\infty(\Omega) \cap C^m(\overline{\Omega}).$$

Of course, constructing the eigenfunctions of  $(-\Delta)^s$  is more difficult. In spite of this, we have some regularity results for them. Precisely, denoting by  $\delta(x) = \text{dist}(x, \partial\Omega)$ ,  $x \in \mathbb{R}^n$ , by [22, Theorems 1.1 and 1.3] and [30, Proposition 4], we have that

$$e_{k, s} / \delta|_\Omega^s \in C^{0, \alpha}(\overline{\Omega}) \quad \text{for some } \alpha \in (0, 1),$$

namely  $e_{k, s} / \delta|_\Omega^s$  has a continuous extension to  $\overline{\Omega}$  which is  $C^{0, \alpha}(\overline{\Omega})$ . In particular,  $e_{k, s}$  is Hölder continuous up to the boundary.

Aim of this section will be to show that *the Hölder regularity is optimal* for the eigenfunctions  $e_{k, s}$  of  $(-\Delta)^s$ . To this purpose, first of all we recall the notion of Poisson kernel of fractional type and, then, we discuss the optimal regularity of the eigenfunctions  $e_{k, s}$ .

**2.1. Poisson kernel of fractional type.** Here we recall the notion of Poisson kernels of fractional type and their relation with the Dirichlet problem (see [20, Chapter I]).

First of all, for any  $r > 0$ ,  $x \in B_r$  (that is the ball of radius  $r$  centered at the origin) and  $y \in \mathbb{R}^n \setminus B_r$ , we define

$$P_r(x, y) := c_o(n, s) \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{1}{|x - y|^n},$$

with  $c_o(n, s) > 0$ . It is known (see [20, Appendix]) that, for any fixed  $x \in B_r$  the function

$$I(x) := \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) dy$$

is constant in  $x$ . Therefore, we normalize  $c_o(n, s)$  in such a way that<sup>3</sup>

$$(2.4) \quad \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) dy = 1.$$

The function  $P_r$  plays the role of a fractional Poisson kernel, namely if  $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  and

$$(2.5) \quad u_g(x) := \begin{cases} \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) g(y) dy & \text{if } x \in B_r \\ g(x) & \text{if } x \in \mathbb{R}^n \setminus B_r, \end{cases}$$

then  $u_g$  is the unique solution of

$$(2.6) \quad \begin{cases} (-\Delta)^s u_g = 0 & \text{in } B_r \\ u_g = g & \text{outside } B_r. \end{cases}$$

For this, see [20, 33].

**2.2. Optimal regularity for the eigenfunctions of  $(-\Delta)^s$ .** In this subsection we prove that the  $C^{0, \alpha}$ -regularity of the eigenfunctions  $e_{k, s}$  is optimal. Precisely, we show that, in general, *the eigenfunctions of  $(-\Delta)^s$  need not to be Lipschitz continuous up to the boundary (i.e. the Hölder regularity is optimal).*

For concreteness, we consider the case

$$(2.7) \quad n > 2s,$$

the domain  $\Omega := B_r$  and the first eigenfunction  $e_{1, s}$  (normalized in such a way that  $\|e_{1, s}\|_{L^2(\mathbb{R}^n)} = 1$  and  $e_{1, s} \geq 0$  in  $\mathbb{R}^n$ , see [28, Proposition 9 and Appendix A]) of  $(-\Delta)^s$  in  $B_r$ , i.e.

$$(2.8) \quad \begin{cases} (-\Delta)^s e_{1, s} = \lambda_{1, s} e_{1, s} & \text{in } B_r \\ e_{1, s} = 0 & \text{in } \mathbb{R}^n \setminus B_r. \end{cases}$$

We prove that

**Proposition 2.** *The function  $e_{1, s}$  given in (2.8) is such that*

$$e_{1, s} \notin W^{1, \infty}(B_r).$$

*Proof.* The proof is by contradiction. We suppose that  $e_{1, s} \in W^{1, \infty}(B_r)$  and so  $e_{1, s} \in W^{1, \infty}(\mathbb{R}^n)$ , that is

$$(2.9) \quad |e_{1, s}(x)| + |\nabla e_{1, s}(x)| \leq M, \quad x \in \mathbb{R}^n$$

for some  $M > 0$ .

From now on, we proceed by steps.

**Step 1.** *The function  $e_{1, s}$  is spherically symmetric and radially decreasing in  $\mathbb{R}^n$ .*

<sup>3</sup>More explicitly, one can choose  $c_o(n, s) := \Gamma(n/2) \sin(\pi s) / \pi^{(n/2)+1}$ , see [20, pages 399–400].

*Proof.* For this, since  $e_{1,s} \geq 0$  in  $\mathbb{R}^n$ , we consider its symmetric radially decreasing rearrangement  $e_{1,s}^*$  (see, e.g., [19, Chapter 2] for the basics of such a rearrangement). We observe that  $e_{1,s}^*$  vanishes outside  $B_r$ , since so does  $e_{1,s}$ . Moreover, we recall that the  $L^2$ -norm is preserved by the rearrangement, while the fractional Gagliardo seminorm decreases, see, e.g. [1, 3, 21]. Then, by this and since  $\lambda_{1,s}$  is obtained by minimizing the fractional Gagliardo seminorm under constraint on the  $L^2$ -norm for functions that vanish outside  $B_r$  (see [28, Proposition 9]), we conclude that the minimum is attained by  $e_{1,s}^*$  (as well as by  $e_{1,s}$ ).

Since  $\lambda_{1,s}$  is a simple eigenvalue (see [28, Proposition 9 and Appendix A]), it follows that  $e_{1,s}^* = e_{1,s}$  and Step 1 is proved.  $\square$

Now, let  $Q$  be the fractional fundamental solution given by

$$Q(x) := c_1(n, s)|x|^{2s-n}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Here the constant  $c_1(n, s) > 0$  is chosen in such a way that  $(-\Delta)^s Q$  is the Dirac's delta  $\delta_0$  centered at the origin (see, e.g., [20, page 44] for the basic properties of fractional fundamental solutions).

We define

$$(2.10) \quad \tilde{v}(x) := \lambda_{1,s} Q * e_{1,s}(x) = \lambda_{1,s} c_1(n, s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y) dy, \quad x \in \mathbb{R}^n$$

and

$$(2.11) \quad v(x) := e_{1,s}(x) - \tilde{v}(x), \quad x \in \mathbb{R}^n.$$

First of all, notice that  $\tilde{v} \geq 0$  in  $\mathbb{R}^n$ , since  $\lambda_{1,s} > 0$ ,  $Q > 0$  and  $e_{1,s} \geq 0$  in  $\mathbb{R}^n$ .

**Step 2.** *The function  $\tilde{v}$  is spherically symmetric and radially decreasing in  $\mathbb{R}^n$ .*

*Proof.* Indeed, if  $\mathcal{R}$  is a rotation, we use Step 1 and the substitution  $\tilde{y} := \mathcal{R}y$  to obtain for any  $x \in \mathbb{R}^n$

$$\begin{aligned} \tilde{v}(x) &= \lambda_{1,s} c_1(n, s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y) dy = \\ &= \lambda_{1,s} c_1(n, s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(\mathcal{R}(x-y)) dy \\ &= \lambda_{1,s} c_1(n, s) \int_{\mathbb{R}^n} |\tilde{y}|^{2s-n} e_{1,s}(\mathcal{R}x - \tilde{y}) d\tilde{y} = \tilde{v}(\mathcal{R}x), \end{aligned}$$

that shows the spherical symmetry of  $\tilde{v}$ .

