ON THE STABILITY OF A CLASS OF NONLINEAR STOCHASTIC SYSTEMS

by

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ABSTRACT

In this thesis we consider the questions of stability of systems described by stochastic nonlinear Volterra integral equations. Particularly the following types of stability are treated in great detail :

(i) stability in the mean $m(m \geq 1)$,

(ii) asymptotic stability in the mean $m(m \geq 1)$,

(iii) almost sure L_p $(p \geq 1)$ stability and

(iv) almost sure asymptotic stability.

In section 2.2 we consider the existence and uniqueness of the solution of the system in a sufficiently general Banach space containing $L_m(\Omega, \beta, \mu)$ $(m \geq 1)$ spaces and establish the stability of the system S (the system under consideration) in the sense (i).

In section 2.4 the above results are then used to prove the asymptotic stability in the mean $m(m \geq 1)$ (ref. ii) of the system S.

For illustration, the results of section 2.2 are used to study the stability of a distributed parameter system with a random boundary condition.

In section 3.2 the question of the existence of a solution of the system S in L_p $(p \geq 1)$ spaces for almost all $\omega \in \Omega$ is considered. This gives us the almost sure $L_p(p \geq 1)$ stability (ref. iii).

In section 3.4 this result is then used to prove the almost sure asymptotic stability (ref. iv) of the system S.

For illustration of the results of sections 2.4 and 3.2 several examples, which include a system described by a Volterra integral equation of the third kind excited by the Wiener process, are presented. An example of a feedback control system containing a nonlinear amplifier with random gain is considered in detail.

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CHAPTER 1

INTRODUCTION

Since Lyapunov presented his fundamental memoir (1892, in Russian) [18], the questions of both deterministic stability and stochastic stability have aroused the interest of many investigators from different disciplines of natural and social sciences. It may be said that Lyapunov is the true founder of modern stability theory. In Lyapunov's fundamental memoir, he dealt with stability by two distinct methods. His so-called first method presupposes an explicit known solution and is only applicable to some restricted but important cases. As against this the second or direct method of Lyapunov is of greater generality and power and above all does not require the knowledge of the solutions themselves.

In recent years a great deal of interest has been given in the study of stability theory by using functional analysis as the principal tool.

Loosely speaking, stability is concerned with the behavior of a dynamical system relative to certain parameters suchas, the initial condition, the time parameter etc. . In general, the concepts of stochastic stability can be derived by a combination of the concepts of deterministic Lyapunov stability and the various modes of convergence found in measure theory. The following modes of convergence are well-known in measure theory,

(i) uniform convergence,

(ii) almost uniform convergence,

(iii) convergence almost everywhere (almost surely),

(iv) convergence in the mean $m (m \geq 1)$, and

(v) convergence in measure (in probability).

It is clear that one has at least five times as many concepts of stochastic stability as there are in the deterministic case. For illustration, a systematic account of various forms of stochastic stability are presented in the following table:

		δ	σ	ν	ξ	ρ
	(Ω, β, μ) / (R_o, A, M) Lebesgue measure space	convergence in probability	convergence in the mean m $(m \geq 1)$	Almost sure convergence	Almost uniform convergence	uniform convergence
α	(Lyapunov) stability	αδ : stability in probability	ασ : * stability in the mean m $(m \geq 1)$	αν : Almost sure stability	αξ : Almost uniform stability	αρ uniform stability
β	$L_p(R_o)$ $(p \geq 1)$ stability	βδ : $L_p(R_o)$ $(p \geq 1)$ stability in probability	βσ : $L_p(R_o)(p \geq 1)$ stability in the mean m $(m \geq 1)$ or; σβ: $L_m(\Omega, \beta, \mu)$ $(m \geq 1)$ stability in the mean p $(p \geq 1)$	* βν : Almost sure $L_p(R_o)$ $(p \geq 1)$ stability	βξ : Almost uniform L_p $(p \geq 1)$ stability	βρ uniform L_p $(p \geq 1)$ stability
γ	Asymptotic (Lyapunov) stability	γδ : Asymptotic stability in probability	γσ : * Asymptotic stability in the mean m $(m \geq 1)$	γν : * Almost sure asymptotic stability	γξ : Almost uniform asymptotic stability	γρ uniform asymptotic stability

Table 1

It is interesting to note (see table) that the concepts of stability for stochastic systems appearing in column ρ are those which are closest to those for deterministic systems.

In this thesis we have only considered the following types of stability:

(i) stability in the mean $m (m \geq 1)$ ($\alpha\sigma$, table 1; definition 2.4),

(ii) asymptotic stability in the mean $m (m \geq 1)$ ($\gamma\sigma$, table 1; definition 2.7),

(iii) almost sure $L_p(R_o)$ $(p \geq 1)$ stability ($\beta\nu$, table 1; definition 3.1),

and (iv) almost sure asymptotic stability ($\gamma\nu$, table 1; definition 3.3).

Furthermore, two new stability concepts for stochastic systems have been introduced. They are stated in definition 2.3 and definition 2.6. The corresponding results are given in theorem 2.1 and theorem 2.5. From these, we derive the results on the stability in the mean $m (m \geq 1)$ and the asymptotic stability in the mean $m (m \geq 1)$.

To study the other concepts of stability for stochastic systems, one can easily derive their corresponding definitions.

In the rest of this chapter, we shall briefly discuss some of the known results in this area and the results that we have obtained.

In 1969 , Tsokos [20] proved the existence and uniqueness of solution of a specific class of stochastic Volterra integral equations of the form,

$$W_1: \quad x(t;\omega) = h(t;\omega) + \int_o^t K(t, \tau;\omega) f(\tau, x(\tau;\omega)) d\tau \qquad t \geq 0$$

in the B-space $C(R_o, L_2(\Omega, \beta, \mu))$ (definition 2.2 with $m = 2$).

However, in many practical situations the nonlinear no-memory part of the system S (chapter 2) is a stochastic operator rather than a deterministic one. Moreover, the system S may have a unique solution in some B-spaces $C_q(R_o, L_m(\Omega, \beta, \mu))$ $m \geq 1$ (definition 2.1) but not in the B-space $C(R_o, L_2(\Omega, \beta, \mu))$ (definition 2.2 with $m = 2$). Therefore in this thesis we have considered system S containing stochastic nonlinear operator f and the stability of the corresponding system in the B-space $C_q(R_o, L_m(\Omega, \beta, \mu))$ $m \geq 1$.

Recently, Ahmed used in his papers [1] and [2] Schauder's fixed point theorem to establish the $L_p (p \geq 1)$ stability of a class of nonlinear deterministic feedback systems described by

$$W_2: \quad x(t) = u(t) - \lambda \int_o^t K(t,\tau) f(\tau, x(\tau)) d\tau$$
$$y(t) = \int_o^t K(t,\tau) f(\tau, x(\tau)) d\tau$$

In [2], he also presented certain sufficient conditions for the asymptotic stability of a class of systems described by

$$W_3: \quad x(t) = u(t) - \lambda \int_o^t K(t, t-\tau) f(\tau, x(\tau)) d\tau$$
$$y(t) = \int_o^t K(t, t-\tau) f(\tau, x(\tau)) d\tau .$$

In 1969, Ahmed [3] used Schauder's fixed point theorem to establish the existence of a random solution in $L_p (p \geq 1)$ spaces of stochastic integral equations of the following class:

$$W_4: \quad u = v + \lambda A_\sigma u = B_\sigma u$$

where

$$(A_\sigma u)(t) = \sum_{n=1}^{\infty} \int_{I^n} \cdots \int L_n(\sigma | t; \tau_1 \ldots \tau_n) \prod_{i=1}^{n} u(\sigma, \tau_i) d\tau_1 \ldots d\tau_n$$
$$t \in I = [t_o, T]$$

and σ is an element of a probability (measure) space (Σ, S, μ).

Moreover, in [3] he also pointed out that under certain conditions a random Banach fixed point theorem can be used to prove the existence and uniqueness of random solution of the random integral equation W_4.

In chapter 3 of this thesis the author has extended part of the results of [1] and [2] to the stochastic case.

Morozan [17] used the method of Lyapunov function to ivestigate certain problems on stability of the trivial solution of a class of stochastic discrete systems described by

$$W_5: \quad x_{n+1}(\omega) = f_n(x_n(\omega), \omega) \qquad n = 1, 2, 3, \ldots .$$

Sufficient conditions for the following results can be found in [17] :

(i) almost surely attractive with respect to $\mathcal{A} \cap L_1(\Omega)$ [theorem 3, 17] ;

(ii) almost surely attractive with respect to $\mathcal{A}$ and also attractive in L_p($p \geq 1$) with respect to $\mathcal{A}$ [theorem 4, 17]

(iii) uniformly stable in $L_2(\Omega)$ with respect to $\mathcal{A} \triangleq \{x : x \in L_2(\Omega), \|x\|_2 \leq H\}$ [theorem 6, 17]

and (iv) asymptotically stable in mean square [theorem 7, 17] .

However, there exist many other interesting results on stability theory as discussed by Sandberg [19], Desoer and Wu [10] and others.

In this thesis the results of our investigation are presented in chapter 2 and chapter 3 as described in the abstract.

Note: Throughout this thesis the following notations are used:

$L_m(\Omega, \beta, \mu)$ signifies the space of β-measurable and mth power μ-integrable functions on Ω.

$L_p(R_o)$ signifies the space of Lebesgue measurable and pth power Lebesgue integrable functions on $R_o \triangleq [o, \infty)$.

CHAPTER 2

STABILITY IN THE MEAN m (m $\geq$ 1) AND ASYMPTOTIC STABILITY IN THE MEAN m(m $\geq$ 1).

2.1: INTRODUCTION

In this chapter we consider the questions of stochastic stability ((i) stability in the mean $m (m \geq 1)$ and, (ii) asymptotic stability in the mean $m (m \geq 1)$) of a class of nonlinear stochastic integral equations of Volterra type of the form:

$$S: \quad x(t;\omega) = h(t;\omega) + \lambda \int_0^t K(t,\tau;\omega)\, f(\tau, x(\tau;\omega),\omega)\, d\tau, \quad t \in R_o \triangleq [o,\infty),$$

where

(i) $\omega \in \Omega$, Ω being the supporting set of the probability space (Ω, β, μ) induced jointly by the random processes h, K and f;

(ii) $x : R_o \otimes \Omega \longrightarrow R$ is the unknown random process, where the notation $\otimes$ is used to denote the Cartesian product;

(iii) $h : R_o \otimes \Omega \longrightarrow R$ is the stochastic free term;

(iv) $K : R_o \otimes R_o \otimes \Omega \longrightarrow R$ is the stochastic kernel ;

(v) $f : R_o \otimes R \otimes \Omega \longrightarrow R$ is a random function;

(vi) λ is a real-valued constant.

