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ON THE STABILITY OF DIFFERENTIALLY ROTATING BODIES

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Summary

A variational principle of great power is derived. It is naturally adapted for computers, and may be used to determine the stability of any fluid flow including those in differentially-rotating, self-gravitating stars and galaxies. The method also provides a powerful theoretical tool for studying general properties of eigen-functions, and the relationships between secular and ordinary stability. In particular we prove the anti-spiral theorem indicating that no stable (or steady) mode can have a spiral structure.

1. *Introduction.* The variational principles of Chandrasekhar (Chandrasekhar 1964) and their generalization to uniformly rotating bodies (Clement 1964) may be used to determine the stability and the eigen functions of bodies that are static in suitably chosen axes. However many important astrophysical problems involve a determination of the stability of differentially rotating bodies. Such problems have arisen recently in the fission theory of binary stars (Roxburgh 1966) the theory of spiral structure in galaxies (Toomre 1964, Goldreich & Lynden-Bell 1965, Lin & Shu 1964), W. A. Fowler's superstar model of quasars (Fowler 1966) and the theory of rapidly rotating white dwarfs (Ostriker, Bodenheimer & Lynden-Bell 1966).

We present here a development of the methods of Chandrasekhar and Clement to include all problems of the stability of steady fluid flow. The mechanical analogues of such problems were first systematically treated in Routh's famous but now neglected Adams prize essay on the Stability of Motion (Routh 1877).

Generalization to fluid flow was considered by Kelvin and Tait (Thompson & Tait 1879) and by Lamb (Lamb 1895). Applications to magnetohydrodynamics were considered recently by Frieman & Rotenberg (Frieman & Rotenberg 1960).

The essential step forward from Clement's variational principle is achieved by using the Lagrangian rather than the Eulerian variation of the equations o motion.

The Lagrangian change operator Δ has been defined by Lebovitz (Lebovit 1961) as follows

$$\Delta(Q) = Q[\mathbf{r} + \boldsymbol{\xi}(\mathbf{r}, t), t)] - Q_0(\mathbf{r}, t) \quad (1$$

Here Q and Q_0 are the values of a physical quantity in the perturbed and ur perturbed flows and $\boldsymbol{\xi}(\mathbf{r}, t)$ is the displacement suffered by that fluid elemer which would have been at $\mathbf{r}$ at time t in the unperturbed flow. Hence $Q(\mathbf{r} + \boldsymbol{\xi},$ is the perturbed value of Q for that element. By contrast the Eulerian differen δ is defined by

$$\delta(Q) = Q(\mathbf{r}, t) - Q_0(\mathbf{r}, t); \quad ($$

So to first order in $\boldsymbol{\xi}$ the two operators are related by

$$\Delta = \delta + \boldsymbol{\xi} \cdot \nabla \tag{3}$$

The greater physical utility of Δ as opposed to δ has already been pointed out by Chandrasekhar & Lebovitz (Chandrasekhar & Lebovitz 1964); its importance here arises because, to first order in $\boldsymbol{\xi}$, the operator Δ and the convective operator (D/Dt) commute. This convective rate of change operator is defined thus:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r}, t) \cdot \nabla \tag{4}$$

while we also define

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}_0(\mathbf{r}, t) \cdot \nabla \tag{5}$$

where $\mathbf{u}(r, t)$ is the fluid velocity and the subscript '0' here and elsewhere designates quantities for the unperturbed flow.

Since the authors differ as to the relative importance of physical and mathematical reasoning we shall give two proofs of the first order commutation of Δ and D/Dt.

Consider the perturbed fluid flow and imagine that it is accompanied in an ideal world by a ghostly flow that is unperturbed. We shall concentrate on that fluid element which is at $\mathbf{r}'$ at time t in the real perturbed flow. The ghost of our fluid element will be at $\mathbf{r}$ where $\mathbf{r}' = \mathbf{r} + \boldsymbol{\xi}(\mathbf{r}, t)$. The physical quantity Q defined in the perturbed flow may be considered either as a function of actual position $\mathbf{r}'$ and time t or alternatively as a function of the position $\mathbf{r}$ of the corresponding ghostly fluid element and time. Thus

$$Q = Q(\mathbf{r}', t) = Q(\mathbf{r} + \boldsymbol{\xi}(r, t), t).$$

When Q is considered as written in the first form, i.e. as a function of $\mathbf{r}'$, we shall write Q' for short. Consider now the rate of change of our element's Q as the flow proceeds. It is

$$\frac{D'Q'}{Dt} \equiv \frac{\partial Q'}{\partial t} + \mathbf{u}(\mathbf{r}') \cdot \nabla' Q' \tag{6}$$

However when our element's Q is considered as a function of the position of the corresponding ghostly element the rate of change must be found by following the ghosts' flow, i.e.

$$\frac{D_0}{Dt} [Q(\mathbf{r} + \boldsymbol{\xi}(r, t), t)] \tag{7}$$

Expressions (6) and (7) are alternative forms for the rate of change of our elements Q so they must be equal. Thus

$$\left[\frac{D'Q'}{Dt}\right]_{\mathbf{r}' = \mathbf{r} + \boldsymbol{\xi}} = \frac{D_0 Q}{Dt} \tag{8}$$

Subtracting $D_0 Q_0/Dt$ and using the definition (1) of Δ we obtain

$$\Delta\left(\frac{DQ}{Dt}\right) = \frac{D_0}{Dt}(\Delta Q). \tag{9}$$

This equation is exact.