As for the fact that  $\tilde{v}$  is radially decreasing in  $\mathbb{R}^n$ , we take  $\rho > 0$  and define

$$(2.12) \quad v_\star(\rho) := -(\lambda_{1,s} c_1(n, s))^{-1} \tilde{v}(0, \dots, 0, \rho) = - \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(-y', \rho - y_n) dy,$$

where we used the notation  $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$  for the coordinates in  $\mathbb{R}^n$ .

The goal is to show that for any  $\rho > 0$

$$(2.13) \quad v'_\star(\rho) \geq 0.$$

For this, first note that

$$\begin{aligned} v_\star(\rho) &= - \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leq r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n) dy \\ &\quad - \int_{\mathbb{R}^n \cap \{|\rho - y_n| > r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n) dy \\ &= - \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leq r\}} |y|^{2s-n} e_{1,s}(-y', \rho - y_n) dy, \end{aligned}$$

since  $\{|\rho - y_n| > r\} \subseteq \{|(-y', \rho - y_n)| > r\}$  and  $e_{1,s}$  vanishes outside  $B_r$ . Also, since the function  $e_{1,s}$  is spherically symmetric and radially decreasing in  $\mathbb{R}^n$  by Step 1, we write  $e_{1,s}(x) = -E(|x|)$  with  $E' \geq 0$  in  $\mathbb{R}^+$ . Thus,

$$v_\star(\rho) = \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leq r\}} |y|^{2s-n} E(|(-y', \rho - y_n)|) dy$$

and so

$$(2.14) \quad v'_\star(\rho) = \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leq r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|} dy.$$

Now, let us consider the following change of variables

$$(2.15) \quad \begin{cases} \tilde{y}' := y' \\ \tilde{y}_n := 2\rho - y_n. \end{cases}$$

First of all, note that if  $\tilde{y}_n - \rho \geq 0$ , then  $-\tilde{y}_n \leq 2\rho - \tilde{y}_n \leq \tilde{y}_n$ , so that

$$(2\rho - \tilde{y}_n)^2 \leq \tilde{y}_n^2$$

and

$$|(\tilde{y}', 2\rho - \tilde{y}_n)| = \sqrt{|\tilde{y}'|^2 + (2\rho - \tilde{y}_n)^2} \leq \sqrt{|\tilde{y}'|^2 + \tilde{y}_n^2} = |\tilde{y}|.$$

As a consequence of this and recalling that  $n \geq 2s$ , we obtain that

$$(2.16) \quad |(\tilde{y}', 2\rho - \tilde{y}_n)|^{2s-n} \geq |\tilde{y}|^{2s-n}.$$

Therefore, by (2.15) and (2.16) we get

$$\begin{aligned} & \int_{\mathbb{R}^n \cap \{0 \leq \rho - y_n \leq r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|} dy \\ &= \int_{\mathbb{R}^n \cap \{0 \leq \tilde{y}_n - \rho \leq r\}} |(\tilde{y}', 2\rho - \tilde{y}_n)|^{2s-n} E'(|(\tilde{y}', \rho - \tilde{y}_n)|) \frac{\tilde{y}_n - \rho}{|(\tilde{y}', \rho - \tilde{y}_n)|} d\tilde{y} \\ &\geq \int_{\mathbb{R}^n \cap \{0 \leq \tilde{y}_n - \rho \leq r\}} |\tilde{y}|^{2s-n} E'(|(\tilde{y}', \rho - \tilde{y}_n)|) \frac{\tilde{y}_n - \rho}{|(\tilde{y}', \rho - \tilde{y}_n)|} d\tilde{y}, \end{aligned}$$

due to the fact that  $E' \geq 0$  in  $\mathbb{R}^+$ .

Hence, recalling (2.14), we get

$$\begin{aligned} v'_\star(\rho) &= \int_{\mathbb{R}^n \cap \{|\rho - y_n| \leq r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|} dy \\ &= \int_{\mathbb{R}^n \cap \{0 \leq \rho - y_n \leq r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|} dy \\ &\quad + \int_{\mathbb{R}^n \cap \{0 \leq y_n - \rho \leq r\}} |y|^{2s-n} E'(|(-y', \rho - y_n)|) \frac{\rho - y_n}{|(-y', \rho - y_n)|} dy \\ &\geq \int_{\mathbb{R}^n \cap \{0 \leq \tilde{y}_n - \rho \leq r\}} |\tilde{y}|^{2s-n} E'(|(\tilde{y}', \rho - \tilde{y}_n)|) \frac{\tilde{y}_n - \rho}{|(\tilde{y}', \rho - \tilde{y}_n)|} d\tilde{y} \\ &\quad + \int_{\mathbb{R}^n \cap \{0 \leq \tilde{y}_n - \rho \leq r\}} |\tilde{y}|^{2s-n} E'(|(-\tilde{y}', \rho - \tilde{y}_n)|) \frac{\rho - \tilde{y}_n}{|(-\tilde{y}', \rho - \tilde{y}_n)|} d\tilde{y} \\ &= 0, \end{aligned}$$

due to the fact that  $|(\tilde{y}', \rho - \tilde{y}_n)| = |(-\tilde{y}', \rho - \tilde{y}_n)|$ . Hence, (2.13) is proved.

Then, by (2.12), the spherical symmetry of  $\tilde{v}$  and the fact that  $\lambda_{1,s}$  and  $c_1(n, s)$  are positive constants, we get that  $\tilde{v}$  is radially decreasing in  $\mathbb{R}^n$ . This concludes the proof of Step 2.  $\square$

Next step will exploit assumption (2.9) taken for the argument by contradiction.

**Step 3.** *The function  $\tilde{v}$  is such that*

$$\tilde{v} \in W^{1,\infty}(B_{2r}).$$

*Proof.* To check this, we observe that for any  $x \in \mathbb{R}^n$

$$\tilde{v}(x) = \lambda_{1,s} c_1(n,s) \int_{\mathbb{R}^n} |y|^{2s-n} e_{1,s}(x-y) dy = \lambda_{1,s} c_1(n,s) \int_{B_r(x)} |y|^{2s-n} e_{1,s}(x-y) dy,$$

since  $e_{1,s}$  vanishes outside  $B_r$  by (2.8). Here,  $B_r(x)$  denotes the ball of radius  $r$  centered at  $x$ .

Now, we notice that if  $x \in B_{2r}$  then  $B_r(x) \subset B_{3r}$ . As a consequence, recalling also (2.9), we obtain that for any  $x \in B_{2r}$

$$\begin{aligned} |\tilde{v}(x)| + |\nabla \tilde{v}(x)| &\leq \lambda_{1,s} c_1(n,s) \int_{B_r(x)} |y|^{2s-n} (|e_{1,s}(x-y)| + |\nabla e_{1,s}(x-y)|) dy \\ &\leq \lambda_{1,s} c_1(n,s) M \int_{B_{3r}} |y|^{2s-n} dy, \end{aligned}$$

which is finite (being  $s > 0$ ). Hence, Step 3 is established.  $\square$

Now we can conclude the proof of Proposition 2.