It may be mentioned that a large class of stochastic ordinary differential equations as well as stochastic partial differential equations can be transformed into a stochastic integral equation of the form S. Further, a large class of stochastic feedback control systems also can be represented by an integral equation S .

2.2. EXISTENCE AND UNIQUENESS OF SOLUTION.

The purpose of this section is to show the existence and uniqueness of the solution of the system S.

It is assumed in this section that (i) the stochastic kernel $K(t,\tau;\omega)$ is a measurable function defined on Ω for all $(t,\tau) \in \Delta \triangleq \{(t,\tau): o \leq \tau \leq t < \infty\}$, and is a measurable function on the triangle Δ for almost all $\omega \in \Omega$; and (ii) $f(t,z,\omega)$ is a measurable function on $R_o \otimes \Omega$ for every fixed $z \in R \triangleq (-\infty,\infty)$.

For the proof of the existence and uniqueness of solution of the system S, we need the following preparation.

Definition 2.1:

$C_q^m \triangleq C_q(R_o, L_m(\Omega, \beta, \mu))$ denotes the Banach space of all measurable functions from R_o into $L_m(\Omega, \beta, \mu)$ $(m \geq 1)$ so that

$\sup_{t \in R_o} \| \frac{x(t;\omega)}{q(t)} \|_m < \infty$, where $q(t)$ is a positive continuous function on R_o and $\| \cdot \|_m \triangleq \{\int_\Omega |\cdot|^m d\mu(\omega)\}^{1/m}$.

The norm in the space C_q^m is defined by

$$\| x \|_{C_q^m} = \sup_{t \in R_o} \{ \frac{1}{q(t)} \cdot \| x(t;\omega) \|_m \}.$$

Remark: It can be verified that for any positive continuous function q and for any $m \geq 1$ C_q^m is a Banach space.

Lemma 2.1: Suppose there exist a positive continuous function q on R_o, a non-negative measurable function g on R_o, a non-negative β-measurable function $A_f(\omega)$ on Ω and a number $\alpha > o$ so that the function f appearing in the definition of the system S satisfies the following properties:

(i) $\sup_{t \in R_o} \{ \int_\Omega | \frac{f(t,o,\omega)}{g(t)} |^m d\mu(\omega)\}^{1/m} < \infty$ $(m \geq 1)$

(ii) for all $x, y \in R$ and for all $t \in R_o$

$$q(t) \, | f(t,x,\omega) - f(t,y,\omega) | \leq A_f(\omega) \, g(t) \, | x-y |$$

and (iii) $\mu\{\omega \in \Omega: A_f(\omega) > \alpha\} = 0$. Then for every $x \in C_q^m$,

$\sup_{t \in R_o} \{ \int_\Omega | \frac{f(t,x(t;\omega),\omega)}{g(t)} |^m d\mu(\omega)\}^{1/m} < \infty$; and for every $t \in R_o$

$$\frac{\| f(t,x(t;\omega),\omega) - f(t,y(t;\omega),\omega) \|_m}{g(t)} \leq \alpha \, \frac{\| x(t;\omega) - y(t;\omega) \|_m}{q(t)}$$

Proof: The proof follows from the well-known Minkowski's inequality.

Lemma 2.2: Let (R_o, A, M) be a Lebesgue measure space; let $x(t;\omega)$ be a measurable function of $t \in R_o$ with values in the Banach space $L_m(\Omega, \beta, \mu)(m \geq 1)$; and let $\|x(t;\omega)\|_m$ be a measurable and M-integrable real valued function on R_o. Then

$$\| \int_B x(t;\omega)dt \|_m \leq \int_B \| x(t;\omega)\|_m dt, \qquad \text{for every } B \in A.$$

Proof: (Yosida [22], pp. 133, Th. 1, Cor. 1) .

Theorem 2.1: Consider the system S and suppose there exist a positive continuous function q on R_o and a non-negative measurable function g on R_o and a number $H > 0$ so that

$$A_h \quad : \quad h \in C_q^m;$$

$$A_K \quad : \quad \int_o^t \| K(t,\tau;\omega)\|_\infty g(\tau)d\tau \leq H\, q(t) \text{ for every } (t,\tau) \in \Delta \triangleq$$

$$\triangleq \{(t,\tau): o \leq \tau \leq t < \infty\};$$

and that f satisfies the hypothesis (i), (ii) and (iii) of lemma 2.1 and suppose that $|\lambda|\, \alpha\, H < 1$. Then the system S has a unique solution $x \in C_q^m$.

Proof: Let us define an operator U on the B-space C_q^m by

$$(Ux)(t;\omega) = h(t;\omega) + \lambda \int_o^t K(t,\tau;\omega)\, f(\tau, x(\tau;\omega),\omega)\, d\tau \quad \text{for } t \in R_o, \qquad (2.1)$$

$\omega \in \Omega$ and $h \in C_q^m$.

We show that the operator $U : C_q^m \longrightarrow C_q^m$ and that it is a contraction.

Using Minkowski's inequality it follows from Lemma 2.2 that for every finite t

$$\| Ux(t;\omega)\|_m \leq \|h(t;\omega)\|_m + |\lambda|\, \gamma \int_o^t \| K(t,\tau;\omega)\|_\infty g(\tau)\, d\tau \qquad (2.2)$$

where $\gamma \triangleq \sup_{t \in R_o} \{ \int_\Omega | \frac{f(t,x(t;\omega),\omega)}{g(t)} |^m \, d\mu(\omega)\}^{1/m}$

Since f satisfies the properties (i), (ii) and (iii) of lemma 2.1, it follows from that lemma that $\gamma < \infty$. Thus by use of the assumption A_K we have for $t \in R_o$

$$\| U x(t;\omega) \|_m \le \|h(t;\omega)\|_m + |\lambda| \gamma H q(t). \qquad (2.3)$$

Therefore

$$\sup_{t\in R_o} \left\| \frac{Ux(t;\omega)}{q(t)} \right\|_m \le \sup_{t\in R_o} \left\| \frac{h(t;\omega)}{q(t)} \right\|_m + |\lambda| \gamma H, \qquad (2.4)$$

where by hypothesis A_h, the first term on the right hand side of the inequality (2.4) is finite and $|\lambda| \gamma H$ is also finite. Thus $U x \in C_q^m$ for every $x \in C_q^m$. Therefore $U: C_q^m \longrightarrow C_q^m$.

For the proof of the contraction property, let us consider $x, y \in C_q^m$. For every $t < \infty$ it follows from lemma 2.2 that

$$\| (Ux - Uy)(t;\omega) \|_m \le |\lambda| \int_0^t \|K(t,\tau;\omega)\|_\infty \|f(\tau, x(\tau;\omega),\omega) - f(\tau, y(\tau;\omega),\omega)\|_m d\tau \qquad (2.5)$$

Clearly

$$\|(Ux - Uy)(t;\omega)\|_m \le |\lambda| \int_0^t \|K(t,\tau;\omega)\|_\infty \cdot \left\| \frac{f(\tau,x(\tau;\omega),\omega) - f(\tau,y(\tau;\omega),\omega)}{g(\tau)} \right\|_m \cdot g(\tau) d\tau. \qquad (2.6)$$

It follows from lemma 2.1 that for every $x, y \in C_q^m$

$$\sup_{t\in R_o} \left\| \frac{f(t,x(t;\omega),\omega) - f(t,y(t;\omega),\omega)}{g(t)} \right\|_m \le \alpha \sup_{t\in R_o} \left\| \frac{x(t;\omega) - y(t;\omega)}{q(t)} \right\|_m. \qquad (2.7)$$

Therefore

$$\|(Ux - Uy)(t;\omega)\|_m \le |\lambda| \alpha \sup_{t\in R_o} \left\{ \left\| \frac{x(t;\omega)-y(t;\omega)}{q(t)} \right\|_m \right\} \cdot \int_0^t \|K(t,\tau;\omega)\|_\infty g(\tau) d\tau \le$$

$$\le |\lambda| \alpha H q(t) \sup_{t\in R_o} \left\{ \left\| \frac{x(t;\omega)-y(t;\omega)}{q(t)} \right\|_m \right\} \qquad (2.8)$$

and consequently

$$\sup_{t\in R_o} \left\{ \left\| \frac{(Ux-Uy)(t;\omega)}{q(t)} \right\|_m \right\} \le |\lambda| \alpha H \sup_{t\in R_o} \left\{ \left\| \frac{x(t;\omega)-y(t;\omega)}{q(t)} \right\|_m \right\}$$

i.e. $$\|Ux - Uy\|_{C_q^m} \le |\lambda| \alpha H \|x - y\|_{C_q^m}. \qquad (2.9)$$

Since by hypothesis $|\lambda| \alpha H < 1$, the inequality (2.9) shows that U is a contraction. Therefore by Banach fixed point theorem [13, pp. 190] there exists one and only one solution of the system S in C_q^m.

Before we present certain special cases of theorem 2.1 we need the following definition.

Definition 2.2 $C^m \triangleq C(R_o, L_m(\Omega, \beta, \mu))$ denotes the Banach space of all measurable functions x from R_o into $L_m(\Omega, \beta, \mu)$ $(m \geq 1)$ so that $\sup_{t \in R_o} \{\int_\Omega |x(t;\omega)|^m d\mu(\omega)\}^{1/m} < \infty$.

The norm in the space C^m is defined by

$$\|x\|_{C^m} = \sup_{t \in R_o} \{ \|x(t;\omega)\|_m \} .$$

Corollary 2.1.1: Consider the random integral equation S and suppose that all the hypotheses of theorem 2.1 hold with $q(t) \equiv 1$. Then there exists a unique solution $x \in C^m$.

Proof: In the case $q(t) \equiv 1$, the B-space C_q^m coincides with the B-space C^m as defined before. Thus the proof follows from that of theorem 2.1.

Remark: For m = 2 and $f(t,x,w) \equiv f(t,x)$ (i.e. when f is a fixed real valued function defined on $R_o \otimes R$) the result of corollary 2.1.1 is equivalent to that of theorem 3.2 of Tsokos [20].

Corollary 2.1.2: Consider the random integral equation S and suppose that all the hypotheses of theorem 2.1 hold with $g(t) \equiv 1$. Then there exists a unique solution $x \in C_q^m$.

Corollary 2.1.3: Consider the random integral equation S and suppose that all the hypotheses of theorem 2.1 hold with $g(t) \equiv 1$ and $q(t) \equiv 1$. Then there exists a unique solution $x \in C^m$.

Remark: For m = 2 and $f(t,x,\omega) \equiv f(t,x)$ the result of corollary 2.1.3 is equivalent to that of corollary 3.2.1 of Tsokos [20].

