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On the Stability and Numerical Scheme of Fractional Differential Equations with Application to Biology

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Abstract: The fractional differential equations involving different types of fractional derivatives are currently used in many fields of science and engineering. Therefore, the first purpose of this study is to investigate the qualitative properties including the stability, asymptotic stability, as well as Mittag–Leffler stability of solutions of fractional differential equations with the new generalized Hattaf fractional derivative, which encompasses the popular forms of fractional derivatives with non-singular kernels. These qualitative properties are obtained by constructing a suitable Lyapunov function. Furthermore, the second aim is to develop a new numerical method in order to approximate the solutions of such types of equations. The developed method recovers the classical Euler numerical scheme for ordinary differential equations. Finally, the obtained analytical and numerical results are applied to a biological nonlinear system arising from epidemiology.

Keywords: stability; Hattaf fractional derivative; fractional differential equations; Lyapunov direct method; numerical method

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1. Introduction

Fractional differential equations (FDEs) are the differential equations with non-integer powers of the differentiation order. In recent years, FDEs have gained importance in both theoretical and applied aspects of several fields of science and engineering such as biology [1,2], epidemiology [3–5], control theory [6], viscoelasticity [7], engineering [8], and bioengineering [9]. For other recent developments in the literature on theoretical and numerical studies of FDEs, see for example [10–12] for the inverse problem associated with FDEs, ref. [13] for Hermite–Hadamard-type inequalities, [14–16] for numerical methods of FDEs, and [17–19] for analytical and numerical solutions of some FDEs.

Currently, stability analysis of FDEs has been investigated by many authors. Li et al. [20] dealt with the stability of nonlinear dynamic systems describing by the Caputo fractional derivative with a singular kernel [21]. Delavari et al. [22] analyzed the stability of fractional-order nonlinear systems. They presented an extension of the Lyapunov direct method for Caputo-type fractional-order systems using Bihari's and Bellman–Gronwall's inequalities [23,24]. The stability of FDEs with Hadamard fractional derivative [25] was investigated by Wang et al. [26] utilizing a new fractional comparison principle. In [27], the authors focused on the Ulam stability of a generalized delayed differential equation of fractional order. A more recent study presented in [28] discussed the Mittag–Leffler stability of FDEs using the new generalized Hattaf fractional (GHF) derivative [29], which includes many fractional derivatives available in the literature such as the Caputo–Fabrizio fractional derivative [30], the Atangana–Baleanu fractional derivative [31], and the weighted Atangana–Baleanu fractional derivative [32]. The stability in the sense of Ulam–Hyers of FDEs with GHF derivative was studied in [33] using a new version of the Gronwall inequality. However, this paper extends the fractional comparison principle to the GHF derivative and the results related to Mittag–Leffler stability given in [28] using class $\mathcal{K}$ functions. Furthermore, the present paper establishes new interesting results concerning the stability, as well as the asymptotic stability of FDEs with the GHF derivative by means of the Lyapunov direct method and class $\mathcal{K}$ functions.

On the other hand, numerical methods have become indispensable tools to find the approximate solutions of both ordinary differential equations (ODEs) and FDEs. Oftentimes, it is impossible or complicated to find the exact solution for many nonlinear systems modeling real phenomena. Therefore, the second main objective of this research is to develop a new numerical method for solving FDEs with the GHF derivative.

The rest of the paper is outlined as follows. Section 2 describes the basic concepts and extends the fractional comparison principle to the GHF derivative. Section 3 analyzes the stability, the asymptotic stability, and the Mittag–Leffler stability of FDEs with the GHF derivative. Section 4 proposes a new numerical method to solve nonlinear FDEs. Section 5 presents an application of our analytical and numerical results to a biological system. Finally, the conclusion is given in Section 6.

2. Preliminaries

In this section, we present the necessary concepts and results related to the GHF derivative that are used throughout this paper.

Definition 1 ([29])**.** *Let $\alpha \in [0,1)$, $\beta,\gamma > 0$, and $f \in H^1(a,b)$. The GHF derivative of order α in the Caputo sense of the function $f(t)$ with respect to the weight function $w(t)$ is defined as follows:*

$$ {}^{C}D^{\alpha,\beta,\gamma}_{a,t,w} f(t) = \frac{N(\alpha)}{1-\alpha}\frac{1}{w(t)}\int_a^t E_\beta[-\mu_\alpha(t-\tau)^\gamma]\frac{d}{d\tau}(wf)(\tau)d\tau, \tag{1} $$

where $w \in C^1(a,b)$, $w, w' > 0$ on [a,b], $N(\alpha)$ is a normalization function obeying $N(0) = N(1) = 1$, $\mu_\alpha = \dfrac{\alpha}{1-\alpha}$, and $E_\beta(t) = \displaystyle\sum_{k=0}^{+\infty}\frac{t^k}{\Gamma(\beta k+1)}$ is the Mittag–Leffler function of parameter β.

Some examples of normalization functions are as follows:

- $N(\alpha) = 1$,
- $N(\alpha) = 1 - \alpha + \dfrac{\alpha}{\Gamma(\alpha)}$.