For this, note that, from (2.9) and Step 3, we get

$$v = e_{1,s} - \tilde{v} \in W^{1,\infty}(B_{2r}),$$

i.e. there exists  $\tilde{M} > 0$  such that

$$(2.17) \quad |v(x) - v(y)| \leq \tilde{M}|x - y|$$

for any  $x, y \in B_{2r}$ .

Also, by (2.10) and the choice of  $Q$

$$(-\Delta)^s \tilde{v} = \lambda_{1,s} e_{1,s} * (-\Delta)^s Q = \lambda_{1,s} e_{1,s} * \delta_0 = \lambda_{1,s} e_{1,s}$$

and so, by (2.8) and (2.11)

$$(-\Delta)^s v = (-\Delta)^s e_{1,s} - (-\Delta)^s \tilde{v} = \lambda_{1,s} e_{1,s} - \lambda_{1,s} e_{1,s} = 0$$

in  $B_r$ . Therefore, we can reconstruct  $v$  by its values outside  $B_r$  via the fractional Poisson kernel, that is, for any  $x \in B_r$ ,

$$(2.18) \quad v(x) = \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) v(y) dy,$$

for this see (2.5) and (2.6).

Since (2.11) holds true and  $e_{1,s} = 0$  outside  $B_r$ , by (2.18) we deduce

$$\begin{aligned} (2.19) \quad v(x) &= \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) v(y) dy \\ &= \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) e_{1,s}(y) dy - \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) \tilde{v}(y) dy \\ &= - \int_{\mathbb{R}^n \setminus B_r} P_r(x,y) \tilde{v}(y) dy. \end{aligned}$$

By (2.11), (2.17), (2.18) and (2.19) we get

$$\begin{aligned}
(2.20) \quad & \left| \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) \tilde{v}(y) dy - \tilde{v}(0, \dots, 0, r) \right| \\
&= \left| - \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) v(y) dy + v(0, \dots, 0, r) - e_{1,s}(0, \dots, 0, r) \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) v(y) dy - v(0, \dots, 0, r) \right| \\
&= |v(0, \dots, 0, r) - v(x)| \\
&\leq \tilde{M} |(0, \dots, 0, r) - x|
\end{aligned}$$

for any  $x \in B_r$ .

If in (2.20) we take  $x := (0, \dots, 0, r - \varepsilon) \in B_r$  for a small  $\varepsilon \in (0, r)$ , recalling (2.4), we deduce that

$$\begin{aligned}
(2.21) \quad & \tilde{M}\varepsilon = \tilde{M} |(0, \dots, 0, r) - x| \\
&\geq \left| \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) \tilde{v}(y) dy - \tilde{v}(0, \dots, 0, r) \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_r} P_r(x, y) (\tilde{v}(y) - \tilde{v}(0, \dots, 0, r)) dy \right| \\
&= c_o(n, s) \int_{|y|>r} \left( \frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^s \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{|x - y|^n} dy \\
&= c_o(n, s) (r^2 - |x|^2)^s \int_{|y|>r} \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{(|y|^2 - r^2)^s |x - y|^n} dy \\
&\geq c_o(n, s) r^s (r - |x|)^s \int_{|y|>r} \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{(|y|^2 - r^2)^s (|y'|^2 + |y_n - r + \varepsilon|^2)^{n/2}} dy \\
&= \varepsilon^s \int_{|y|>r} f_\varepsilon(y) dy,
\end{aligned}$$

where

$$f_\varepsilon(y) := c_o(n, s) r^s \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{(|y|^2 - r^2)^s (|y'|^2 + |y_n - r + \varepsilon|^2)^{n/2}}.$$

We remark that  $f_\varepsilon(y) \geq 0$  for any  $|y| > r$ , since

$$(2.22) \quad \tilde{v}(0, \dots, 0, r) \geq \tilde{v}(y) \text{ for any } |y| > r,$$

thanks to Step 1. Moreover

$$\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon(y) = c_o(n, s) r^s \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{(|y|^2 - r^2)^s (|y'|^2 + |y_n - r|^2)^{n/2}}.$$

So, we divide by  $\varepsilon^s$  the inequality obtained in (2.21) and we use Fatou's Lemma: we conclude that

$$\begin{aligned}
0 &= \liminf_{\varepsilon \rightarrow 0^+} \tilde{M}\varepsilon^{1-s} \geq \liminf_{\varepsilon \rightarrow 0^+} \int_{|y|>r} f_\varepsilon(y) dy \\
&= c_o(n, s) r^s \int_{|y|>r} \frac{\tilde{v}(0, \dots, 0, r) - \tilde{v}(y)}{(|y|^2 - r^2)^s (|y'|^2 + |y_n - r|^2)^{n/2}} dy.
\end{aligned}$$

This and (2.22) yield that  $\tilde{v}(y)$  is constantly equal to  $\tilde{v}(0, \dots, 0, r)$  for any  $|y| > r$ , so that, in particular, if  $x^* := (0, \dots, 2r)$  we have that

$$(2.23) \quad \partial_n \tilde{v}(x^*) = 0.$$



On the other hand, by (2.10),

$$\begin{aligned} \frac{1}{\lambda_{1,s}c_1(n,s)}\partial_n\tilde{v}(x^*) &= \frac{\partial}{\partial x_n} \int_{B_r} |x-z|^{2s-n}e_{1,s}(z) dz \Big|_{x=x^*} \\ &= (2s-n) \int_{B_r} |x^*-z|^{2s-n-2}(x_n^*-z_n)e_{1,s}(z) dz \\ &= (2s-n) \int_{B_r} (|z'|^2+|2r-z_n|^2)^{(2s-n-2)/2}(2r-z_n)e_{1,s}(z) dz, \end{aligned}$$

which is strictly negative, by (2.7). This is a contradiction with (2.23) and hence Proposition 2 is proved.  $\square$

### 3. THE SPECTRUM OF $A_s$ AND $(-\Delta)^s$

In this section we focus on the spectrum of the operators  $A_s$  and  $(-\Delta)^s$ . In what follows, we will denote by

$$0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

the divergent sequence of the eigenvalues of the Laplace operator  $-\Delta$  in  $\Omega$  with Dirichlet homogeneous boundary data, while by  $\lambda_{k,A_s}$  the sequence of eigenvalues of problem (2.1) and, finally, by  $\lambda_{k,s}$  the eigenvalues of (2.2).

By definition of  $A_s$ , it easily follows that the eigenvalues  $\lambda_{k,A_s}$  are exactly the  $s$ -power of the ones of the Laplacian, that is

$$(3.1) \quad \lambda_{k,A_s} = \lambda_k^s, \quad k \in \mathbb{N}.$$

As for  $\lambda_{k,s}$ , we refer to [28, Proposition 9 and Appendix A], [25, Proposition 5] and [30, Proposition 4] for their variational characterizations and some basic properties. In particular, we recall that for  $k \in \mathbb{N}$

$$(3.2) \quad \lambda_{k,s} = \frac{c(n,s)}{2} \min_{u \in \mathbb{P}_{k,s} \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dx dy}{\int_{\Omega} |u(x)|^2 dx},$$

where

$$\mathbb{P}_{1,s} = X_0(\Omega) := \{u \in H^s(\mathbb{R}^n) \text{ s.t. } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}$$

and

$$(3.3) \quad \mathbb{P}_{k,s} := \{u \in X_0(\Omega) \text{ s.t. } \langle u, e_{j,s} \rangle_{X_0(\Omega)} = 0 \quad \forall j = 1, \dots, k-1\}, \quad k \geq 2$$

with

$$\langle u, v \rangle_{X_0(\Omega)} = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dx dy.$$

In what follows we will show that  $A_s$  and  $(-\Delta)^s$  have different eigenvalues. Of course, at this purpose we will use properties (3.1) and (3.2), but the main ingredient will be the extension of the operator  $A_s$ , carried on in the forthcoming Section 4.