Theorem 2.2: Consider the system S and suppose there exist a number $H > 0$ and a positive continuous function q on R_o so that

A_k : for every $t \in R_o$, $\| K(t, \tau ; \omega) \|_\infty \leq H \cdot q(t)$
for all $\tau \in [0,t]$,
and, A_h : $h \in C_q^m$.

Furthermore, suppose that the function f satisfies the hypotheses (i), (ii) and (iii) of lemma 2.1. Then if the function g appearing in the lemma 2.1 belongs to $L_1(R_o)$ and λ is such that $|\lambda| \alpha H \int_0^\infty g(t)dt < 1$, the system S has a unique solution $x \in C_q^m$.

Proof: This proof is analogous to that of theorem 2.1.

Corollary 2.2.1: Consider the system S and suppose that all of the hypothesis of the theorem 2.2 hold with $q(t) \equiv 1$. Then there exists a unique solution $x \in C^m$.

Remark: For $m = 2$ and $f(t,x,\omega) \equiv f(t,x)$ the result of corollary 2.2.1 is equivalent to that of corollary 3.2.2 of Tsokos [20] .

Theorem 2.3: Consider the system S and suppose there exist two positive numbers H and δ so that for every $(t, \tau) \in \Delta$, $\|K(t, \tau;\omega)\|_\infty \leq H \cdot e^{-\delta(t-\tau)}$ and; $h \in C^m$. Moreover, if the function f satisfies the hypotheses (i), (ii) and (iii) of lemma 2.1 with $q(t) \equiv 1$ and the function g appearing in the lemma 2.1 is bounded almost everywhere on R_o, then the system S has a unique solution $x \in C^m$ if $|\lambda| \alpha$ is sufficiently small.

Proof: The proof is similar to that given for theorem 2.1.

In order to study the questions of stability of the system S, we need the following definitions:

Definition 2.3: The system S is said to be stable in the sense of $C_q^m \triangleq C_q(R_o, L_m(\Omega, \beta, \mu))$ $(m \geq 1)$ if the solution $x \in C_q^m$ whenever the stochastic free term $h \in C_q^m$.

Definition 2.4: The system S is said to be stable in the mean $m(m \geq 1)$ if the solution $x \in C^m$ whenever the stochastic free term $h \in C^m$.

It is clear that the system S is stable in the mean $m(m \geq 1)$ if it has a solution $x \in C_q^m$ with q(t) being any positive bounded continuous function. Note that it is not essential that $q(t) \equiv 1$.

In the rest of this section we generalize the main result of Tsokos [20]. In his main theorem, Tsokos considered the existence and uniqueness of solution of the system S in the B-space D contained in a locally convex topological space $E^2 \triangleq E\ (R_o, L_2(\Omega, \beta, \mu))$ under the assumption that the nonlinear operator f is deterministic. We consider the same problem and present a proof of the existence and uniqueness of the solution of the system S in a B-space $D \subset E\ (R_o, L_m(\Omega, \beta, \mu))\ m \geq 1$ without requiring f to be deterministic. Before we discuss this generalized result we need the following preparation.

Definition 2.5: $E^m \triangleq E(R_o, L_m(\Omega, \beta, \mu))$ denotes the space of all continuous functions from R_o into $L_m(\Omega, \beta, \mu)$ $(m \geq 1)$ with the topology of uniform convergence on every finite interval $[0, T]$ for $T > 0$. This topology is defined by a family of semi-norms $\{N_n\}$ in which

$$N_n(x) = \|x(t;\omega)\|_n = \sup_{0 \leq t \leq n} \left\{ \int_\Omega |x(t;\omega)|^m d\mu(\omega) \right\}^{1/m}, \quad n = 1, 2, \ldots .$$

Equipped with this family of semi-norms the space E^m becomes a locally convex linear topological space [22, pp. 24-26].

Lemma 2.3: Let T be a continuous linear operator from E^m $(m \geq 1)$ into itself. Suppose that B and D are Banach spaces such that $B, D \subset E^m$ and that $TB \subset D$. Then T is a continuous linear operator from B to D.

The lemma follows easily from the closed-graph theorem [12, pp. 217].

Remark: Since the continuity of the linear operator T implies its boundedness, we can find a constant $L > 0$ such that

$$\|Tx\|_D \leq L \|x\|_B .$$

Theorem 2.4: Consider the system S and suppose that the following conditions are satisfied: (i) the stochastic kernel $K(t,\tau;\omega)$ is an essentially bounded μ-measurable function on Ω for every $(t,\tau) \in \Delta \triangleq$ $\triangleq \{(t,\tau): 0 \le \tau \le t < \infty\}$, (ii) B and D are Banach spaces such that $B, D \subset E^m$ and that $TB \subset D$, where T is defined by

$(Tu)(t;\omega) = \int_0^t K(t,\tau;\omega)\, u(\tau;\omega)\, d\tau$ for $u \in B$, and (iii) $f_\omega : x(t;\omega) \longrightarrow f(t, x(t;\omega), \omega)$ is an operator on D with values in B, and satisfies the following Lipschitz condition

$$\| f(t,x(t;\omega),\omega) - f(t,y(t;\omega),\omega) \|_B \le \alpha \, \| x(t;\omega) - y(t;\omega) \|_D .$$

for $x, y \in D$. Then the system S has a unique solution $x \in D$ for every $h \in D$ and every λ such that $|\lambda| \alpha L < 1$, where L is the norm of the operator T.

Proof: Let us define an operator U on the B-space D by

$$(Ux)(t;\omega) = h(t;\omega) + \lambda \int_0^t K(t,\tau;\omega)\, f(\tau, x(\tau;\omega), \omega)\, d\tau \tag{2.10}$$

for $t \in R_o$, $\omega \in \Omega$ and $h \in D$.

We show that the operator $U: D \longrightarrow D$ and that it is a contraction.

By assumption (iii), $f(\tau, x(\tau;\omega),\omega) \in B$ for every $x \in D$. It is easily shown that T is continuous from E^m into E^m. Thus, from lemma 2.3 we see that T is a continuous operator from the Banach space B into D, which implies that we can find a constant $L > 0$ such that

$$\| Tf \|_D \le L \, \| f \|_B$$

That is,

$$\left\| \int_0^t K(t,\tau;\omega)\, f(\tau, x(\tau;\omega),\omega)\, d\tau \right\|_D \le L \, \| f(t,x(t;\omega),\omega) \|_B . \tag{2.11}$$

Using condition (iii) and the hypothesis that $h \in D$ it follows from Minkowski's inequality and the inequality (2.11) that

$$\| Ux \|_D \le \| h \|_D + L\, |\lambda|\, \alpha \, \| x \|_D + L\, |\lambda| \, \| f(t,o,\omega) \|_B < \infty \tag{2.12}$$

Therefore $U: D \longrightarrow D$.

For the proof of the contraction property, let us consider $x, y \in D$, thus we have

$$(Ux)(t;\omega)-(Uy)(t;\omega)= \lambda \int_0^t K(t, \tau;\omega)[f(\tau, x(\tau;\omega),\omega)-f(\tau, y(\tau;\omega),\omega)]\,d\tau \quad (2.13)$$

By assumptions (ii) and (iii), $[f(\tau,x(\tau;\omega),\omega)-f(\tau,y(\tau;\omega),\omega)] \in B$. From lemma 2.3 we have

$$\| Ux-Uy \|_D \le L \; |\lambda| \; \| f(t,x(t;\omega),\omega) - f(t, y(t;\omega),\omega) \|_B . \quad (2.14)$$

By the Lipschitz condition given in (iii), we have

$$\| Ux-Uy \|_D \le |\lambda| \; La \; \| x - y \|_D . \quad (2.15)$$

The condition $a \, |\lambda| \, L < 1$ implies that the operator U is a contraction operator. Therefore by Banach fixed-point theorem there exists a unique solution $x \in D$.

2.3: ILLUSTRATIVE EXAMPLES.

For illustration, some examples are presented below. Example 2.1 illustrates stability in the sense of $C_q^m (m \ge 1)$ and examples 2.2 and 2.3 illustrate stability in the mean $m (m \ge 1)$.

Example 2.1 Consider one-dimensional heat-conduction in a semi-infinite medium governed by the equation [7, pp 235-236] .

$$L \frac{\partial^2 x_\omega}{\partial \ell^2} = \frac{\partial x_\omega}{\partial t} , \quad 0 \le \ell < \infty , \; t \ge 0 , \quad (2.16)$$

T :

$$x_\omega(\ell, 0) = 0, \qquad 0 \le \ell < \infty , \quad (2.17)$$

$$\left. \frac{\partial x_\omega}{\partial \ell} \right|_{\ell = 0} = - G_\omega(t, x_\omega), \text{ for } t > 0, \quad (2.18)$$

where

L = thermal diffusion coefficient of the solid,

x_ω = temperature of the surface of the body measured from absolute zero,

$\frac{\partial x_\omega}{\partial \ell}$ = thermal gradient at the surface, evaluated in the direction of the interior normal of the solid.

The function $G_\omega(t,x)$ is assumed to be a measurable function on $\Omega \otimes R_o$ for every fixed $x \in R$, where Ω is the supporting set of the probability space (Ω, β, μ) induced by the random boundary condition (2.18). It is a result that the solution of (2.16) satisfying the initial condition (2.17) and the boundary condition (2.18) is given by

$$x_\omega(\ell,t) = \frac{1}{(\pi L)^{1/2}} \int_o^t \frac{G_\omega(\tau, x_\omega(o,\tau))}{(t-\tau)^{1/2}} e^{-\frac{\ell^2}{4L(t-\tau)}} d\tau . \qquad (2.19)$$

Hence $x_\omega(\ell,t)$ is known for all values of ℓ and t if $x_\omega(o,t)$ is known. A nonlinear random integral equation for $x_\omega(o,t)$ is obtained by setting $\ell = o$ in (2.19)

$$x_\omega(o,t) = \frac{1}{(\pi L)^{1/2}} \int_o^t \frac{G_\omega(\tau, x_\omega(o,\tau))}{(t-\tau)^{1/2}} d\tau \qquad (2.20)$$

Suppose that the random function $G_\omega(t,x)$ satisfies the following conditions:

(i) there exists a β-measurable function $a(\omega)$ independent of $x, y \in R$ so that

$$|G_\omega(t,x) - G_\omega(t,y)| \leq a(\omega) \; |x-y| \quad \text{for all } t \in R_o \qquad (2.21)$$

and (ii) there exist $o < \alpha < \infty$ and $b > o$ so that

$\mu\{\omega: a(\omega) > \alpha\} = 0$ and that $\| G_\omega(t,x) \|_{C_q^m} < \infty$ with $q(t) = e^{bt}$ $t \geq o$.

Then the system T has a unique random solution $x_\omega(\ell, t) \in C_q^m$ if $\frac{\alpha}{\sqrt{Lb}} < 1$.