The main properties of the GHF derivative defined by (1) are given in detail in [29,34]. Furthermore, (1) is reduced to the Caputo–Fabrizio fractional derivative [30] when $w(t) = 1$ and $\beta = \gamma = 1$, to the Atangana–Baleanu fractional derivative [31] when $w(t) = 1$ and $\beta = \gamma = \alpha$, as well as to the weighted Atangana–Baleanu fractional derivative [32] when $\beta = \gamma = \alpha$.

Now, denote ${}^{C}D^{\alpha,\beta,\beta}_{a,t,w}$ by $\mathcal{D}^{\alpha,\beta}_{a,w}$. By [29], the generalized fractional integral associated with $\mathcal{D}^{\alpha,\beta}_{a,w}$ is given by the following definition.

Definition 2 ([29])**.** *The generalized fractional integral operator associated with $\mathcal{D}^{\alpha,\beta}_{a,w}$ is defined by*

$$ \mathcal{I}^{\alpha,\beta}_{a,w} f(t) = \frac{1-\alpha}{N(\alpha)} f(t) + \frac{\alpha}{N(\alpha)}\, {}^{RL}\mathcal{I}^{\beta}_{a,w} f(t), \tag{2} $$

where ${}^{RL}\mathcal{I}^{\beta}_{a,w}$ is the standard weighted Riemann–Liouville fractional integral of order β defined by

$$ {}^{RL}\mathcal{I}^{\beta}_{a,w} f(t) = \frac{1}{\Gamma(\beta)}\frac{1}{w(t)}\int_a^t (t-\tau)^{\beta-1} w(\tau) f(\tau) dx. \tag{3} $$

Now, we recall an important theorem that we will need in the following. This theorem extends the Newton–Leibniz formula introduced in [35,36].

Theorem 1 ([34]). *Let $\alpha \in [0,1)$, $\beta > 0$ and $f \in H^1(a,b)$. Then, we have the following properties:*

$$\mathcal{I}^{\alpha,\beta}_{a,w}(\mathcal{D}^{\alpha,\beta}_{a,w}f)(t) = f(t) - \frac{w(a)f(a)}{w(t)}, \tag{4}$$

and

$$\mathcal{D}^{\alpha,\beta}_{a,w}(\mathcal{I}^{\alpha,\beta}_{a,w}f)(t) = f(t) - \frac{w(a)f(a)}{w(t)}. \tag{5}$$

In addition, we will need the following definition and lemma.

Definition 3. *A continuous function $\psi : [0,+\infty) \to [0,+\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\psi(0) = 0$.*

Lemma 1. *(Fractional comparison principle) Let $x(t)$ and $y(t)$ two functions defined on $[t_0,+\infty)$ with $\mathcal{D}^{\alpha,\beta}_{t_0,w}x(t) \geq \mathcal{D}^{\alpha,\beta}_{t_0,w}y(t)$ and $x(t_0) \geq y(t_0)$. Then, $x(t) \geq y(t)$, for all $t \geq t_0$.*

Proof. We have $\mathcal{D}^{\alpha,\beta}_{t_0,w}x(t) \geq \mathcal{D}^{\alpha,\beta}_{t_0,w}y(t)$. By applying the fractional Hattaf integral to both sides of this inequality and using (4), we obtain

$$x(t) - \frac{x(t_0)w(t_0)}{w(t)} \geq y(t) - \frac{y(t_0)w(t_0)}{w(t)},$$

which leads to

$$x(t) \geq y(t) + \frac{w(t_0)\big(x(t_0) - y(t_0)\big)}{w(t)}.$$

Since $x(t_0) \geq y(t_0)$, we deduce that

$$x(t) \geq y(t), \text{ for all } t \geq t_0.$$

This completes the proof □

Remark 1. *Lemma 1 extends two results, one presented in Theorem 2.4 of [22] with the Riemann–Liouville fractional derivative and the other in Lemma 6.1 of [20] with the Caputo fractional derivative.*

3. Stability of FDEs with the GHF Derivative

In this section, we study the stability of the following nonautonomous FDE with the GHF derivative expressed by

$$\mathcal{D}^{\alpha,\beta}_{0,w}x(t) = f(t, x(t)), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the pseudo-state variable, $f : [0,+\infty) \times \Omega \to \mathbb{R}^n$ is a continuous locally Lipschitz function satisfying in particular $f(t,0) = 0$, and Ω is a domain of $\mathbb{R}^n$ that contains the origin $x = 0$. When $f(t,x) = f(x)$, (6) becomes an autonomous FDE of the form

$$\mathcal{D}^{\alpha,\beta}_{0,w}x(t) = f(x(t)). \tag{7}$$

Since $\alpha \in [0,1)$, we chose $x(0) = x_0$ as the initial condition for (6) and (7).

First, we give some definitions of stabilities that will be used in the remainder of this paper. We begin with the definition of stability and asymptotic stability.