### 4. ONE-DIMENSIONAL ANALYSIS

In this section we perform an ODE analysis related to the extension of the operator  $A_s$ , as it will be clear in the forthcoming Section 5.

This analysis is not new in itself (see also [7, Section 3.2] and [34, Section 3.1]): similar results were obtained, for instance, in [34] by using a conjugate equation and suitable special functions such as different kinds of Bessel and Hankel functions. Here, we use an elementary and self-contained approach.

Given  $a \in (-1, 1)$  in what follows we denote by  $W_a^{1,2}(\mathbb{R}^+)$  the following Sobolev space

$$W_a^{1,2}(\mathbb{R}^+) := \left\{ g \in W_{\text{loc}}^{1,1}(\mathbb{R}^+) : \int_{\mathbb{R}^+} t^a |g(t)|^2 dt < +\infty \text{ and } \int_{\mathbb{R}^+} t^a |\dot{g}(t)|^2 dt < +\infty \right\}$$

endowed with the norm

$$(4.1) \quad \|g\|_{W_a^{1,2}(\mathbb{R}^+)} := \left( \int_{\mathbb{R}^+} t^a |g(t)|^2 dt + \int_{\mathbb{R}^+} t^a |\dot{g}(t)|^2 dt \right)^{1/2}.$$

Here, as usual, we used the notation  $\mathbb{R}^+ := (0, +\infty)$ . We also denote by  $W_{1,a}^{1,2}(\mathbb{R}^+)$  the closure, with respect to the norm in (4.1), of the set of all functions  $g \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$  with bounded support and  $g(0) = 1$ .

It is useful to point out that  $W_a^{1,2}(\mathbb{R}^+)$  and  $W_{1,a}^{1,2}(\mathbb{R}^+)$  are contained in a classical Sobolev space. Precisely, denoting by  $W^{1,p}((0, \kappa))$ ,  $p \geq 1$  and  $\kappa > 0$ , the classical Sobolev space endowed with the norm

$$\|g\|_{W^{1,p}((0, \kappa))} = \left( \|g\|_{L^p((0, \kappa))}^p + \|\dot{g}\|_{L^p((0, \kappa))}^p \right)^{1/p},$$

the following result holds true:

**Lemma 3.** *Fix  $a \in (-1, 1)$  and  $\kappa > 0$ . Then,*

$$W_a^{1,2}(\mathbb{R}^+) \subseteq W^{1,p}((0, \kappa))$$

for any  $p \in [1, a^*]$ , with

$$a^* = \begin{cases} 2/(a+1) & \text{if } a \in (0, 1) \\ 2 & \text{if } a \in (-1, 0]. \end{cases}$$

Moreover, there exists  $C_\kappa > 0$  such that

$$\|g\|_{W^{1,p}((0, \kappa))} \leq C_\kappa \|g\|_{W_a^{1,2}(\mathbb{R}^+)}$$

for any  $g \in W_a^{1,2}(\mathbb{R}^+)$ .

*Proof.* Let  $a \in (-1, 1)$ ,  $g \in W_a^{1,2}(\mathbb{R}^+)$  and  $p \in [1, a^*]$ . We use the Hölder Inequality with exponents  $2/(2-p)$  and  $2/p$  (note that both these exponents are greater than 1, thanks to the choice of  $p$ ) to see that

$$\begin{aligned} \|g\|_{L^p((0, \kappa))}^p &= \int_0^\kappa t^{-pa/2} t^{pa/2} |g(t)|^p dt \\ &\leq \left[ \int_0^\kappa t^{-pa/(2-p)} dt \right]^{(2-p)/2} \left[ \int_0^\kappa t^a |g(t)|^2 dt \right]^{p/2} \\ &= \left[ \frac{2-p}{2-p(1+a)} \kappa^{(2-p(1+a))/(2-p)} \right]^{(2-p)/2} \left[ \int_{\mathbb{R}^+} t^a |g(t)|^2 dt \right]^{p/2} < +\infty, \end{aligned}$$

again since  $p < a^*$ .

A similar inequality holds if we replace  $g$  with  $\dot{g}$ , and this proves the desired result.  $\square$

Hence, as a consequence of Lemma 3, the functions in  $W_{1,a}^{1,2}(\mathbb{R}^+)$  are uniformly continuous in any interval, by the standard Sobolev embedding, and have a distributional derivative which is well-defined a.e.

Now, for any  $\lambda > 0$  and any  $g \in W_{1,a}^{1,2}(\mathbb{R}^+)$ , we consider the functional

$$G_\lambda(g) := \int_{\mathbb{R}^+} t^a \left( |g(t)|^2 dt + \lambda |\dot{g}(t)|^2 \right) dt.$$

The minimization problem of  $G_\lambda$  is described in detail by the following result:

**Theorem 4.** *There exists a unique  $g_\lambda \in W_{1,a}^{1,2}(\mathbb{R}^+)$  such that*

$$(4.2) \quad m_\lambda := \inf_{g \in W_{1,a}^{1,2}(\mathbb{R}^+)} G_\lambda(g) = G_\lambda(g_\lambda),$$

that is, the above infimum is attained.

Moreover,  $g_\lambda \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$  and it satisfies

$$(4.3) \quad \begin{cases} \ddot{g}_\lambda(t) + \frac{a}{t} \dot{g}_\lambda(t) - \lambda g_\lambda(t) = 0 & \text{for any } t \in \mathbb{R}^+ \\ g_\lambda(0) = 1, \end{cases}$$

and

$$(4.4) \quad \lim_{t \rightarrow 0^+} t^a \dot{g}_\lambda(t) = -m_1 \lambda^{(a+1)/2}.$$

Finally,

$$(4.5) \quad g_\lambda(t) \in [0, 1] \text{ and } \dot{g}_\lambda(t) \leq 0 \text{ for all } t \in \mathbb{R}^+ \text{ and } \lim_{t \rightarrow +\infty} g_\lambda(t) = 0.$$

*Proof.* By plugging a smooth and compactly supported function in  $G_\lambda$ , we see that  $m_\lambda \in [0, +\infty)$ , so we can take a minimizing sequence  $g_j$  in  $W_{1,a}^{1,2}(\mathbb{R}^+)$ , that is a sequence  $g_j$  such that

$$G_\lambda(g_j) \rightarrow m_\lambda$$

as  $j \rightarrow +\infty$ .