<u>Remark:</u> It is interesting to note that for the example under consideration we can prove the existence and uniqueness of solution of the system S in C_q^m for only an appropriate positive continuous function q on R_o.

<u>Example 2.2:</u> Consider the stochastic system described by the Volterra integral equation of the third kind of the form:

$$W: \quad \rho(t) \cdot x(t;\omega) = h(t;\omega) + \lambda \int_o^t K(t,\tau;\omega) f(\tau, x(\tau;\omega), \omega) d\tau$$

for $t \in R_o$ and $\omega \in \Omega$.

Suppose that the stochastic kernel K has the form

$K(t,\tau;\omega) = a(\omega)(t-\tau) \sin\left[\frac{t-\tau}{a(\omega)}\right]$ or $a(\omega) \cdot \sin\left[\frac{t-\tau}{a(\omega)}\right]$, (the second being the impulse response of a random oscillator), and that the function f has the form $f(t,z,\omega) = \tan^{-1}[b(\omega)\cdot z]$ for any $z \in R$.

Suppose there exists a non-negative measurable function ρ on R_o satisfying the following properties:

(i) $\sup_{t \in R_o} \{ \frac{t^{1/2}}{\rho(t)} \} < \infty$ and ;

(ii) $\sup_{t \in R_o} \{ \frac{t^2}{\rho(t)} \} < \infty$. Furthermore, let x be the unknown random process defined on $R_o \otimes \Omega$ where (Ω, β, μ) is the probability space corresponding to the process h and the random variables a and b; let h be the Wiener process satisfying the property that $h(0;\omega) = 0$ with probability one; and let λ be a sufficiently small real number representing the gain of the feedback loop.

The random variables $a(\omega)$ and $b(\omega)$ are assumed to be essentially bounded β-measurable functions defined on Ω.

Since the random function $f(t,z,\omega)$ is taken as $\tan^{-1}[b(\omega)\cdot z]$,it follows from the mean value theorem that for all $u, v \in R$ and every $t \in R_o$,

$$|f(t,u,\omega) - f(t,v,\omega)| \leq |b(\omega)| \cdot |u-v| . \tag{2.22}$$

If the stochastic kernel $K(t,\tau;\omega)$ is taken as $a(\omega)\cdot(t-\tau)\cdot \sin\left[\frac{t-\tau}{a(\omega)}\right]$ with $a(\omega)$ as definedabove, then

$$\| K(t,\tau;\omega) \|_\infty \leq \| a(\omega) \|_\infty \cdot (t-\tau) . \tag{2.23}$$

Thus

$$\int_o^t \| K(t,\tau;\omega) \|_\infty \cdot d\tau \leq \frac{1}{2} \| a(\omega) \|_\infty \cdot t^2 . \tag{2.24}$$

Since $h(t;\omega)$ is the Wiener process satisfying the property that $h(o;\omega) = 0$ with probability one, we have for every $t \in R_o$,

$$\left\| \frac{h(t;\omega)}{\rho(t)} \right\|_m = \left(\frac{2^{\frac{m}{2}}}{\sqrt{\pi}} \cdot \Gamma\left(\frac{1}{2}(m+1)\right) \right)^{1/m} \cdot \frac{t^{1/2}}{\rho(t)} \tag{2.25}$$

Therefore, by the assumption that $\sup_{t \in R_o} \{ \frac{t^{1/2}}{\rho(t)} \} < \infty$, we have

$$\sup_{t \in R_o} \{ \| \frac{h(t;\omega)}{\rho(t)} \|_m \} < \infty . \qquad (2.26)$$

Then it can be shown that the operator $\mathcal{U}$ defined on the B-space C^m by

$$(\mathcal{U}x)(t;\omega) = \frac{h(t;\omega)}{\rho(t)} + \lambda \cdot \frac{1}{\rho(t)} \int_o^t K(t,\tau;\omega)\, f(\tau;\, x(\tau;\omega),\omega)\, d\tau \qquad (2.27)$$

has the properties :

(i) $\mathcal{U}: C^m \longrightarrow C^m$

and (ii) $\| \mathcal{U}x - \mathcal{U}y \|_{C^m} \leq \alpha_o \| x-y \|_{C^m}$

where $\alpha_o \triangleq \{ \frac{|\lambda| \cdot \| b(\omega) \|_\infty \cdot \| a(\omega) \|_\infty}{2} \sup_{t \in R_o} \frac{t^2}{\rho(t)} \}$.

Since all the quantities in the parenthesis except λ are finite, we can find a λ sufficiently small so that $\alpha_o < 1$. This leads to the contraction property for the operator $\mathcal{U}$. Therefore by Banach fixed point theorem there exists a unique solution in C^m. Thus the system W is stable in the mean $m (m \geq 1)$ (definition 2.4)

Example 2.3: Consider a simple example of a time varying system (Fig. 1).

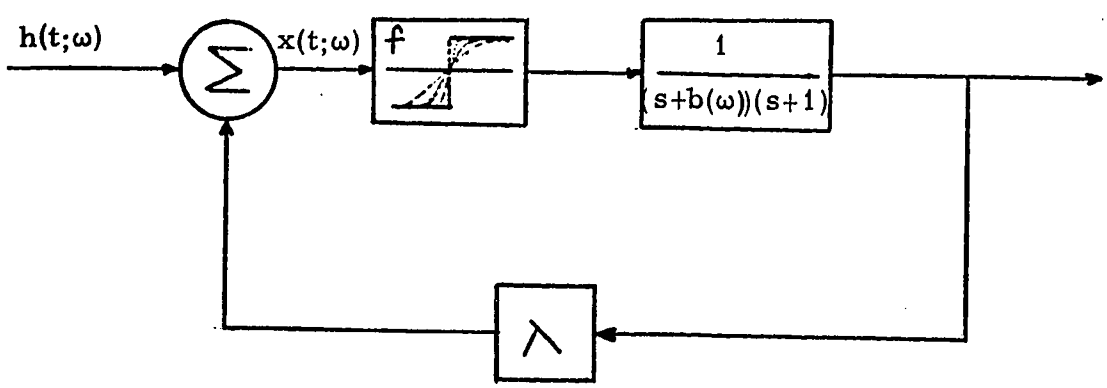

Fig. 1

Suppose that the zero-memory nonlinear element f is an amplifier with random gain which can degenerate into a relay, (for example, $f(t,x,\omega) = \tan^{-1}[a(\omega) \cdot x]$) and that the stochastic kernel K is of the form

$$K(t, \tau;\omega) = \frac{1}{(b(\omega)-1)} [e^{-(t-\tau)} - e^{-b(\omega)(t-\tau)}] \quad \text{for} \quad 0 \le \tau \le t < \infty$$

$$= 0 \qquad \text{elsewhere.}$$

Moreover, we assume that $h \in C^m$ (that is $\sup_{t \ge 0} \int_\Omega | h(t,\omega)|^m d\mu(\omega) < \infty$), and that λ is a sufficiently small real number.

The random variable $a(\omega)$ is asumed to be an essentially bounded β-measurable function, and the random variable b is assumed to have the following uniform probability density function

$$P_b(y) = 1 \qquad \text{for all } y \in [1,2]$$
$$P_b(y) = 0 \qquad \text{for } \quad y \notin [1,2] \tag{2.28}$$

Using the mean value theorem the random function f has the following properties:

$$| \tan^{-1} a(\omega)\cdot x - \tan^{-1} a(\omega)\cdot y | \le |a(\omega)| \cdot |x-y| \tag{2.29}$$

for all $x, y \in R$, where $a(\omega)$ is as described above. For every u and $v \in C^m$, it follows from the property of the function $a(\omega)$ and the inequality (2.29) that

$$\| \tan^{-1} a(\omega)\cdot u(t;\omega) - \tan^{-1} a(\omega)\cdot v(t;\omega) \|_m \le \|a(\omega)\|_\infty \cdot \|u(t;\omega)-v(t;\omega)\|_m \tag{2.30}$$

for every fixed $t \in R_o$, where $\| a(\omega) \|_\infty < \infty$.

Since the stochastic kernel K is of the form

$K(t,\tau;\omega) = \frac{1}{b(\omega)-1} [e^{-(t-\tau)} - e^{-b(\omega)(t-\tau)}]$ with b having the probability density function as defined in (2.28), we have

$$\| K(t,\tau;\omega) \|_\infty = e^{-(t-\tau)} \left\| \frac{1 - e^{-(b(\omega)-1)(t-\tau)}}{b(\omega) - 1} \right\|_\infty$$

$$\le (t-\tau) \cdot e^{-(t-\tau)} \tag{2.31}$$

Thus

$$\int_o^t \| K(t,\tau;\omega) \|_\infty d\tau \le \int_o^\infty \xi \cdot e^{-\xi} d\xi = 1. \tag{2.32}$$

Therefore all the hypothesis of corollary (2.1.3) hold, thus it follows from that corollary that the system under consideration has a unique solution $x \in C^m$. Therefore, by definition 2.4, the system is stable in the mean $m(m \geq 1)$.

2.4: ASYMPTOTIC STABILITY IN THE MEAN m $(m \geq 1)$.

In many problems the asymptotic behavior of the solution of the system is of greater practical interest. In this section we present sufficient conditions for asymptotic stability in the mean $m(m \geq 1)$ of the system S. For this, we need the following preparation.

<u>Definition 2.6</u>: The system S is said to be asymptotically stable in $C_q^m \triangleq C_q(R_o, L_m(\Omega, \beta, \mu))(m \geq 1)$ if the solution $x \in C_q^m$ and has the property that $\lim_{t \to \infty} \{ \int_\Omega | \frac{x(t;\omega)}{q(t)} |^m d\mu(\omega) \}^{1/m} = 0$ whenever the stochastic free term $h \in C_q^m$ and has the property that $\lim_{t \to \infty} \{ \int_\Omega | \frac{h(t;\omega)}{q(t)} |^m \cdot d\mu(\omega) \}^{1/m} = 0$.

<u>Definition 2.7:</u> The system S is said to be asymptotically stable in the mean $m(m \geq 1)$ if the solution $x \in C^m \triangleq C(R_o, L_m(\Omega, \beta, \mu))$ $(m \geq 1)$ and has the property that $\lim_{t \to \infty} \{ \int_\Omega | x(t;\omega) |^m d\mu(\omega) \}^{1/m} = 0$ whenever the stochastic free term $h \in C^m$ and has the property that $\lim_{t \to \infty} \{ \int_\Omega | h(t;\omega) |^m d\mu(\omega) \}^{1/m} = 0$.

It is clear that the system S is asymptotically stable in the mean $m(m \geq 1)$ if it has a solution $x \in C_q^m$ and has the property that $\lim_{t \to \infty} \{ \int_\Omega | \frac{x(t;\omega)}{q(t)} |^m d\mu(\omega) \}^{1/m} = 0$ with $q(t)$ being any positive bounded continuous function. Note that it is not essential that $q(t) \equiv 1$.