Definition 4. *Let $x = 0$ be an equilibrium point for the system (6):*

(i) *The equilibrium point $x = 0$ is said to be stable if, for any $\epsilon > 0$, there exists a $\eta > 0$ such that for each initial condition $x(t_0) = x_0$ satisfying $\|x_0\| < \eta$, the solution $x(t)$ of (6) satisfies $\|x(t)\| < \epsilon$ for all $t \geq t_0$. Otherwise, we say that $x = 0$ is unstable.*

(ii) *The equilibrium point $x = 0$ is said to be asymptotically stable if it is stable and $\lim\limits_{t\to+\infty} x(t) = 0$.*

For the Mittag–Leffler stability, we have the following definition.

Definition 5. *The trivial solution of (6) is said to be Mittag–Leffler stable if*

$$\|x(t)\| \leq [m(x(t_0))E_\beta(-\lambda(t-t_0)^\beta)]^\nu,$$

where t_0 is the initial time, $\lambda \geq 0$, $\nu > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{R}^n$ with the Lipschitz constant m_0.

It is very important to note that the Mittag–Leffler stability implies asymptotic stability.

Theorem 2. *Let $x = 0$ be an equilibrium point for the system (6). If there exist a continuously differentiable function $V(t,x) : [0,+\infty) \times \Omega \to \mathbb{R}$ and a class $\mathcal{K}$ function ψ satisfying the following conditions:*

$$V(t,x) \geq \psi(\|x\|), \quad (8)$$
$$\mathcal{D}^{\alpha,\beta}_{0,w}V(t,x) \leq 0, \quad (9)$$

then $x = 0$ is locally stable. If (8) and (9) hold globally on $\mathbb{R}^n$, then $x = 0$ is globally stable.

Proof. According to (9) and Theorem 1, we have

$$V(t,x(t)) \leq \frac{V(0,x(0))w(0)}{w(t)}.$$

Since $w(0) \leq w(t)$ for all $t \geq 0$, we obtain:

$$V(t,x(t)) \leq V(0,x(0)), \forall t \geq 0.$$

It follows from (8) that

$$\|x(t)\| \leq \psi^{-1}(V(0,x(0))), \forall t \geq 0. \quad (10)$$

Then, the equilibrium $x = 0$ is stable. □

Remark 2. *Theorem 2 extends the asymptotical stability result with the Caputo derivative given in Theorem 3.2 of [37] to the GHF derivative.*

Theorem 3. *Let $x = 0$ be an equilibrium point for the system (6). If there exist a continuously differentiable function $V(t,x) : [0,+\infty) \times \Omega \to \mathbb{R}$ and class $\mathcal{K}$ functions ψ_i (i = 1, 2, 3) satisfying*

$$\psi_1(\|x\|) \leq V(t,x) \leq \psi_2(\|x\|), \quad (11)$$
$$\mathcal{D}^{\alpha,\beta}_{0,w}V(t,x) \leq -\psi_3(\|x\|), \quad (12)$$

then $x = 0$ is asymptotically stable.

Proof. By (11) and (12), we have

$$\mathcal{D}^{\alpha,\beta}_{0,w}V(t,x) \leq -\psi_3 \circ \psi_2^{-1}(V(t,x)).$$

It follows from the fractional comparison principle presented in Lemma 1 that $V(t,x)$ is bounded by the nonnegative solution of the following FDE:

$$\begin{cases} \mathcal{D}^{\alpha,\beta}_{0,w}y(t) = -\psi_3 \circ \psi_2^{-1}(y(t)), \\ y(0) = V(0,x(0)). \end{cases} \quad (13)$$

We have $\mathcal{D}^{\alpha,\beta}_{0,w}y(t) \leq 0$. Then,

$$y(t) \leq y(0), \forall t \geq 0. \tag{14}$$

If $y(0) = 0$, then $y(t) = 0$ for all $t \geq 0$.

If $y(0) \neq 0$, then $\lim_{t\to+\infty} y(t) = 0$. Assume the contrary, then there exists a $\epsilon > 0$ such that $y(t) \geq \epsilon$ for all $t \geq 0$. We have

$$\begin{aligned} \mathcal{D}^{\alpha,\beta}_{0,w}y(t) &= -\psi_3 \circ \psi_2^{-1}(y(t)) \\ &\leq -\psi_3 \circ \psi_2^{-1}(\epsilon) \\ &= -\frac{\psi_3 \circ \psi_2^{-1}(\epsilon)}{y(0)} y(0). \end{aligned}$$

Then

$$\mathcal{D}^{\alpha,\beta}_{0,w}y(t) \leq -\lambda y(t),$$

where $\lambda = \frac{\psi_3 \circ \psi_2^{-1}(\epsilon)}{y(0)} > 0$. By applying Corollary 1 of [28], we obtain

$$y(t) \leq y(0) E_\beta\big(-\frac{\alpha \lambda t^\beta}{N(\alpha) + (1-\alpha)}\big), \forall t \geq 0.$$