In particular,  $G_\lambda(g_j) \leq m_\lambda + 1$ . As a consequence of this,  $\|g_j\|_{W_a^{1,2}(\mathbb{R}^+)}$  is bounded uniformly in  $j$ . Hence, there exists  $g_\lambda \in W_{1,a}^{1,2}(\mathbb{R}^+)$  such that  $g_j \rightarrow g_\lambda$  weakly in  $W_{1,a}^{1,2}(\mathbb{R}^+)$  as  $j \rightarrow +\infty$ . Also, for any  $k \in \mathbb{N}$ ,  $k \geq 2$ , we have that

$$\tilde{C}_k \int_{1/k}^k (|g_j(t)|^2 + |\dot{g}_j(t)|^2) dt \leq \int_{1/k}^k t^a (|g_j(t)|^2 + |\dot{g}_j(t)|^2) dt \leq m_\lambda + 1,$$

where  $\tilde{C}_k = (1/k)^a$  if  $a \geq 0$ , while  $\tilde{C}_k = k^a$  if  $a < 0$ . Namely,  $\|g_j\|_{W^{1,2}([1/k,k])}$  is bounded uniformly in  $j$ .

Now, we perform a diagonal compactness argument over the index  $k$ . Namely, we take an increasing function  $\phi_k : \mathbb{N} \rightarrow \mathbb{N}$  and we use it to extract subsequences. We have a subsequence  $g_{\phi_2(j)}$  that converges a.e. in  $[1/2, 2]$  to  $g_\lambda$  with  $\dot{g}_{\phi_2(j)}$  converging to  $\dot{g}_\lambda$  weakly in  $L^2([1/2, 2])$  as  $j \rightarrow +\infty$ . Then, we take a further subsequence  $g_{\phi_3(\phi_2(j))}$  that converges a.e. in  $[1/3, 3]$  to  $g_\lambda$  with  $\dot{g}_{\phi_3(\phi_2(j))}$  converging to  $\dot{g}_\lambda$  weakly in  $L^2([1/3, 3])$  as  $j \rightarrow +\infty$ . Iteratively, for any  $k \in \mathbb{N}$ , we get a subsequence  $g_{\phi_k \circ \dots \circ \phi_2(j)}$  that converges a.e. in  $[1/k, k]$  to  $g_\lambda$  with  $\dot{g}_{\phi_k \circ \dots \circ \phi_2(j)}$  converging to  $\dot{g}_\lambda$  weakly in  $L^2([1/k, k])$  as  $j \rightarrow +\infty$ .

Hence we look at the diagonal sequence  $\bar{g}_j := g_{\phi_j \circ \dots \circ \phi_2(j)}$ . By construction  $\bar{g}_j$  converges to  $g_\lambda$  a.e. in  $\mathbb{R}^+$  as  $j \rightarrow +\infty$  and therefore, by Fatou Lemma,

$$(4.6) \quad \liminf_{j \rightarrow +\infty} \int_{\mathbb{R}^+} t^a |\bar{g}_j(t)|^2 dt \geq \int_{\mathbb{R}^+} t^a |g_\lambda(t)|^2 dt.$$

On the other hand, by the weak convergence of  $\dot{\bar{g}}_j$  to  $\dot{g}_\lambda$  in  $L^2([1/k, k])$  as  $j \rightarrow +\infty$ , we have that  $\dot{g}_\lambda \in L^2([1/k, k])$  and so  $\psi(t) := t^a \dot{g}_\lambda(t)$  is also in  $L^2([1/k, k])$ , which gives

$$\lim_{j \rightarrow +\infty} \int_{1/k}^k \dot{\bar{g}}_j(t) \psi(t) dt = \int_{1/k}^k \dot{g}_\lambda(t) \psi(t) dt,$$

that is

$$\lim_{j \rightarrow +\infty} \int_{1/k}^k t^a \dot{\bar{g}}_j(t) \dot{g}_\lambda(t) dt = \int_{1/k}^k t^a |\dot{g}_\lambda(t)|^2 dt$$

for any  $k \in \mathbb{N}$ ,  $k \geq 2$ . As a consequence of this, we obtain that

$$\begin{aligned}
(4.7) \quad & 0 \leq \liminf_{j \rightarrow +\infty} \int_{1/k}^k t^\alpha |\dot{\bar{g}}_j(t) - \dot{g}_\lambda(t)|^2 dt \\
& = \liminf_{j \rightarrow +\infty} \left( \int_{1/k}^k t^\alpha |\dot{\bar{g}}_j(t)|^2 dt + \int_{1/k}^k t^\alpha |\dot{g}_\lambda(t)|^2 dt - 2 \int_{1/k}^k t^\alpha \dot{\bar{g}}_j(t) \cdot \dot{g}_\lambda(t) dt \right) \\
& = \liminf_{j \rightarrow +\infty} \int_{1/k}^k t^\alpha |\dot{\bar{g}}_j(t)|^2 dt - \int_{1/k}^k t^\alpha |\dot{g}_\lambda(t)|^2 dt
\end{aligned}$$

for any  $k \in \mathbb{N}$ ,  $k \geq 2$ .

By (4.6), (4.7) and the positivity of  $\lambda$  we get

$$\begin{aligned}
m_\lambda &= \lim_{j \rightarrow +\infty} G_\lambda(g_j) \\
&= \lim_{j \rightarrow +\infty} \left( \int_{\mathbb{R}^+} t^\alpha |\bar{g}_j(t)|^2 dt + \lambda \int_{\mathbb{R}^+} t^\alpha |\dot{\bar{g}}_j(t)|^2 dt \right) \\
&\geq \liminf_{j \rightarrow +\infty} \left( \int_{\mathbb{R}^+} t^\alpha |\bar{g}_j(t)|^2 dt + \lambda \int_{1/k}^k t^\alpha |\dot{\bar{g}}_j(t)|^2 dt \right) \\
&\geq \int_{\mathbb{R}^+} t^\alpha |g_\lambda(t)|^2 dt + \lambda \int_{1/k}^k t^\alpha |\dot{g}_\lambda(t)|^2 dt
\end{aligned}$$

for any  $k \in \mathbb{N}$ ,  $k \geq 2$ . By taking  $k \rightarrow +\infty$ , we deduce that  $m_\lambda \geq G_\lambda(g_\lambda)$ . This proves that the infimum in (4.2) is attained at  $g_\lambda$ . The uniqueness of the minimizer is due to the convexity of the functional  $G_\lambda$ . This completes the proof of (4.2).

Now, notice that, since  $g_\lambda \in W_{1,a}^{1,2}(\mathbb{R}^+)$ , then  $g_\lambda(0) = 1$  and  $g_\lambda \in W^{1,p}((0, \kappa))$  for any  $p \in [1, a^*)$  and any  $\kappa > 0$ , by Lemma 3. Hence, it is uniformly continuous on  $(0, \kappa)$  for any  $\kappa > 0$ , by the standard Sobolev embedding, and so it can be extended with continuity at 0, that is the function  $g_\lambda \in C^0(\overline{\mathbb{R}^+})$ .

Moreover, by taking standard perturbation of the functional  $G_\lambda$  at  $g_\lambda + \varepsilon\phi$ , with  $\phi \in C_0^\infty(\mathbb{R}^+)$  and  $\varepsilon \in \mathbb{R}$  small, one obtains that

$$(4.8) \quad \int_{\mathbb{R}^+} t^\alpha \left( g_\lambda(t)\phi(t) + \lambda \dot{g}_\lambda(t)\dot{\phi}(t) \right) dt = 0.$$

Hence,  $g_\lambda$  satisfies weakly an ODE and so  $g_\lambda \in C^\infty(\mathbb{R}^+)$  by uniformly elliptic regularity theory (see for instance<sup>4</sup> [16, Theorem 8.10]). Moreover, integrating by parts in (4.8) it easily follows that  $g_\lambda$  solves problem (4.3).