<u>Lemma 2.4:</u> Consider the function $D(t)$ on R_o given by

$$D(t) = \int_0^t K(t, t-\tau) V(\tau) d\tau . \tag{2.33}$$

Suppose that $\{\int_o^t |K(t,\xi)|^{q_2} d\xi\}^{1/q_2} \triangleq \hat{K}(t) \in L_1(R_o)$ and is bounded uniformly on R_o and that $V(t) \in L_{p_2}(R_o)$ where $\frac{1}{p_2} + \frac{1}{q_2} = 1$ $(p_2 > 1)$. Then $\lim_{t\to\infty} |D(t)| = 0$.

Proof: Since for all $t \in R_o$, $\{\int_o^t |K(t,\xi)|^{q_2} d\xi\}^{1/q_2} \triangleq \hat{K}(t) \in L_1(R_o)$ and is bounded uniformly on R_o and $V(t) \in L_{p_2}(R_o)$, there exists, for every $\epsilon > 0$, a $T_o = T_o(\epsilon) \in R_o$ such that for all $t > T_o$

$$\int_{T_o}^t |K(t,\xi)|^{q_2} d\xi \le \left(\frac{\epsilon}{2P}\right)^{q_2}$$

$$\int_{T_o}^t |V(\xi)|^{p_2} d\xi \le \left(\frac{\epsilon}{2Q}\right)^{p_2} \tag{2.34}$$

where

$$P \triangleq \{\int_o^\infty |V(\xi)|^{p_2} d\xi\}^{1/p_2}$$

$$Q \triangleq \sup_{t \ge T_o} \{\int_o^{t-T_o} |K(t,\xi)|^{q_2} d\xi\}^{1/q_2}$$

Therefore by the estimates (2.34) and the fact that Q is bounded it follows from Hölder's inequality that for $t > 2T_o$ we have

$$\begin{aligned} |D(t)| &= \left|\int_o^t K(t,t-\tau)V(\tau)\,d\tau\right| \\ &\le \{(\int_{t-T_o}^t |K(t,\xi)|^{q_2} d\xi)^{1/q_2} (\int_o^{T_o} |V(\xi)|^{p_2} d\xi)^{1/p_2} + \\ &+ (\int_o^{t-T_o} |K(t,\xi)|^{q_2} d\xi)^{1/q_2} (\int_{T_o}^t |V(\xi)|^{p_2} d\xi)^{1/p_2}\} \\ &\le \epsilon \end{aligned} \tag{2.35}$$

Since $\epsilon > o$ is arbitrary, this completes the proof of the lemma.

For convenience we introduce the following notations:

(i) $\{\int_\Omega |x(t;\omega)|^m d\mu(\omega)\}^{1/m} = \|x(t;\omega)\|_m \triangleq \hat{x}(t)$ for $t \in R_o$ and $x \in C_q^m$;

(ii) $\{\int_\Omega |h(t;\omega)|^m d\mu(\omega)\}^{1/m} = \|h(t;\omega)\|_m \triangleq \hat{h}(t)$ for $t \in R_o$ and $x \in C_q^m$;

and (iii) $\{\int_\Omega |f(t,x(t;\omega),\omega)|^m d\mu(\omega)\}^{1/m} = \|f(t,x(t;\omega),\omega)\|_m \triangleq V_x(t)$ for $t \in R_o$ and $x \in C_q^m$.

The above abbreviations are used throughout this section without further mention.

Theorem 2.5: Consider the system S and suppose that all the hypotheses of theorem 2.1 hold, and that there exists a real valued measurable function $\alpha(t;\omega)$ on $R_o \otimes \Omega$ satisfying the following properties:

(i) $\| \alpha(t;\omega) \|_m \triangleq \theta(t) \in L_{p_2}(R_o)$ with $m \geq 1$, $p_2 > 1$;

and (ii) for almost all $\omega \in \Omega$ and for almost all $t \in R_o$

$$|f(t,z,\omega)| \leq |\alpha(t;\omega)| + |\eta(\omega)| \cdot |z|$$

where $z \in R$ and $\eta(\omega)$ is an essentially bounded measurable function defined on Ω. Further, assume that the functions h,K and g, satisfy the following properties:

$A_{\hat{h}}$: $\hat{h}(t) \in L_{p_2}(R_o)$

A_{K^o} : $\|K(t,\tau,\omega)\|_\infty = K^o(t,t-\tau)$

$A_{\hat{K}}$: $\{\int_c^t |K^o(t,\xi)|^{q_2} d\xi\}^{1/q_2} \triangleq \hat{K}(t) \in L_{q_1}(R_o)$ and is bounded uniformly on R_o

$A_{\hat{g}}$: $\{\int_o^t |g(\tau)|^{p_2} d\tau\}^{1/p_2} \triangleq \hat{g}(t) \in L_{p_1}(R_o)$

where $\dfrac{1}{p_1} + \dfrac{1}{q_1} = \dfrac{1}{p_2}$; $\dfrac{1}{p_2} + \dfrac{1}{q_2} = 1$ and $(p_1 \geq p_2 > 1)$.

Then $\lim_{t\to\infty} \{\dfrac{\hat{h}(t)}{q(t)}\} = 0$ implies that $\lim_{t\to\infty} \{\dfrac{\hat{x}(t)}{q(t)}\} = 0$.

Proof: By theorem 2.1, the system S has a unique solution $x \in C_q^m$ such that

$$x(t;\omega) = h(t;\omega) + \lambda \int_o^t K(t,\tau;\omega)\, f(\tau, x(\tau;\omega),\omega)\, d\tau .$$

Therefore, in case $\lim_{t\to\infty} q(t) = 0$, we have $\lim_{t\to\infty} \hat{x}(t) = 0$, thus we exclude this case.

Using hypothesis $A_{\hat{K}}$ and $A_{\hat{g}}$ we have for each $t \in R_o$

$K^o(t,\xi) \in L_{q_2}[o,t)$ and $g(t) \in L_{p_2}[o,t)$. Thus by lemma 2.2 we have for every $t \in R_o$.

$$\hat{x}(t) \le \hat{h}(t) + |\lambda| \ \gamma \hat{K}(t). \ \hat{g}(t) \tag{2.36}$$

where $\gamma \triangleq \sup_{t \in R_o} \{ \frac{\|f(t,x(t;\omega),\omega)\|_m}{g(t)} \} < \infty$ (lemma 2.1).

Since $\hat{h}(t) \in L_{p_2}(R_o)$, $\hat{K}(t) \in L_{q_1}(R_o)$ and $\hat{g}(t) \in L_{p_1}(R_o)$,

we have

$$\{\int_o^\infty |\hat{x}(t)|^{p_2} dt\}^{1/p_2} \le \{\int_o^\infty |\hat{h}(t)|^{p_2} dt\}^{1/p_2} + |\lambda|\gamma \{\int_o^\infty |\hat{K}(t)|^{q_1} dt\}^{1/q_1} \cdot \{\int_o^\infty |\hat{g}(t)|^{p_1} dt\}^{1/p_1} < \infty . \tag{2.37}$$

Thus for this $x \in C_q^m$ it follows from the definition for V_x, the hypothesis on f(i and ii of this theorem) and the estimate (2.37) that

$$\{\int_o^\infty |V_x(t)|^{p_2} \cdot dt\}^{1/p_2} \le \{\int_o^\infty |\theta(t)|^{p_2} dt\}^{1/p_2} + \|\eta(\omega)\|_\infty \{\int_o^\infty |\hat{x}(t)|^{p_2} dt\}^{1/p_2} < \infty . \tag{2.38}$$

This implies that $V_x(t) \in L_{p_2}(R_o)$ for the solution x of the system S satisfying the inequality (2.37).

Using Minkowski's inequality, lemma 2.2 and hypothesis A_{K^o} it is easily verified that for every finite t,

$$\hat{x}(t) \le \hat{h}(t) + |\lambda| . \ D(t) \tag{2.39}$$

where $D(t) = \int_o^t K^o(t,t-\tau) \cdot V_x(\tau)\, d\tau$.

By hypothesis $A_{\hat{K}}$ and the fact that $V_x \in L_{p_2}(R_o)$ it follows from lemma 2.4 that $\lim_{t\to\infty} D(t) = 0$. Therefore if $\lim_{t\to\infty} \frac{\hat{h}(t)}{q(t)} = 0$ then

$$\lim_{t\to\infty} \frac{\hat{x}(t)}{q(t)} = 0 .$$

Remark: It is important to note that the system S under the hypothesis of theorem 2.5 is only asymptotically stable in C_q^m $(m \geq 1)$. This is due to the fact that $\lim_{t\to\infty} \{\frac{\hat{x}(t)}{q(t)}\} = 0$ does not imply that $\lim_{t\to\infty} \hat{x}(t) = 0$ since q need not be finite.

Corollary 2.5.1: Consider the system S and suppose that all the hypotheses of theorem 2.5 hold with $q(t) \equiv 1$. Then $\lim_{t\to\infty} \hat{x}(t) = 0$ if $\lim_{t\to\infty} \hat{h}(t) = 0$.

Remark: Corollary 2.5.1 implies that the system S is asymptotically stable in the mean $m (m \geq 1)$. Moreover, the result of theorem 2.5 is based on theorem 2.1 and the result of corollary 2.5.1 is based on corollary 2.1.1.

Corollary 2.5.2: Consider the system S and suppose that all the hypotheses of corollary 2.1.2 hold, and that there exists a real valued measurable function $\alpha(t;\omega)$ on $R_o \otimes \Omega$ satisfying the following properties:

(i) $\| \alpha(t;\omega) \|_m \triangleq \theta(t) \in L_{p_2}(R_o)$ with $m \geq 1$, $p_2 > 1$;

and (ii) for almost all $\omega \in \Omega$ and for almost all $t \in R_o$

$$|f(t,z,\omega)| \leq |\alpha(t;\omega)| + |\eta(\omega)| \cdot |z|$$

where $z \in R$ and $\eta(\omega)$ is an essentially bounded measurable function defined on Ω. Further, assume that the functions h and K satisfy the following additional properties:

$A_{\hat{h}}$: $\hat{h}(t) \in L_{p_2}(R_o)$;

A_{K^o} : $\| K(t,\tau;\omega) \|_\infty = K^o(t,t-\tau)$ and $\{ \int_o^t |K^o(t,\xi)| d\xi \} \in L_{p_2}(R_o)$;

$A_{\hat{K}}$: $\{ \int_o^t |K^o(t,\xi)|^{q_2} d\xi \}^{1/q_2} \triangleq \hat{K}(t) \in L_1(R_o)$ and is bounded uniformly on R_o

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$ $(p_2 > 1)$.