This is a contradiction. Therefore, it follows from the fractional comparison principle that

$$V(t, x(t)) \leq y(t).$$

By (11), we obtain

$$\|x(t)\| \leq \psi_1^{-1}(y(t)),$$

which implies that

$$\lim_{t\to+\infty} \|x(t)\| = 0. \tag{15}$$

Then, $x = 0$ is asymptotically stable. □

Remark 3. *Theorem 3 extends the result of the asymptotic stability with the Caputo fractional derivative introduced in Theorem 6.2 of [20] to the GHF derivative with a nonsingular kernel.*

Theorem 4. *Let $x = 0$ be an equilibrium point for the system (6). If there exist a continuously differentiable function Let $V(t, x) : [0, +\infty) \times \Omega \to \mathbb{R}$ such that is locally Lipschitz with respect to x and a class $\mathcal{K}$ function ψ satisfying:*

$$k_1\|x\|^p \leq V(t, x) \leq k_2\psi(\|x\|), \tag{16}$$

$$\mathcal{D}^{\alpha,\beta}_{0,w}V(t, x) \leq -k_3\psi(\|x\|), \tag{17}$$

where k_1, k_2, k_3, and p are arbitrary positive constants, then $x = 0$ is Mittag–Leffler stable. If (16) and (17) hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag–Leffler stable.

Proof. It follows from (16) and (17) that

$$\mathcal{D}^{\alpha,\beta}_{0,w}V(t, x) \leq -\frac{k_3}{k_2} V(t, x).$$

According to Corollary 1 of [28], we obtain

$$V(t, x) \leq V(0, x(0)) E_\beta\big(-\eta t^\beta\big),$$

where $\eta = \frac{k_3 \alpha}{k_2 N(\alpha) + k_3(1-\alpha)}$. Using (16), we obtain

$$k_1 \|x(t)\|^p \leq V(0, x(0)) E_\beta(-\eta t^\beta),$$

which leads to

$$\|x(t)\| \leq \left[m(x(0)) E_\beta(-\eta t^\beta) \right]^{\frac{1}{p}}, \tag{18}$$

where $m(x) = \frac{V(0,x)}{k_1}$. Therefore, the equilibrium $x = 0$ is Mittag–Leffler stable. □

Remark 4. *Theorem 4 generalizes the result concerning the Mittag–Leffler stability presented in Theorem 3 of [28]. It suffices to take $\psi(z) = z^q$, where $z \in [0, +\infty)$ and q is an arbitrary positive constant.*

Theorem 5. *Let $x = 0$ be an equilibrium point for the autonomous system (7) and $V(x)$ be a continuously differentiable function in a neighborhood $U \subset \mathbb{R}^n$ of the origin satisfying the following conditions:*

(i) $V(0) = 0$ *and* $V(x) > 0$ *for all* $x \in U \backslash \{0\}$;

(ii) $\mathcal{D}^{\alpha,\beta}_{0,w} V(x) \leq 0$ *for all* $x \in U \backslash \{0\}$.

Then, $x = 0$ is stable.

Proof. Let $\epsilon > 0$ such that $\overline{B(0,\epsilon)} \subset U$, where $\overline{B(0,\epsilon)}$ denotes the closed ball with center 0 and radius ϵ be defined by $\overline{B(0,\epsilon)} = \{x \in \mathbb{R}^n : \|x\| \leq \epsilon\}$. Furthermore, we define the open ball with center 0 and radius ϵ by $B(0,\epsilon) = \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$.
Since V is continuous on the compact subset $S(0,\epsilon) = \{x \in \mathbb{R}^n : \|x\| = \epsilon\}$, we deduce there exists a $\hat{x} \in S(0,\epsilon)$ such that

$$V(\hat{x}) = \inf_{x \in S(0,\epsilon)} V(x) = \varrho. \tag{19}$$

According to (i), we have $\varrho > 0$. Consider the following subset of U:

$$U_1 = \{x \in B(0,\epsilon) : V(x) < \varrho\}. \tag{20}$$

Then, U_1 is a neighborhood of the origin because it is an open ball containing 0.

Let $\phi(t, x_0)$ be a solution of (7) with initial condition $\phi(0, x_0) = x_0 \in U_1$. According to (ii), we have $\mathcal{D}^{\alpha,\beta}_{0,w} V\big(\phi(t, x_0)\big) \leq 0$. Then,

$$V\big(\phi(t, x_0)\big) \leq V\big(\phi(0, x_0)\big) = V(x_0), \text{ for all } t \geq 0.$$

Hence,

$$\phi(t, x_0) \notin S(0,\epsilon), \text{ for all } t \geq 0. \tag{21}$$

Indeed, suppose the contrary. Therefore, there exists a $t_1 \geq 0$ such that $\phi(t_1, x_0) \in S(0,\epsilon)$. This implies that $V\big(\phi(t_1, x_0)\big) \geq \varrho$. Then, $V(x_0) \geq \varrho$, which is contradicted with $x_0 \in U_1$.