Now, we prove (4.4). For this, it is convenient to reduce to the case  $\lambda = 1$ , by noticing that if  $g^{(\lambda)}(t) := g(t/\sqrt{\lambda})$ , we have that

$$G_\lambda(g^{(\lambda)}) = \lambda^{(a+1)/2} G_1(g)$$

and therefore

$$(4.9) \quad m_\lambda = \lambda^{(a+1)/2} m_1 \quad \text{and} \quad g_\lambda(t) = g_1(t/\sqrt{\lambda}).$$

Let us fix  $\phi \in C_0^\infty([-1, 1])$  with  $\phi(0) = 1$  and let

$$\gamma(t) := t^\alpha (g_1(t)\phi(t) + \dot{g}_1(t)\dot{\phi}(t)).$$

By the Cauchy–Schwarz Inequality, we have that

$$\int_{\mathbb{R}^+} \gamma(t) dt \leq G_1(g_1) G_1(\phi) < +\infty,$$

<sup>4</sup>In further detail,  $g_\lambda$  satisfies [16, Equation (8.2)] with  $n = 1$ ,  $a^{ij} = a^{11} = \lambda t^\alpha$ ,  $b^i = b^1 = 0$ ,  $c^i = c^1 = 0$ ,  $d = t^\alpha$  and this equation is uniformly elliptic in bounded domains separated from 0: so we can apply [16, Theorem 8.10] with  $f = 0$  and obtain that  $g_\lambda \in W^{k,2}(b_1, b_2)$  for any  $b_2 > b_1 > 0$  and any  $k \in \mathbb{N}$ .

so that  $\gamma \in L^1(\mathbb{R}^+)$ . Therefore, by the absolute continuity of the Lebesgue integral, for any fixed  $\varepsilon > 0$  there exists  $\delta_\varepsilon > 0$  such that if  $0 < t_1 < t_2 < \delta_\varepsilon$  then

$$\int_{t_1}^{t_2} \gamma(\tau) d\tau < \varepsilon.$$

As a consequence, the function

$$\Gamma(t) := \int_t^{+\infty} \gamma(\tau) d\tau$$

is uniformly continuous in  $(0, 1)$  and therefore it may be extended with continuity at 0 as follows

$$(4.10) \quad \Gamma(0) = \int_0^{+\infty} \gamma(\tau) d\tau = \int_0^{+\infty} \tau^\alpha \left( g_1(\tau)\phi(\tau) + \dot{g}_1(\tau)\dot{\phi}(\tau) \right) d\tau.$$

By (4.3) with  $\lambda = 1$  it is easy to see that for any  $t \in \mathbb{R}^+$

$$t^\alpha g_1(t) = \frac{d}{dt} (t^\alpha \dot{g}_1(t)).$$

So, by this and recalling that  $\phi(0) = 1$  and  $\phi(t) = 0$  if  $t \geq 1$ , we get

$$(4.11) \quad \begin{aligned} \Gamma(0) &= \int_0^1 \tau^\alpha \left( g_1(\tau)\phi(\tau) + \dot{g}_1(\tau)\dot{\phi}(\tau) \right) d\tau \\ &= \int_0^1 \left[ \frac{d}{d\tau} \left( \tau^\alpha \dot{g}_1(\tau) \right) \phi(\tau) + \tau^\alpha \dot{g}_1(\tau)\dot{\phi}(\tau) \right] d\tau \\ &= \lim_{t \rightarrow 0^+} \int_t^1 \frac{d}{d\tau} \left( \tau^\alpha \dot{g}_1(\tau) \phi(\tau) \right) d\tau \\ &= - \lim_{t \rightarrow 0^+} t^\alpha \dot{g}_1(t) \phi(t) \\ &= - \lim_{t \rightarrow 0^+} t^\alpha \dot{g}_1(t). \end{aligned}$$

Note that the computation carried on in (4.11) has also shown that the above limit exists.

Now, to compute explicitly such limit, we consider the perturbation

$$g_{1,\varepsilon} := (g_1 + \varepsilon\phi)/(1 + \varepsilon)$$

with  $\varepsilon \in \mathbb{R}$  small. First of all, notice that  $g_{1,\varepsilon} = g_1 + \varepsilon\phi - \varepsilon g_1 + o(\varepsilon)$  and so

$$|g_{1,\varepsilon}|^2 = |g_1|^2 + 2\varepsilon g_1\phi - 2\varepsilon |g_1|^2 + o(\varepsilon),$$

and similarly if we replace  $g_{1,\varepsilon}$  with  $\dot{g}_{1,\varepsilon}$ . It follows that

$$G_1(g_{1,\varepsilon}) = G_1(g_1) + 2\varepsilon \int_{\mathbb{R}^+} \tau^\alpha \left( g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot{g}_1(\tau)\dot{\phi}(\tau) - |\dot{g}_1(\tau)|^2 \right) d\tau + o(\varepsilon).$$

Then, the minimality condition implies that

$$\int_{\mathbb{R}^+} \tau^\alpha \left( g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot{g}_1(\tau)\dot{\phi}(\tau) - |\dot{g}_1(\tau)|^2 \right) d\tau = 0.$$

Hence, by this, (4.10) and the definition of  $m_1$  we deduce

$$\begin{aligned} 0 &= \int_{\mathbb{R}^+} \tau^\alpha \left( g_1(\tau)\phi(\tau) - |g_1(\tau)|^2 + \dot{g}_1(\tau)\dot{\phi}(\tau) - |\dot{g}_1(\tau)|^2 \right) d\tau \\ &= \int_{\mathbb{R}^+} \tau^\alpha \left( g_1(\tau)\phi(\tau) + \dot{g}_1(\tau)\dot{\phi}(\tau) \right) d\tau - \int_{\mathbb{R}^+} \tau^\alpha \left( |g_1(\tau)|^2 + |\dot{g}_1(\tau)|^2 \right) d\tau \\ &= \Gamma(0) - m_1. \end{aligned}$$

This and (4.11) prove (4.4) for  $\lambda = 1$ . In general, recalling (4.9), we obtain

$$\lim_{t \rightarrow 0^+} t^\alpha \dot{g}_\lambda(t) = \lambda^{-1/2} \lim_{t \rightarrow 0^+} t^\alpha \dot{g}_1(t\lambda^{-1/2}) = \lambda^{(a+1)/2} \lim_{t \rightarrow 0^+} (t\lambda^{-1/2})^a \dot{g}_1(t\lambda^{-1/2}) = -m_1 \lambda^{(a+1)/2},$$

thus establishing (4.4) for any  $\lambda > 0$ .

Now, let us prove (4.5) For this we first observe that  $G_1(|g_1|) = G_1(g_1)$ , which implies, by the uniqueness of the minimizer, that  $g_1 = |g_1|$  and so  $g_1 \geq 0$  in  $\mathbb{R}^+$ .