Then $\lim_{t\to\infty} \frac{\hat{h}(t)}{q(t)} = 0$ implies that $\lim_{t\to\infty} \frac{\hat{x}(t)}{q(t)} = 0$.

Corollary 2.5.3: Consider the system S and suppose that all the hypotheses of corollary 2.5.2 hold with $q(t) \equiv 1$. Then $\lim_{t\to\infty} \hat{x}(t) = 0$ if $\lim_{t\to\infty} \hat{h}(t) = 0$.

Remark: Corollary 2.5.3 implies that the system S is asymptotically stable in the mean $m (m \geq 1)$. Moreover, the result of corollary 2.5.2 is based on corollary 2.1.2 and the result of corollary 2.5.3 is based on corollary 2.1.3.

Theorem 2.6 Consider the system S and suppose that all the hypotheses of theorem 2.2 hold, and that there exists a real valued measurable function $\alpha(t;\omega)$ on $R_o \otimes \Omega$ satisfying the following properties:

(i) $\| \alpha(t;\omega) \|_m \triangleq \theta(t) \in L_{p_2}(R_o)$ with $m \geq 1$, $p_2 > 1$;

and (ii) for almost all $\omega \in \Omega$ and for almost all $t \in R_o$

$$|f(t,z,\omega)| \leq |\alpha(t;\omega)| + |\eta(\omega)| \cdot |z|$$

where $z \in R$ and $\eta(\omega)$ is an essentially bounded measurable function defined on Ω. Further, assume that the functions h, K and g satsify the following additional properties:

$A_{\hat{h}}$: $\hat{h}(t) \in L_{p_2}(R_o)$;

A_{K^o} : $\| K(t,\tau;\omega) \|_\infty = K^o(t, t-\tau)$;

$A_{\hat{K}}$: $\{\int_o^t |K(t,\xi)|^{q_2} d\xi\}^{1/q_2} \triangleq \hat{K}(t) \in L_{q_1}(R_o)$ and is bounded uniformly on R_o ;

$A_{\hat{g}}$: $\{\int_o^t |g(\tau)|^{p_2} d\tau\}^{1/p_2} \triangleq \hat{g}(t) \in L_{p_1}(R_o)$

where $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2}$; $\frac{1}{p_2} + \frac{1}{q_2} = 1$ and $(p_1 \geq p_2 > 1)$.

Then $\lim_{t \to \infty} \{\frac{\hat{h}(t)}{q(t)}\} = 0$ implies that $\lim_{t \to \infty} \{\frac{\hat{x}(t)}{q(t)}\} = 0$.

Proof: This proof is analogous to that of theorem 2.5.

Corollary 2.6.1: Consider the random integral equation S and suppose that all the hypotheses of theorem 2.6 hold with $q(t) \equiv 1$. Then $\lim_{t\to\infty} \hat{x}(t) = 0$ if $\lim_{t\to\infty} \hat{h}(t) = 0$.

Remark: Corollary 2.6.1 implies that the system S is asymptotically stable in the mean $m(m \geq 1)$. Moreover, the result of theorem 2.6 is based on theorem 2.2 and the result of corollary 2.6.1 is based on corollary 2.2.1.

Theorem 2.7: Consider the random integral equation S and suppose that all the hypotheses of theorem 2.3 hold, and that there exists a real valued measurable function $\alpha(t;\omega)$ on $R_o \otimes \Omega$ satisfying the following properties:

(i) $\| \alpha(t;\omega) \|_m \triangleq \theta(t) \in L_{p_2}(R_o)$ with $m \geq 1$, $p_2 > 1$;

and (ii) for almost all $\omega \in \Omega$ and for almost all $t \in R_o$

$$| f(t,z,\omega) | \leq |\alpha(t;\omega)| + |\eta(\omega)| \cdot |z|$$

where $z \in R$ and $\eta(\omega)$ is an essentially bounded measurable function defined on Ω. Further, assume that the functions h and K satisfy the following additional properties:

$$A_{\hat{h}} : \hat{h}(t) \in L_{p_2}(R_o) ;$$

$$A_{K^o} : \| K(t,\tau;\omega) \|_\infty \triangleq K^o(t,\tau) \text{ and } \{ \int_o^t | K^o(t,\tau) | \, d\tau \} \in L_{p_2}(R_o) ;$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1 \qquad p_2 > 1$.

Then $\lim_{t\to\infty} \hat{h}(t) = 0$ implies that $\lim_{t\to\infty} \hat{x}(t) = 0$.

Proof: The proof is similar to that given for theorem 2.5.

Remark: Theorem 2.7 implies that the system S is asymptotically stable in the mean $m(m \geq 1)$. Moreover, the result of theorem 2.7 is based on theorem 2.3.

2.5: AN ILLUSTRATIVE EXAMPLE.

Example 2.4: Consider the stochastic system W as in example 2.2 and suppose that all the hypothesis of example 2.2 hold with $\rho(t) = (\delta + t^{\alpha})$, where $\delta > 0$ and $\alpha > 2$. Then we can prove that the system under consideration is asymptotically stable in the mean $m (m \geq 1)$.

In example 2.2, we have shown that the system W has a unique solution $x \in C^m$. Thus, by use of Minkowski's inequality and lemma 2.2, it is easily verified that for any $t \in R_o$ we have

$$\hat{x}(t) \leq \hat{h}(t) + |\lambda| \cdot \frac{1}{\delta + t^{\alpha}} \cdot H(t), \tag{2.40}$$

where $\hat{h}(t) \triangleq \dfrac{\| h(t;\omega) \|_m}{\delta + t^{\alpha}}$ and

$$H(t) \triangleq \int_o^t \| K(t, \tau;\omega) \|_{\infty} \| f(\tau, x(\tau;\omega), \omega) \|_m d\tau .$$

Using inequality (2.24), we have for any $t \in R_o$

$$H(t) \leq \gamma \cdot \frac{\| a(\omega) \|_{\infty}}{2} \cdot t^2 \tag{2.41}$$

where $\| a(\omega) \|_{\infty} < \infty$ and $\gamma \triangleq \sup_{t \in R_o} \{ \| f(t, x(t;\omega), \omega) \|_m \} < \infty$ for $x \in C^m$ as already shown in example 2.2. Thus, it can be easily verified that $\lim_{t \to \infty} \hat{x}(t) = 0$. Therefore, by definition 2.7, the system W is asymptotically stable in the mean $m (m \geq 1)$.

CHAPTER 3

ALMOST SURE $L_p (p \geq 1)$ STABILITY AND ALMOST SURE ASYMPTOTIC STABILITY.

3.1: INTRODUCTION

In this chapter we consider the questions of (i) almost sure $L_p (p \geq 1)$ stability ; and (ii) almost sure asymptotic stability of the same system as considered in the previous chapter.

3.2: ALMOST SURE $L_p(p \geq 1)$ STABILITY

In this section we are interested in the question of the existence of a solution $x(t;\omega) \in L_p(R_o)$ for almost all $\omega \in \Omega$, where Ω is the supporting set of the probability space (Ω, β, μ) as defined before.

It is assumed in this section that (i) the kernel $K(t, \tau;\omega)$ is a measurable function on the triangle $\Delta \triangleq \{(t, \tau): o \leq \tau \leq t < \infty\}$ for almost all $\omega \in \Omega$ and is a measurable function on Ω for all $(t, \tau) \in \Delta \triangleq \{(t, \tau) : o \leq \tau \leq t < \infty\}$; and (ii) the function $f(t,x,\omega)$ is measurable in t on R_o for almost every $\omega \in \Omega$ and for each fixed $x \in R \triangleq (-\infty, +\infty)$ and is continuous in x on R for almost all $t \in R_o$ and for almost all $\omega \in \Omega$ and is measurable in ω on Ω for almost all $t \in R_o$ and for each $x \in R$ (Carathéodory condition [15, pp. 20]).

For convenience we introduce the following notations:

(i) $x(t;\omega) \triangleq x_\omega(t)$

(ii) $h(t;\omega) \triangleq h_\omega(t)$

(iii) $f(t,x,\omega) \triangleq f_\omega(t,x)$

and (iv) $K(t, \tau;\omega) \triangleq K_\omega(t, \tau)$.

For the proof of the almost sure $L_p (p \geq 1)$ stability of the system S, we need the following preparation.

Definition 3.1: The system S is said to be almost surely $L_p(p \geq 1)$ stable if its solution x has the property that $\mu\{\omega \in \Omega: \int_{R_o} |x(t;\omega)|^p dt < \infty\} = 1$

whenever the stochastic free term h has the property that

$$\mu\ \{\omega \in \Omega : \int_{R_o} |\ h(t;\omega)|^p\, dt < \infty\} = 1.$$

Lemma 3.1: Suppose there exist real numbers $a \geq 1$ and $p_2 \geq 1$, an essentially bounded β-measurable function $N(\omega)$ on Ω and a real valued non-negative measurable function $r(t;\omega)$ on $R_o \otimes \Omega$ satisfying the following properties:

(i) $\int_{R_o} |r(t;\omega)|^{p_2} dt$ is an essentially bounded β-measurable function on Ω,

and (ii) for almost all $\omega \in \Omega$ and for almost all $t \in R_o$

$|f_\omega(t,z)| \leq r(t;\omega) + |N(\omega)|\,|z|^a$ for any $z \in R$. Then for almost all $\omega \in \Omega$ the random operator f_ω maps $L_{p_1}(R_o)$ into $L_{p_2}(R_o)$ and is continuous and bounded, where $p_1 = a\, p_2$.

Proof: Since all the hypotheses of lemma 3.1 hold for almost all $\omega \in \Omega$, there exists a set $\Omega_o \subset \Omega$ so that $\mu(\Omega_o) = \mu(\Omega)$ and that all the hypotheses of this lemma hold for every $\omega \in \Omega_o$. Thus for every $\omega \in \Omega_o$, we proceed with the proof of the lemma as follows:

For all $\omega \in \Omega_o$ the first and the third assertions of the lemma follows from the inequality

$$\{\int_{R_o} |f_\omega(t,z(t))|^{p_2} dt\}^{1/p_2} \leq \{\int_{R_o} |r(t;\omega)|^{p_2} dt\}^{1/p_2} + |N(\omega)|\,\{\int_{R_o} |z(t)|^{p_1} dt\}^{a/p_1} \tag{3.1}$$

which implies that for all $\omega \in \Omega_o$, f_ω: $L_{p_1}(R_o) \to L_{p_2}(R_o)$ and is bounded.

The first in turn implies continuity (Krasnoselskii, [15, th 2.1, pp. 22]) of the operator f_ω for every $\omega \in \Omega_o$. Therefore, for almost all $\omega \in \Omega$ the random operator f_ω maps $L_{p_1}(R_o)$ into $L_{p_2}(R_o)$ and is continuous and bounded.