Now, we prove that

$$\phi(t, x_0) \in \overline{B(0,\epsilon)}, \text{ for all } t \geq 0. \tag{22}$$

In fact, assume the contrary. Then, there exists a $t_2 \geq 0$ such that $\phi(t_2, x_0) \notin \overline{B(0,\epsilon)}$, which implies that $\|\phi(t_2, x_0)\| > \epsilon$. Let $\Lambda = \{t \geq 0 : \|\phi(t, x_0)\| > \epsilon\}$. Hence, there exists a sequence (t_n) in $\{t \geq 0 : \|\phi(t, x_0)\| > \epsilon\}$ such that $\lim_{n \to +\infty} t_n = \Lambda$. Thus, $\lim_{n \to +\infty} \|\phi(t_n, x_0)\| \geq \epsilon$. Therefore,

$$\|\phi(\Lambda, x_0)\| \geq \epsilon. \tag{23}$$

For $t < \Lambda$, we have $\|\phi(t, x_0)\| \leq \epsilon$. Then, $\lim_{t \to \Lambda^-} \|\phi(t, x_0)\| \leq \epsilon$, which implies that

$$\|\phi(\Lambda, x_0)\| \leq \epsilon. \tag{24}$$

From (23) and (24), we obtain $\|\phi(\Lambda, x_0)\| = \epsilon$. This contradicts (21). We conclude that if any initial condition of System (7) satisfies $x_0 \in U_1$, then the solution of (7) satisfies $\phi(t, x_0) \in \overline{B(0, \epsilon)}$, for all $t \geq 0$. This implies that the equilibrium $x = 0$ of System (7) is stable. □

4. Numerical Scheme

In this section, we introduce a numerical method to approximate the solution of the FDE with the GHF derivative given in (6).

From Theorem 1, Equation (6) can be converted to the following fractional integral equation:

$$x(t) - \frac{x(0)w(0)}{w(t)} = \frac{1-\alpha}{N(\alpha)} f(t, x(t)) + \frac{\alpha}{N(\alpha)\Gamma(\beta)} \frac{1}{w(t)} \int_0^t (t-\tau)^{\beta-1} w(\tau) f(\tau, x(\tau)) d\tau. \tag{25}$$

Let $t_n = nh$, where $n \in \mathbf{N}$ and h is the time step duration. We have

$$\begin{aligned} x(t_{n+1}) &= \frac{x_0 w(0)}{w(t_n)} + \frac{1-\alpha}{N(\alpha)} f(t_n, x(t_n)) \\ &\quad + \frac{\alpha}{N(\alpha)\Gamma(\beta)w(t_n)} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\beta-1} w(\tau) f(\tau, x(\tau)) d\tau. \end{aligned} \tag{26}$$

By applying the rectangular integration to the integral in the right-hand side of (26), we obtain

$$\begin{aligned} \int_0^{t_{n+1}} (t_{n+1}-\tau)^{\beta-1} w(\tau) f(\tau, x(\tau)) d\tau &= \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} (t_{n+1}-\tau)^{\beta-1} w(\tau) f(\tau, x(\tau)) d\tau \\ &\simeq \sum_{k=0}^{n} w(t_k) f(t_k, x(t_k)) \int_{t_k}^{t_{k+1}} (t_{n+1}-\tau)^{\beta-1} d\tau \\ &= \frac{h^\beta}{\beta} \sum_{k=0}^{n} w(t_k) f(t_k, x(t_k)) \mathcal{A}_{n,k}^{\beta}, \end{aligned}$$

where

$$\mathcal{A}_{n,k}^{\beta} = (n-k+1)^\beta - (n-k)^\beta. \tag{27}$$

Therefore, we obtain the following numerical scheme:

$$\begin{aligned} x_{n+1} &= \frac{x_0 w(0)}{w(t_n)} + \frac{1-\alpha}{N(\alpha)} f(t_n, x_n) \\ &\quad + \frac{\alpha h^\beta}{N(\alpha)\Gamma(\beta+1)w(t_n)} \sum_{k=0}^{n} w(t_k) f(t_k, x_k) \mathcal{A}_{n,k}^{\beta}. \end{aligned} \tag{28}$$

Remark 5. *If $\alpha = \beta = 1$ and $w(t) = 1$, then (28) becomes*

$$x_{n+1} = x_n + h f(t_n, x_n). \tag{29}$$

Hence, the classical Euler numerical scheme for ODEs is recovered. Indeed,

$$\begin{aligned} x_{n+1} &= x_0 + \frac{h}{N(1)\Gamma(2)} \sum_{k=0}^{n} f(t_k, x_k) \mathcal{A}_{n,k}^{1} \\ &= x_0 + h \sum_{k=0}^{n} f(t_k, x_k) \\ &= x_0 + h \sum_{k=0}^{n-1} f(t_k, x_k) + hf(t_n, x_n) \\ &= x_n + hf(t_n, x_n). \end{aligned}$$

5. Application to Biology

In this section, we apply our main analytical and numerical results to the biological system describing the dynamics of an epidemic disease:

$$\begin{cases} \mathcal{D}_{0,1}^{\alpha,\beta} S(t) &= A - \mu S(t) - \kappa S(t) I(t), \\ \mathcal{D}_{0,1}^{\alpha,\beta} E(t) &= \kappa S(t) I(t) - (\mu + \varepsilon) E(t), \\ \mathcal{D}_{0,1}^{\alpha,\beta} I(t) &= \varepsilon E(t) - (\mu + r) I(t), \\ \mathcal{D}_{0,1}^{\alpha,\beta} R(t) &= r I(t) - \mu R(t), \end{cases} \tag{30}$$

where $S(t)$, $I(t)$, and $R(t)$ are the fractions of susceptible, exposed, infectious, and recovered individuals at time t, respectively. The biological meanings of the parameters are presented in Table 1.