We start showing that

$$(4.12) \quad \dot{g}_1 \leq 0 \text{ in the whole of } \mathbb{R}^+.$$

By contradiction, if  $g_1$  was increasing somewhere, there would exist  $t_2 > t_1 \geq 0$  such that  $0 \leq g_1(t_1) < g_1(t_2)$ . Let  $b := (g_1(t_1) + g_1(t_2))/2 \in (g_1(t_1), g_1(t_2))$ . Notice that there exists  $t_3 > t_2$  such that  $g(t_3) = b$ : otherwise, by continuity, we would have that  $g(t) > b > 0$  for any  $t > t_2$  and so, using that  $a \in (-1, 1)$ ,

$$G_1(g_1) \geq \int_{t_2}^{+\infty} t^a |g_1(t)|^2 dt \geq b^2 \int_{t_2}^{+\infty} t^a dt = +\infty,$$

which is against our contraction.

Having established the existence of the desired  $t_3$ , we use the Weierstrass Theorem to obtain  $t_\star \in [t_1, t_3]$  in such a way that

$$g_1(t_\star) = \max_{t \in [t_1, t_3]} g_1(t).$$

Note that, by definition of  $b$ ,

$$g_1(t_\star) \geq g_1(t_2) > b > g_1(t_1).$$

Hence,  $t_\star \neq t_1$  and also  $t_\star \neq t_3$ , being  $g_1(t_3) = b$ . Thus,  $t_\star$  is an interior maximum for  $g_1$ . Accordingly  $\dot{g}_1(t_\star) = 0$  and  $\ddot{g}_1(t_\star) \leq 0$ . Thus, by (4.3),

$$0 = \ddot{g}_1(t_\star) + \frac{a}{t_\star} \dot{g}_1(t_\star) - g_1(t_\star) \leq 0 + 0 - b = -b < 0.$$

This is a contradiction and it proves (4.12).

A consequence of (4.12) is also that  $g_1(t) \leq g_1(0) = 1$  for any  $t \in \mathbb{R}^+$ . Moreover, it implies that the limit

$$\ell := \lim_{t \rightarrow +\infty} g_1(t) \in [0, 1]$$

exists. Necessarily, it must be

$$\ell = 0.$$

Otherwise, if  $\ell > 0$ , it would follow that  $g(t) \geq \ell/2$  for any  $t \geq t_o$ , for a suitable  $t_o > 0$ . This yields that (using also that  $a \in (-1, 1)$ )

$$G_1(g_1) \geq \int_{t_o}^{+\infty} t^a |g_1(t)|^2 dt \geq (\ell/2)^2 \int_{t_o}^{+\infty} t^a dt = +\infty,$$

which is against our contraction. All these considerations imply (4.5) for  $\lambda = 1$ , and thus for any  $\lambda > 0$ , thanks to (4.9).  $\square$

## 5. A RELATION BETWEEN THE FIRST EIGENVALUE OF $A_s$ AND THAT OF $(-\Delta)^s$

This section is devoted to the study of the relation between the first eigenvalue of  $A_s$  and of  $(-\Delta)^s$ , that is between  $\lambda_{1, A_s}$  and  $\lambda_{1, s}$ . Precisely, in this framework our main result is the following:

**Proposition 5.** *The relation between the first eigenvalue of  $(-\Delta)^s$  and the one of  $A_s$  is given by*

$$\lambda_{1, s} < \lambda_{1, A_s}.$$

*Proof.* Let us take  $a := 2s - 1 \in (-1, 1)$  and for any  $(x, t) \in \Omega \times \mathbb{R}^+$ , set

$$E_1(x, t) := g_{\lambda_1}(t) e_1(x),$$

where the setting of Theorem 4 is in use,  $\lambda_1$  is the first eigenvalue of the Laplacian  $-\Delta$  and  $e_1 = e_{1, A_s}$  is the first eigenfunction of the operator  $A_s$  (see Section 2).

Notice that  $E_1$  may be thought as an extension of  $e_1$  in the half-space  $\mathbb{R}^n \times \mathbb{R}^+$  that vanishes in  $(\mathbb{R}^n \setminus \Omega) \times \mathbb{R}^+$ . However, we point out that  $E_1$  does not verify  $\operatorname{div}(\nabla E_1) = 0$  in the whole of  $\mathbb{R}^n \times \mathbb{R}^+$ .

Also, note that the function  $E_1 \in C^\infty(\Omega \times \mathbb{R}^+) \cap C(\overline{\Omega \times \mathbb{R}^+})$ , since  $e_1 \in C^\infty(\Omega) \cap C^m(\overline{\Omega})$  for any  $m \in \mathbb{N}$  (see formula (2.3)) and  $g_{\lambda_1} \in C^\infty(\mathbb{R}^+) \cap C^0(\overline{\mathbb{R}^+})$  by Theorem 4. Also,

$$\lim_{t \rightarrow 0^+} t^a \partial_t E_1(x, t) = \lim_{t \rightarrow 0^+} t^a \dot{g}_{\lambda_1}(t) e_1(x) = -m_1 \lambda_1^{(a+1)/2} e_1(x),$$

thanks to (4.4).

Furthermore, since  $G_{\lambda_1}(g_{\lambda_1})$  is finite by Theorem 4, we have that

$$L^1(\mathbb{R}^+) \ni t^a |g_{\lambda_1}(t)|^2 + t^a |\dot{g}_{\lambda_1}(t)|^2 \geq 2t^a |g_{\lambda_1}(t) \dot{g}_{\lambda_1}(t)|$$

and, therefore, there exists a diverging sequence of  $R$  for which

$$(5.1) \quad \lim_{R \rightarrow +\infty} R^a |g_{\lambda_1}(R) \dot{g}_{\lambda_1}(R)| = 0.$$

Now, note that, using<sup>5</sup> the definition of  $E_1$ , the fact that  $e_1$  is the first eigenfunction of  $-\Delta$  (for this see Section 2), for any  $(x, t) \in \Omega \times \mathbb{R}^+$  we have

$$(5.2) \quad \begin{aligned} t^a |\nabla E_1(x, t)|^2 &= \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right) - at^{a-1} E_1(x, t) \partial_t E_1(x, t) \\ &\quad - t^a E_1(x, t) \Delta E_1(x, t) \\ &= \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right) - at^{a-1} E_1(x, t) \dot{g}_{\lambda_1}(t) e_1(x) \\ &\quad - t^a E_1(x, t) g_{\lambda_1}(t) \Delta_x e_1(x) - t^a E_1(x, t) \ddot{g}_{\lambda_1}(t) e_1(x) \\ &= \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right) - at^{a-1} E_1(x, t) \dot{g}_{\lambda_1}(t) e_1(x) \\ &\quad + \lambda_1 t^a E_1(x, t) g_{\lambda_1}(t) e_1(x) - t^a E_1(x, t) \ddot{g}_{\lambda_1}(t) e_1(x) \\ &= \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right) \\ &\quad + t^a E_1(x, t) e_1(x) \left( -at^{-1} \dot{g}_{\lambda_1}(t) + \lambda_1 g_{\lambda_1}(t) - \ddot{g}_{\lambda_1}(t) \right) \\ &= \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right), \end{aligned}$$

thanks to (4.3).