With the help of this lemma, we can prove the following result; we need the following:

Definition 3.2: An operator A mapping a Banach space B_1 into a Banach space B_2 is said to be completely continuous on $D \subset B_1$ if it is continuous on D and AD is a compact subset of B_2 whenever D is bounded .

Theorem 3.1: Let the kernel $K_\omega(t,\tau)$ satisfy the following properties

(i) $K_\omega(t,\tau)$ is a measurable function on $\Delta \triangleq \{(t,\tau): 0 \leq \tau \leq t < \infty\}$ for almost all $\omega \in \Omega$ and is a measurable function on Ω for any $(t,\tau) \in \Delta$; and (ii) for $\frac{1}{p_2} + \frac{1}{q_2} = 1$,

$$\hat{K}_\omega(t) \triangleq \{\int_0^t |K_\omega(t,\tau)|^{q_2} d\tau\}^{1/q_2} \in L_{p_1}(R_0) \text{ for almost all } \omega \in \Omega;$$

and let f_ω satisfy the hypotheses of lemma 3.1. Then the operator A_ω defined by $(A_\omega x)(t) = \int_0^t K_\omega(t,\tau) f_\omega(\tau, x(\tau))\, d\tau$ maps $L_{p_1}(R_0)$ into itself for almost all $\omega \in \Omega$ and that it is completely continuous on $L_{p_1}(R_0)$ for almost all $\omega \in \Omega$.

Proof: Since all the hypotheses of this theorem hold for almost all $\omega \in \Omega$, there exists a common set $\Omega_1 \subset \Omega$ so that $\mu(\Omega_1) = \mu(\Omega)$.
Then the hypotheses of this theorem hold for every $\omega \in \Omega_1$. Thus we proceed with the proof of the theorem as follows:

For every $\omega \in \Omega_1$, $f_\omega : L_{p_1}(R_0) \longrightarrow L_{p_2}(R_0)$ (lemma 3.1).

Further,

$$\{\int_{R_0} |(A_\omega x)(t)|^{p_1} dt\}^{1/p_1} \leq \{\int_{R_0} |\hat{K}_\omega(t)|^{p_1} dt\}^{1/p_1} \cdot \{\int_{R_0} |f_\omega(t,x(t))|^{p_2} dt\}^{1/p_2} \tag{3.2}$$

for every $\omega \in \Omega_1$ and for all $x \in L_{p_1}(R_0)$. Thus the operator A_ω maps $L_{p_1}(R_0)$ into itself for every $\omega \in \Omega_1$ or in other words, the operator A_ω maps $L_{p_1}(R_0)$ into itself almost surely.

For the proof of complete continuity we must prove that for every $\omega \in \Omega_1$, A_ω is continuous (on $L_{p_1}(R_0)$) and compact (i.e. maps every bounded set in $L_{p_1}(R_o)$ into a compact set in $L_{p_1}(R_o)$). For every $\omega \in \Omega_1$ the continuity of the operator A_ω follows from that of the operator f_ω and the inequality

$$\{\int_{R_o} |(A_\omega x)(t)-(A_\omega y)(t)|^{p_1} dt\}^{1/p_1} \leq \{\int_{R_o} |\hat{K}_\omega(t)|^{p_1} dt\}^{1/p_1} \cdot \{\int_{R_o} |(f_\omega x)(t)-(f_\omega y)(t)|^{p_2} dt\}^{1/p_2}. \tag{3.3}$$

It remains to prove the compactness. Let $D_1 \subset L_{p_1}(R_o)$ be bounded. Since for all $\omega \in \Omega_1$ by lemma 3.1 $f_\omega : L_{p_1}(R_o) \rightarrow L_{p_2}(R_o)$ and is bounded, there exists $d_1 \in [o,\infty)$ so that $\sup_{x \in D_1} \{\int_{R_o} |(f_\omega x)(t)|^{p_2} dt\}^{1/p_2} \leq d_1$.

Define $D_2 = \{z_\omega = f_\omega x : x \in D_1\}$. Clearly $D_2 \subset L_{p_2}(R_o)$ is bounded and since for $p_2 > 1$, L_{p_2} is a reflexive Banach space D_2 is weakly compact for every $\omega \in \Omega_1$.

If $p_2 = 1$ then we may assume that the operator f_ω satisfies the additional property that for every $\epsilon > o$ there exists a $\delta > o$ such that

$$\int_E |f_\omega(t,x(t))|^{p_2} dt < \epsilon \text{ for all } x \in D_1 \text{ and all } \omega \in \Omega_1$$

whenever the Lebesgue measure of the set $E \subset R_o$ is less than δ. Under this situation, for every $\omega \in \Omega_1$, D_2 is a weakly (sequentially) compact subset of $L_{p_2}(R_o) = L_1(R_o)$ (Dunford [11, th. 9, pp. 292]). Thus for every $\omega \in \Omega_1$, D_2 is weakly sequentially compact. Now for every $\omega \in \Omega_1$, let $\{z_\omega^n = f_\omega x^n : x^n \in D_1\} \subset D_2$ be any sequence. It is clear that for almost every $t \in R_o$ and for every $\omega \in \Omega_1$

$$y_\omega^n(t) = \int_o^t K_\omega(t,\tau)\, z_\omega^n(\tau)\, d\tau \tag{3.4}$$

is defined. Further it follows from the assumptions on K that for almost every $t \in R_o$ and for every $\omega \in \Omega_1$, $K_\omega(t,\tau) \in L_{q_2}[o,t)$.

Therefore, since for every $\omega \in \Omega_1$ D_2 is weakly (sequentially) compact, there is a subsequence $\{z_\omega^m\}$ ($m = n_1, n_2, \ldots$) of $\{z_\omega^n\}$ and $z_\omega^o \in L_{p_2}(R_o)$ so that for almost all $t \in R_o$ and for every $\omega \in \Omega_1$

$$y_\omega^m(t) \triangleq \int_o^t K_\omega(t,\tau)\, z_\omega^m(\tau)\, d\tau \longrightarrow \int_o^t K_\omega(t,\tau)\, z_\omega^o(\tau)\, d\tau \triangleq y_\omega^o(t). \quad (3.5)$$

Therefore for every $\omega \in \Omega_1$, $\lim\limits_{m\to\infty} (y_\omega^m(t) - y_\omega^o(t)) = 0$ almost everywhere on R_o and consequently for every $\omega \in \Omega_1$

$$\lim_{m\to\infty} |y_\omega^m(t) - y_\omega^o(t)|^{p_1} = 0 \qquad \text{a.e on } R_o. \quad (3.6)$$

Further, for every $\omega \in \Omega_1$, $|y_\omega^m(t) - y_\omega^o(t)| \le \hat{K}_\omega(t)(d_1 + \|z_\omega^o\|_{p_2})$ (3.7)

uniformly with respect to m and for almost all $t \in R_o$. By hypotheses, $\hat{K}_\omega \in L_{p_1}(R_o)$ for every $\omega \in \Omega_1$ and therefore $\{y_\omega^m - y_\omega^o\} \in L_{p_1}(R_o)$ for every $\omega \in \Omega_1$.

Thus, for every $\omega \in \Omega_1$, the function on the right hand side of the inequality (3.7) provides the dominating function required for the well known Lebesgue dominated convergence theorem to hold. Therefore by application of this theorem we have for every $\omega \in \Omega_1$

$$\lim_{m\to\infty} \int_{R_o} |y_\omega^m(t) - y_\omega^o(t)|^{p_1} dt = \int_{R_o} \lim_{m\to\infty} |y_\omega^m(t) - y_\omega^o(t)|^{p_1} dt = 0. \quad (3.8)$$

Therefore $y_\omega^m \longrightarrow y_\omega^o$ for every $\omega \in \Omega_1$ in the strong topology of $L_{p_1}(R_o)$ and it follows from the equality $y_\omega^o = y_\omega^o - y_\omega^m + y_\omega^m$ and the uniform boundedness of the sequence $\{z_\omega^m\} \subset L_{p_2}(R_o)$ and consequently that of $\{y_\omega^m\} \subset L_{p_1}(R_o)$ that $y_\omega^o \in L_{p_1}(R_o)$. Thus for every $\omega \in \Omega_1$, $A_\omega D_1 \subset L_{p_1}(R_o)$ is compact whenever $D_1 \subset L_{p_1}(R_o)$ is bounded. Therefore the random operator A_ω is completely continuous on $L_{p_1}(R_o)$ almost surely.

Using lemma 3.1 and theorem 3.1, we prove the following.

<u>Theorem 3.2:</u> Consider the system S and suppose that all the hypotheses of theorem 3.1 hold. Furthermore, for an arbitrary but fixed $r \in (o, \infty)$ suppose that the ball $S_r = \{x \in L_{p_1}(R_o): [\int_{R_o} |x(t)|^{p_1} dt]^{1/p_1} \leq r\}$ is given and that for almost all $\omega \in \Omega$ the function $a_\omega(r) \triangleq \sup_{x \in S_r} \{\int_{R_o} |A_\omega x(t)|^{p_1} dt\}^{1/p_1}$ is defined. Suppose there exists an $r^* > 0$ such that $\bar{\lambda}_r{}^*$ defined by $\bar{\lambda}_{r*} \triangleq \sup \{|\lambda| : \mu\{\omega: a_\omega(r^*) < \frac{r^*}{|\lambda|}\} = 1\}$ is greater than zero. Then for every $\theta < r^* - \bar{\lambda}_{r*} a_\omega(r^*)$ μ a.e. and for any stochastic free term $h_\omega \in S_\theta$ μ a.e., for $S_\theta \subset L_{p_1}(R_o)$, the system S has a solution $x_\omega \in S_{r*}$ for almost all $\omega \in \Omega$.

<u>Proof:</u> Since all the hypotheses of this theorem hold for almost all $\omega \in \Omega$, we can find a common set $\Omega_2 \subset \Omega$ so that $\mu(\Omega_2) = \mu(\Omega)$ and that for every $\omega \in \Omega_2$ all the hypotheses of this theorem hold.
Therefore for a positive finite number r^* as defined in the hypotheses of the theorem, there exists a $\bar{\lambda}_{r*} > 0$ so that for all $\omega \in \Omega_2$ the random function $a_\omega(r^*)$ satisfies the property that $a_\omega(r^*) < \frac{r^*}{\bar{\lambda}_{r*}}$.
Consequently for every $h_\omega \in S_\theta$ (for all $\omega \in \Omega_2$) with $\theta < r^* - \bar{\lambda}_{r*} a_\omega(r^*)$ (for all $\omega \in \Omega_2$), the operator $B_\omega(h_\omega, \cdot)$ defined on $L_{p_1}(R_o)$ by $B_\omega(h_\omega, x) \triangleq h_\omega + \lambda A_\omega x$ has the property that $\{\int_{R_o} |B_\omega(h_\omega, x)|^{p_1} dt\}^{1/p_1} < r^*$ for all $x \in S_{r*} \subset L_{p_1}(R_o)$, any $|\lambda| \leq \bar{\lambda}_{r*}$ and all $\omega \in \Omega_2$. Thus, for all $\omega \in \Omega_2$, the operator $B_\omega(h_\omega, \cdot)$ maps the ball $S_{r*} \subset L_{p_1}(R_o)$ into a subset of S_{r*} for every $h_\omega \in S_\theta$ (for all $\omega \in \Omega_2$) and for any $|\lambda| \leq \bar{\lambda}_{r*}$.