Table 1. Biological meanings of the parameters of model (30).

Parameter	Biological Meaning
A	Natality or recruitment rate
μ	Natural death rate
κ	Transmission rate of disease
ε	Transfer rate from class E to class I
r	Recovery rate of the infectious individuals

Since the first three equations of (30) do not depend on the last one, System (30) can be reduced to the following model:

$$\begin{cases} \mathcal{D}_{0,1}^{\alpha,\beta} S(t) &= A - \mu S(t) - \kappa S(t) I(t), \\ \mathcal{D}_{0,1}^{\alpha,\beta} E(t) &= \kappa S(t) I(t) - (\mu + \varepsilon) E(t), \\ \mathcal{D}_{0,1}^{\alpha,\beta} I(t) &= \varepsilon E(t) - (\mu + r) I(t). \end{cases} \tag{31}$$

In fact, when the variable $I(t)$ is determined by (31), then we easily obtain $R(t)$ from the last equation of (30).

Clearly, Model (31) has a unique disease-free equilibrium $Q^0 = (S^0, 0, 0)$, where $S^0 = \dfrac{A}{\mu}$. Furthermore, the basic reproduction number of (31) is given by

$$\mathcal{R}_0 = \frac{\varepsilon \kappa S^0}{(\mu + \varepsilon)(\mu + r)}. \tag{32}$$

By a simple computation, Model (31) has a unique endemic $Q^* = (S^*, E^*, I^*)$, where $S^* = \dfrac{A}{\mu \mathcal{R}_0}$, $E^* = \dfrac{\mu(\mu + r)(\mathcal{R}_0 - 1)}{\varepsilon \kappa}$ and $I^* = \dfrac{\mu(\mathcal{R}_0 - 1)}{\kappa}$.

Let $\Omega = \{(S,E,I) \in \mathbb{R}_+^3 : S \leq S^0\}$. For $\mathcal{R}_0 < 1$, construct a Lyapunov function as follows:

$$V(S,E,I) = \rho(S^0 - S) + E + \frac{\mu+\varepsilon}{2\varepsilon} I,$$

where $\rho = \frac{2-\mathcal{R}_0}{2\mathcal{R}_0}$. We have

$$\begin{aligned} \mathcal{D}_{0,1}^{\alpha,\beta} V(S,E,I) &= -\rho \mathcal{D}_{0,1}^{\alpha,\beta} S + \mathcal{D}_{0,1}^{\alpha,\beta} E + \frac{\mu+\varepsilon}{2\varepsilon} \mathcal{D}_{0,1}^{\alpha,\beta} I \\ &\leq -\rho\mu(S^0 - S) - \frac{\mu+\varepsilon}{2} E - \frac{(\mu+\varepsilon)(\mu+r)}{2\varepsilon}(1-\mathcal{R}_0) I. \end{aligned}$$

Thus,

$$\mathcal{D}_{0,1}^{\alpha,\beta} V(S,E,I) \leq -\sigma_1 V(S,E,I), \tag{33}$$

where $\sigma_1 = \min\{\mu, \frac{\mu+\varepsilon}{2}, (\mu+r)(1-\mathcal{R}_0)\}$.

Let $X = (S,E,I) \in \mathbb{R}^3$ with the norm $\|X\| = |S| + |E| + |I|$. Hence,

$$\sigma_2 \|X - Q^0\| \leq V(S,E,I) \leq \sigma_3 \|X - Q^0\|, \tag{34}$$

where $\sigma_2 = \min\{\rho, 1, \frac{\mu+\varepsilon}{2\varepsilon}\}$ and $\sigma_3 = \max\{\rho, 1, \frac{\mu+\varepsilon}{2\varepsilon}\}$. By applying Theorem 4, we deduce that the disease-free equilibrium Q^0 of (31) is Mittag–Leffler stable in Ω when $\mathcal{R}_0 < 1$.

For $\mathcal{R}_0 > 1$, consider the following Lyapunov function:

$$L(S,E,I) = S^*\Phi(\frac{S}{S^*}) + E^*\Phi(\frac{E}{E^*}) + \frac{\mu+\varepsilon}{\varepsilon} I^*\Phi(\frac{I}{I^*}),$$

where $\Phi(z) = z - 1 - \ln z$, for $z > 0$. It is obvious that $\Phi(z)$ attains its global minimum at $z = 1$ and $\Phi(1) = 0$. Then, $\Phi(z) \geq 0$ for all $z > 0$. Hence, $L(S,E,I) \geq 0$ with $L(S^*,E^*,I^*) = 0$.