By (5.2) and the Divergence Theorem, we have that

$$(5.3) \quad \begin{aligned} \iint_{\Omega \times \mathbb{R}^+} t^a |\nabla E_1(x, t)|^2 dx dt &= \lim_{R \rightarrow +\infty} \iint_{\Omega \times (0, R)} t^a |\nabla E_1(x, t)|^2 dx dt \\ &= \lim_{R \rightarrow +\infty} \iint_{\Omega \times (0, R)} \operatorname{div} \left( t^a E_1(x, t) \nabla E_1(x, t) \right) dx dt \\ &= \lim_{R \rightarrow +\infty} \int_{\Omega} \left( t^a E_1(x, t) \partial_t E_1(x, t) \right) \Big|_{t=R} - \left( t^a E_1(x, t) \partial_t E_1(x, t) \right) \Big|_{t=0} dx, \end{aligned}$$

<sup>5</sup>We remark that here  $\nabla$  is the vector collecting all the derivatives, both in  $x$  and in  $t$ . Similarly,  $\Delta = \Delta_x + \partial_t^2$ .

since for any  $t \in \mathbb{R}^+$ ,  $E_1(\cdot, t) = 0$  on  $\partial\Omega$ , being  $e_1 = 0$  outside  $\Omega$ . Hence, by (5.3) and using again the definition of  $E_1$  we deduce that

$$\begin{aligned}
& \iint_{\Omega \times \mathbb{R}^+} t^a |\nabla E_1(x, t)|^2 dx dt \\
&= \lim_{R \rightarrow +\infty} \int_{\Omega} \left[ \left( t^a E_1(x, t) \partial_t E_1(x, t) \right) \Big|_{t=R} - \left( t^a E_1(x, t) \partial_t E_1(x, t) \right) \Big|_{t=0} \right] dx \\
&= \lim_{R \rightarrow +\infty} \int_{\Omega} \left[ R^a E_1(x, R) \partial_t E_1(x, R) - \left( t^a E_1(x, t) \partial_t E_1(x, t) \right) \Big|_{t=0} \right] dx \\
&= \lim_{R \rightarrow +\infty} \int_{\Omega} \left[ R^a g_{\lambda_1}(R) \dot{g}_{\lambda_1}(R) |e_1(x)|^2 - \left( t^a g_{\lambda_1}(t) \dot{g}_{\lambda_1}(t) |e_1(x)|^2 \right) \Big|_{t=0} \right] dx \\
&= \lim_{R \rightarrow +\infty} \int_{\Omega} \left( R^a g_{\lambda_1}(R) \dot{g}_{\lambda_1}(R) |e_1(x)|^2 + m_1 \lambda_1^{(a+1)/2} |e_1(x)|^2 \right) dx \\
&= \lim_{R \rightarrow +\infty} \left( R^a g_{\lambda_1}(R) \dot{g}_{\lambda_1}(R) + m_1 \lambda_1^{(a+1)/2} \right) \|e_1\|_{L^2(\Omega)}^2 \\
&= m_1 \lambda_1^{(a+1)/2} \\
&= m_1 \lambda_1^s,
\end{aligned}$$

thanks to (4.4), the fact that  $g_{\lambda}(0) = 1$ , (5.1) and the choice of  $a$ . As a consequence,

$$\begin{aligned}
(5.4) \quad & \inf_{\substack{U \in C(\overline{\mathbb{R}^n \times \mathbb{R}^+}) \\ \|U(\cdot, 0)\|_{L^2(\Omega)} = 1 \\ U(\cdot, 0) = 0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla U(x, t)|^2 dx dt \leq \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla E_1(x, t)|^2 dx dt \\
&= \iint_{\Omega \times \mathbb{R}^+} t^a |\nabla E_1(x, t)|^2 dx dt \\
&= m_1 \lambda_1^s,
\end{aligned}$$

since  $E_1(\cdot, t) = e_1(\cdot) g_{\lambda}(t) = 0$  in  $\mathbb{R}^n \setminus \Omega$  for any  $t \in \mathbb{R}^+$ .

Now, we use a result in [7] to relate the first term in (5.4) to  $\lambda_{1,s}$  (which, roughly speaking, says the optimal  $U$  is realized by the so called  $a$ -harmonic extension of  $u := U(\cdot, 0)$ ). Namely, by [7, formula (3.7) and its proof at page 1250] and [10, Proposition 3.4], we get

$$\begin{aligned}
& \inf_{\substack{U \in C(\overline{\mathbb{R}^n \times \mathbb{R}^+}) \\ \|U(\cdot, 0)\|_{L^2(\Omega)} = 1 \\ U(\cdot, 0) = 0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla U(x, t)|^2 dx dt = m_1 \inf_{\substack{u \in C(\mathbb{R}^n) \\ \|u\|_{L^2(\Omega)} = 1 \\ u=0 \text{ outside } \Omega}} \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \\
&= m_1 \frac{c(n, s)}{2} \inf_{\substack{u \in C(\mathbb{R}^n) \\ \|u\|_{L^2(\Omega)} = 1 \\ u=0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy, \\
&\geq m_1 \frac{c(n, s)}{2} \min_{\substack{u \in X_0(\Omega) \\ \|u\|_{L^2(\Omega)} = 1}} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \\
&= m_1 \lambda_{1,s},
\end{aligned}$$

thanks also to the variational characterization of  $\lambda_{1,s}$  given in (3.2). Here  $\hat{u}$  denotes the Fourier transform of  $u$ .

Thus,

$$\inf_{\substack{U \in C(\overline{\mathbb{R}^n \times \mathbb{R}^+}) \\ \|U(\cdot, 0)\|_{L^2(\Omega)} = 1 \\ U(\cdot, 0) = 0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla U(x, t)|^2 dx dt \geq m_1 \lambda_{1,s}.$$

This and (5.4) give that

$$\lambda_{1,s} \leq \lambda_1^s.$$



We claim that the strict inequality occurs. If, by contradiction, equality holds here, then it does in (5.4), namely

$$E_1 \in \arg \min \left\{ \begin{array}{l} \inf_{\substack{U \in C(\overline{\mathbb{R}^n \times \mathbb{R}^+}) \\ \|U(\cdot, 0)\|_{L^2(\Omega)} = 1 \\ U(\cdot, 0) = 0 \text{ outside } \Omega}} \iint_{\mathbb{R}^n \times \mathbb{R}^+} t^a |\nabla U(x, t)|^2 dx dt \end{array} \right\}.$$

We remark that such minimizers are continuous up to  $\overline{\mathbb{R}^n \times \mathbb{R}^+}$ , and they solve the associated elliptic partial differential equation in  $\mathbb{R}^n \times \mathbb{R}^+$ , see [12]: in particular  $E_1$  would solve an elliptic partial differential equation in  $\mathbb{R}^n \times \mathbb{R}^+$  and it vanishes in a nontrivial open set (just take a ball  $B$  outside  $\Omega$  and consider  $B \times (1, 2)$ ).

As a consequence of this and of the Unique Continuation Principle (see [18]),  $E_1$  has to vanish identically in  $\Omega \times \mathbb{R}^+$  and so, by taking  $t \rightarrow 0^+$ , we would have that  $e_1(x) = 0$  for any  $x \in \Omega$  (here we use also the fact that  $g_{\lambda_1}(0) = 1$  by (4.3)). This is a contradiction and it establishes that  $\lambda_{1,s} < \lambda_1^s = \lambda_{1,A_s}$ .  $\square$

Our main result, i.e. Theorem 1, is now a direct consequences of Propositions 2 and 5.

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