Since for every $\omega \in \Omega_2$ the operator A_ω is completely continuous (theorem 3.1) on $L_{p_1}(R_o)$, it is clear that for every $\omega \in \Omega_2$, the operator $B_\omega(h_\omega, \cdot)$ is also completely continuous for every fixed h_ω belonging to the set S_θ for all $\omega \in \Omega_2$. Thus by Schauder fixed point principle (Krasnoselskii and Rutickii [16, pp. 209]) there is at least one solution

$x_\omega \in S_{r*}$ for every $\omega \in \Omega_2$. In other words, for almost all $\omega \in \Omega$ the system S has a solution $x_\omega \in S_{r*}$.

Thus, it follows from definition 3.1 and theorem 3.2 that the system S is almost surely L_p $(p \geq 1)$ stable.

3.3: AN ILLUSTRATIVE EXAMPLE

Example 3.1:

Let us consider one dimensional heat-conduction in a semi-infinite medium governed by the equations (2.16), (2.17) and (2.18) (example 2.1).

As in example 2.1, the solution of (2.16) satisfying the initial condition (2.17) and the boundary condition (2.18) is given by the expression (2.19). It is assumed in this example that the expression (2.18) satisfies the random Stefan-Boltzmann condition [7, pp 235-236],

i.e. $$G_\omega(t,x) = N\,[q_\omega^4(t) - x^4] \qquad (3.10)$$

for almost all $\omega \in \Omega$, almost all $t \in R_o$ and any $x \in R$.

where

(i) N is a given constant

(ii) $q_\omega(t)$ is the temperature of the surrounding medium and is assumed to be a measurable function on $R_o \otimes \Omega$ such that for every finite interval $I \subset R_o$ $\int_I |q_\omega(t)|^{p_1} dt \leq b < \infty$ for almost all $\omega \in \Omega$, where $p_1 \in (8, \infty)$. It is not difficult to verify that for almost all $\omega \in \Omega$ the random operator A_ω defined by $(A_\omega x)(t) = \int_o^t \frac{G_\omega(\tau, x(\tau))}{(t-\tau)^{1/2}} d\tau$ maps $L_{p_1}(I)$ $(8 < p_1 < \infty)$ into itself and is completely continuous on $L_{p_1}(I)$ (theorem 3.1).

Suppose there exists an arbitrary but fixed $r \in (o, \infty)$ such that $\mu\{\omega \in \Omega: \frac{r}{a_\omega(r)} > \frac{1}{(\pi L)^{1/2}}\} = 1$, where for almost all $\omega \in \Omega$

$a_\omega(r) \triangleq \sup_{x \in S_r} \{ \int_I |A_\omega x(t)|^{p_1} dt \}^{1/p_1}$ with $S_r \triangleq \{x \in L_{p_1}(I)$: $[\int_I |x(t)|^{p_1} dt]^{1/p_1} \leq r < \infty \}$ and L is as in example 2.1, then by use of theorem 3.2, equation 2.20 has a solution $x_\omega(o,t) \in S_r \subset L_{p_1}(I)$ $(8 < p_1 < \infty)$ for almost all $\omega \in \Omega$. Therefore, it is clear that the system under consideration also has a solution $x_\omega(\ell, t) \in S_r \subset L_{p_1}(I)$ for each $\ell \in [o, \infty)$ and for almost all $\omega \in \Omega$.

3.4: ALMOST SURE ASYMPTOTIC STABILITY

It appears to the author that the almost sure asymptotic stability is by far the most important of all the concepts of stochastic stability. This is due to the fact that when one observes the motion of a random dynamical system it is an individual sample path which is observed rather than its mean.

In order to study the question of the almost sure asymptotic stability of the system S, we need the following preparation.

Definition 3.3 : The system S is said to be almost surely asymptotically stable if its solution x has the property that $\mu \{\omega \in \Omega: \overline{\lim}_t |x(t;\omega)| = 0\} = 1$ whenever the free random process h has the property that

$\mu \{\omega \in \Omega: \overline{\lim}_t |h(t;\omega)| = 0\} = 1.$

Lemma 3.2: Consider the random function $D(t;\omega)$ on $R_o \otimes \Omega$ given by

$$D(t;\omega) = \int_o^t K_\omega(t, t-\tau) V_\omega(\tau) d\tau . \qquad (3.13)$$

Suppose that for almost all $\omega \in \Omega$, $\hat{K}_\omega(t) \triangleq \{\int_o^t |K_\omega(t,\tau)|^{q_2} d\tau\}^{1/q_2} \in L_1(R_o)$ and is bounded uniformly on R_o and that for almost all $\omega \in \Omega, V_\omega(t) \in L_{p_2}(R_o)$, where $\frac{1}{p_2} + \frac{1}{q_2} = 1$ $(p_2 > 1)$. Then $\lim_{t \to \infty} |D_\omega(t)| = 0$ for almost all $\omega \in \Omega$.

Proof: Since all the hypotheses of this lemma hold for almost all $\omega \in \Omega$, there exists a set $\Omega_3 \subset \Omega$ so that $\mu(\Omega_3) = \mu(\Omega)$ and that the hypotheses of this lemma hold for every $\omega \in \Omega_3$. Thus for a fixed but arbitrary $\omega \in \Omega_3$, the proof follows from lemma 2.4. Since this is true for every $\omega \in \Omega_3$, the proof is complete.

For convenience we recall the following notations introduced in 3.2:

(i) $x(t;\omega) \triangleq x_\omega(t)$

(ii) $h(t;\omega) \triangleq h_\omega(t)$

(iii) $f(t, x(t;\omega), \omega) \triangleq f_\omega(t, x_\omega(t))$

and (iv) $K(t, \tau, \omega) \triangleq K_\omega(t, \tau)$.

Theorem 3.3: Consider the system S and suppose that all the hypotheses of theorem 3.2 hold and that for almost all $\omega \in \Omega$

$$\hat{K}_\omega(t) \triangleq \left\{\int_0^t |K_\omega(t,\xi)|^{q_2} d\xi\right\}^{1/q_2} \in L_{p_1}(R_o)$$ and is bounded uniformly

on R_o and further $K_\omega(t, \tau)$ is of the form $K(t, t-\tau; \omega) \triangleq K_\omega(t, t-\tau)$.

Then if the free random process h has the property that $\mu\{\omega \in \Omega: \overline{\lim}_t |h_\omega(t)| = 0\} = 1$ the corresponding solution x has the property that $\mu\{\omega \in \Omega: \overline{\lim}_t |x_\omega(t)| = 0\} = 1$.

Proof: Since all the hypotheses of theorem 3.2 are satisfied the system S has a solution $x_\omega \in S_{r^*}$ for almost all $\omega \in \Omega$.

By lemma 3.1, for almost all $\omega \in \Omega$, the random operator f_ω maps $L_{p_1}(R_o)$ into $L_{p_2}(R_o)$ and is continuous and bounded, where $p_1 = a\, p_2$.

Let

$$D_\omega(t) \triangleq \int_o^t K_\omega(t,\tau)\; f(\tau, x(\tau;\omega), \omega)\, d\tau .$$

Using the hypotheses on the stochastic kernel K, it follows from lemma 3.2 that $\lim_{t \to \infty} | D_\omega(t) | = 0$ for almost all $\omega \in \Omega$. Since the stochastic free term h has the property that $\mu\, \{\omega \in \Omega: \overline{\lim}_t\ |h(t;\omega)| = 0\} = 1$, the solution x of the system S has the property that $\mu\, \{\omega \in \Omega: \overline{\lim}_t\ |x(t;\omega)| = 0\} = 1$.

It follows from the definition 3.2 and the theorem 3.3 that the system S is almost surely asymptotically stable.

Remarks: It is interesting to mention that the function f appearing in the system S satisfies only the conditions stated in lemma 3.1, it is not necessary to impose a sector condition [6, pp. 710] or a Lipschitz condition.

In fact given the system S with f and K satisfying only the conditions as stated in the lemma 3.1 and theorem 3.1 the only choice left to the designer is the value of the feedback gain factor λ.

Thus, it is more appropriate to express the stability criterion of a system in terms of a certain admissibility criteria. Precisely, corresponding to a given system S, the quadruple $\{S, V, V_i, V_o\}$ is said to be admissible if there exists a nonempty (in general) linear topological vector space V and two non-empty classes $V_i, V_o \subset V$ so that the map $S: V_i \longrightarrow V_o$ is defined.

Therefore, for the feedback control system S with f and K satisfying only the conditions of lemma 3.1 and theorem 3.1, it follows from theorem 3.2 that the quadruple $\{S, L_{p_1}(R_o), S_\theta, S_{r*}\}$ is admissible, where ,

the set $S_\theta \triangleq \{h \in L_{p_1}(R_o): \{\int_{R_o} |h(t)|^{p_1} dt\}^{1/p_1} \leq \theta\}$

and the set $S_{r*} \triangleq \{x \in L_{p_1}(R_o) : \{\int_{R_o} |x(t)|^{p_1} dt\}^{1/p_1} \leq r*\}$.

CONCLUSIONS

In this thesis we considered the questions of stability of systems described by stochastic nonlinear Volterra integral equations. Particularly the following types of stability were treated in great detail:

(i) stability in the mean $m(m \geq 1)$,

(ii) asymptotic stability in the mean $m(m \geq 1)$,

(iii) almost sure $L_p(p \geq 1)$ stability

and (iv) almost sure asymptotic stability.

It is important to mention that a large class of stochastic ordinary differential equations and stochastic partial differential equations can be reduced to the stochastic integral equation considered in this thesis. Furthermore, the results of this thesis can be extended to multivariable systems without much difficulty.

It is interesting to mention that $L_p(p \geq 1)$ stability in the mean $m(m \geq 1)$ (β σ, table 1, chapter 1) may also be of great practical interest. Similarly the questions of stability in the senses of stability in probability (α δ, table 1, chapter 1); $L_p(p \geq 1)$ stability in probability (β δ, table 1, chapter 1) and asymptotic stability in probability (γ δ, table 1, chapter 1) may have theoretical interest. These questions were not considered in this thesis.

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