By applying Corollary 2 of [34], we obtain

$$\mathcal{D}_{0,1}^{\alpha,\beta} L(S,E,I) \leq (1 - \frac{S^*}{S}) \mathcal{D}_{0,1}^{\alpha,\beta} S + (1 - \frac{E^*}{E}) \mathcal{D}_{0,1}^{\alpha,\beta} E + \frac{\mu+\varepsilon}{\varepsilon}(1 - \frac{I^*}{I}) \mathcal{D}_{0,1}^{\alpha,\beta} I.$$

Using $A = \mu S^* + (\mu+\varepsilon)E^*$, $\kappa S^* I^* = (\mu+\varepsilon)E^*$ and $\frac{\mu+r}{\varepsilon} = \frac{E^*}{I^*}$, we obtain

$$\begin{aligned} \mathcal{D}_{0,1}^{\alpha,\beta} L(S,E,I) &\leq -\frac{\mu}{S}(S - S^*)^2 + (\mu+\varepsilon)E^*\left(3 - \frac{S^*}{S} - \frac{EI^*}{E^*I} - \frac{SE^*I}{S^*EI^*}\right) \\ &= -\frac{\mu}{S}(S - S^*)^2 - (\mu+\varepsilon)E^*\left(\Phi(\frac{S^*}{S}) + \Phi(\frac{EI^*}{E^*I}) + \Phi(\frac{SE^*I}{S^*EI^*})\right). \end{aligned}$$

Hence, $\mathcal{D}_{0,1}^{\alpha,\beta} L(S,E,I) \leq 0$ when $\mathcal{R}_0 > 1$. It follows from Theorem 5 that the endemic equilibrium Q^* of (31) is stable when $\mathcal{R}_0 > 1$.

In the absence of disease, System (31) reduces to the following linear system:

$$\mathcal{D}_{0,1}^{\alpha,\beta} S(t) = A - \mu S(t). \tag{35}$$

From Lemma 2 of [28], the exact solution of (35) is given by

$$S(t) = \frac{A}{\mu} + \frac{N(\alpha)}{a_\alpha}(S(0) - \frac{A}{\mu}) E_\beta(-\frac{\alpha\mu}{a_\alpha} t^\beta), \tag{36}$$

$a_\alpha = N(\alpha) + \mu(1-\alpha)$. Now, we apply the numerical scheme presented in (28) in order to approximate the solution of (35). For all numerical simulations, we chose $A = 0.01$, $\mu = 0.01$, and the normalization function as follows:

$$N(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}. \tag{37}$$

The comparison between the exact and numerical (approximate) solutions of (35) is displayed in Figure 1 for different values of α, β, and h.

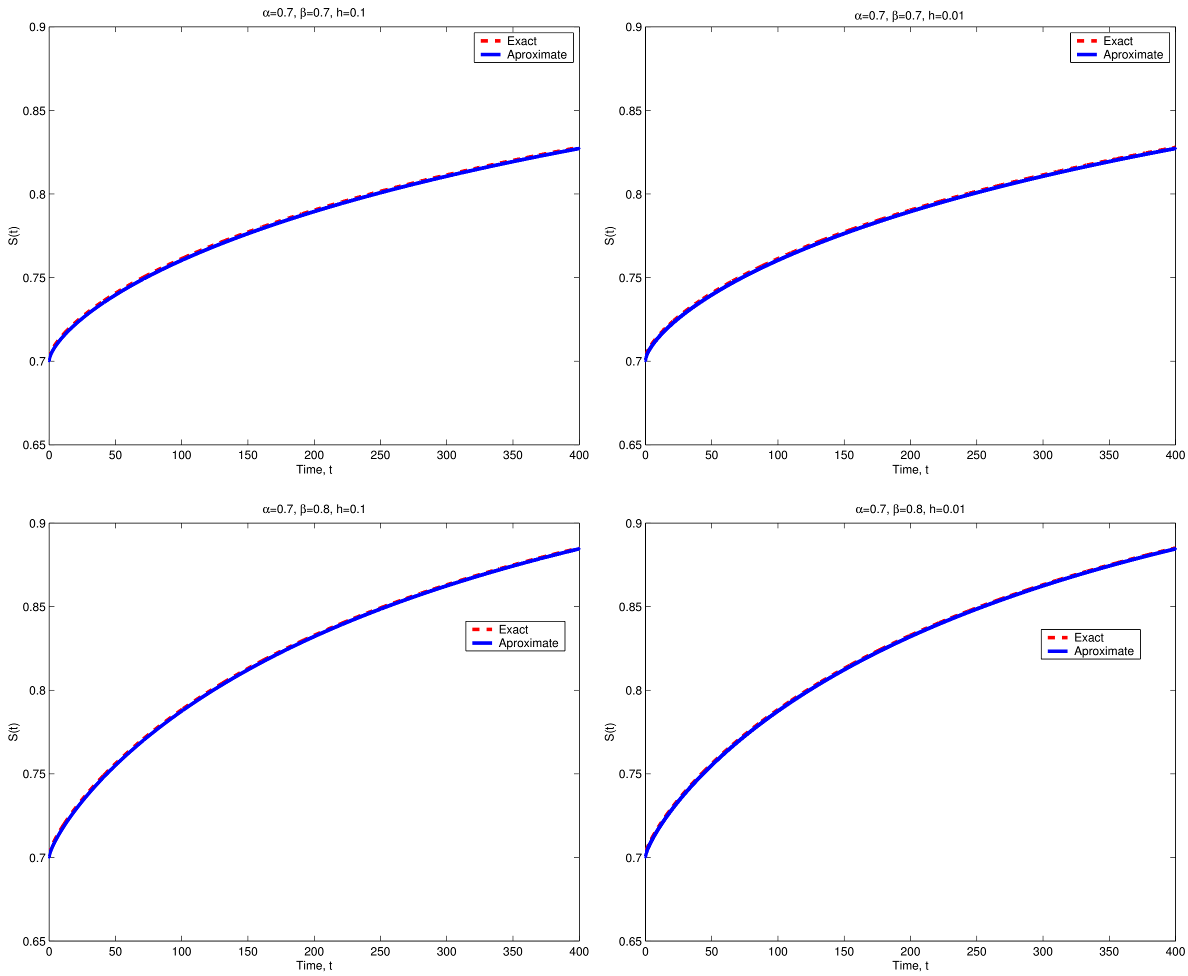

Figure 1. The exact and numerical solutions of (35) for different values of α, β, and h.

In the presence of disease, System (31) cannot be solved analytically. Based on our numerical method, we approximate the solution of (31). Therefore, we chose $\kappa = 0.4$, $\varepsilon = 0.085$, $r = 0.095$, and $\beta = 0.8$. By computation, we have $\mathcal{R}_0 = 3.4085 > 1$. In this case, the solution of (31) converges to the endemic equilibrium $Q^*(0.2934, 0.0744, 0.0602)$, which biologically means that the disease persists in the population. Figure 2 illustrates this observation for different values of α.

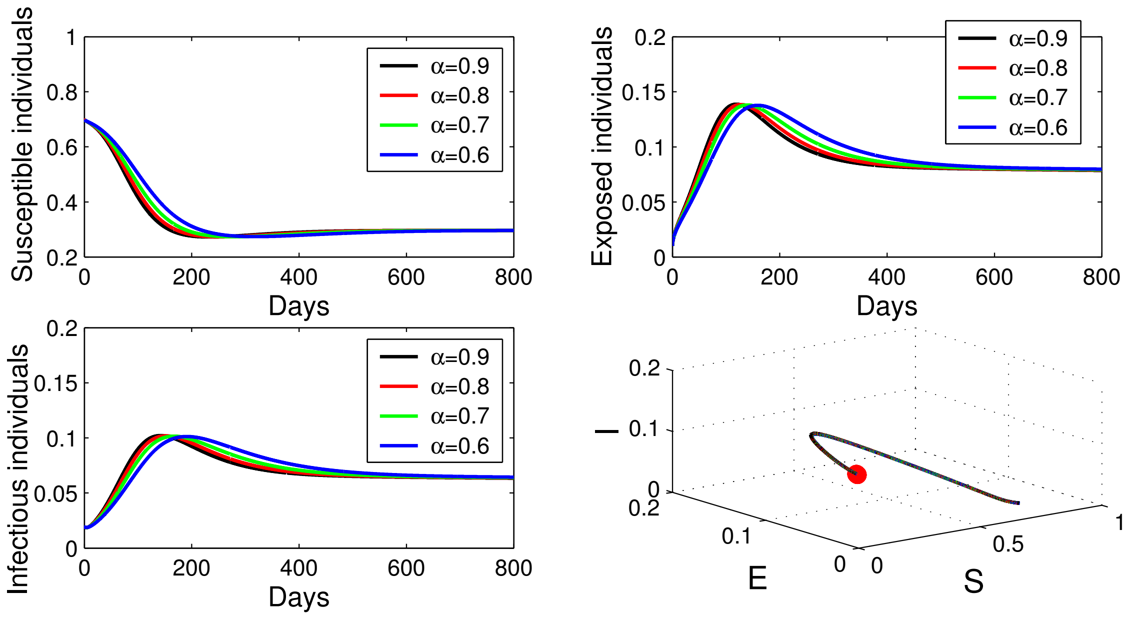

Figure 2. The numerical solution of (31) for different values of α.

6. Conclusions

In this paper, we first investigated the qualitative properties of solutions of FDEs with the new generalized Hattaf fractional derivative, which includes several forms of fractional derivatives with non-singular kernels such as the Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. In addition, we proposed a new numerical method to approximate the solutions of such types of FDEs. The obtained results extend and improve many results existing in the literature concerning the fractional comparison principle, stability, asymptotic stability, as well as Mittag–Leffler stability. Furthermore, the proposed numerical method includes the classical Euler numerical scheme, and it was applied to a nonlinear system describing the dynamics of an epidemic disease, such as COVID-19.

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