We now approximate by assuming that $\boldsymbol{\xi}$ is so small that its square is negligible and obtain

$$\Delta\left(\frac{DQ}{Dt}\right)=\frac{D(\Delta Q)}{Dt}=\Delta\left(\frac{D_0Q}{Dt}\right)=\frac{D_0(\Delta Q)}{Dt}. \tag{10}$$

This demonstrates that Δ commutes with both D/Dt and D_0/Dt to this order.

The mathematical proof of equation (8) is now presented

$$\begin{aligned}\frac{D_0}{Dt}[Q(\mathbf{r}+\boldsymbol{\xi}(\mathbf{r},t),t)]&=\left(\frac{\partial}{\partial t}Q(\mathbf{r}+\boldsymbol{\xi}(\mathbf{r},t),t)\right)_{\mathbf{r}}+((\mathbf{u}_0(\mathbf{r}).\nabla)Q(\mathbf{r}+\boldsymbol{\xi}(\mathbf{r},t),t))_t\\&=\left[\frac{\partial Q(\mathbf{r}',t)}{\partial t}\right]_{\mathbf{r}'=\mathbf{r}+\boldsymbol{\xi}}+\frac{\partial\boldsymbol{\xi}}{\partial t}.\nabla'Q'+\mathbf{u}_0(\mathbf{r}).\nabla'Q'\\&\qquad+[(\mathbf{u}_0(\mathbf{r}).\nabla)\boldsymbol{\xi}].\nabla'Q'\\&=\left(\frac{\partial Q'}{\partial t}\right)_{\mathbf{r}'}+\mathbf{u}_0(\mathbf{r}).\nabla'Q'+\frac{D_0\boldsymbol{\xi}(\mathbf{r},t)}{Dt}.\nabla'Q'.\end{aligned} \tag{11}$$

In the above dashed quantities are functions of $\mathbf{r}'$ and $\mathbf{r}'$ is set equal to $\mathbf{r}+\boldsymbol{\xi}$ after the differentiations have been carried out.

Now from the definition of $\boldsymbol{\xi}$

$$\frac{D_0}{Dt}[\boldsymbol{\xi}(\mathbf{r},t)]=\mathbf{u}(\mathbf{r}+\boldsymbol{\xi},t)-\mathbf{u}_0(\mathbf{r},t)=\Delta\mathbf{u}(\mathbf{r},t). \tag{12}$$

Substituting equation (12) into equation (11) two terms cancel and we are left with

$$\frac{D_0Q}{Dt}=\left(\frac{\partial Q'}{\partial t}\right)_{\mathbf{r}'}+\mathbf{u}(\mathbf{r}+\boldsymbol{\xi},t).\nabla'Q'=\left(\frac{D'Q'}{Dt}\right)_{\mathbf{r}'=\mathbf{r}+\boldsymbol{\xi}}$$

which is equation (8).

2. *Derivation of the variational principle*

2.1. *The basic equation.* We wish to consider the most general steady flows. Since these flows, like Jacobi's ellipsoid, may only appear steady when viewed from uniformly rotating axes we write the equations of motion in the form

$$\left[\frac{D\mathbf{u}}{Dt}+2\boldsymbol{\Omega}\times\mathbf{u}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right]=\nabla\psi-\frac{1}{\rho}\nabla p+\frac{\mathbf{F}}{\rho} \tag{13}$$

where

$$\psi(\mathbf{r})=G\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{14}$$

the gravitational potential, p is the pressure, ρ the density and Ω the constan angular velocity of a coordinate system in which the unperturbed flow is stead} $\mathbf{F}$ is the force density of those forces not explicitly included such as frictional an electromagnetic forces.

Taking Δ of equation (13) and multiplying by ρ_0 we obtain

$$\rho_0\left[\Delta\left(\frac{D\mathbf{u}}{Dt}\right)+2\boldsymbol{\Omega}\times\Delta\mathbf{u}+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\Delta\mathbf{r})\right]=\rho_0\Delta\left[\nabla\psi-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\mathbf{F}\right] \tag{1}$$

Now

$$\left.\begin{aligned} &\Delta\mathbf{r}=\boldsymbol{\xi}, \quad \Delta\mathbf{u}=\frac{D_0\boldsymbol{\xi}}{Dt} \\ &\Delta\frac{D\mathbf{u}}{Dt}=\frac{D_0}{Dt}(\Delta\mathbf{u})=\frac{D_0{}^2\boldsymbol{\xi}}{Dt^2} \end{aligned}\right\} \tag{16}$$

and

So equation (15) may be written in the form

$$\rho_0\frac{D_0{}^2\boldsymbol{\xi}}{Dt^2}+2\rho_0\boldsymbol{\Omega}\times\frac{D_0\boldsymbol{\xi}}{Dt}+\rho_0\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{\xi})=\rho_0\Delta\left[\nabla\psi-\frac{1}{\rho}\nabla p+\frac{1}{\rho}\mathbf{F}\right] \tag{17}$$

Accompanying this equation of motion for perturbations we have the equation of continuity

$$\delta\rho+\operatorname{div}(\rho_0\boldsymbol{\xi})=0 \tag{18}$$

which may be written in terms of Lagrangian changes as

$$\Delta\rho+\rho_0\operatorname{div}\boldsymbol{\xi}=0. \tag{19}$$

By contrast with equation (17) which is exact equations (18) and (19) are only true to first order. Our future working is only true to this order.

To determine the evolution of the perturbations we must find the perturbations of the forces, gravity, pressure, etc. We now do this.

Gravity. We find the gravitational potential of the perturbed flow by considering an element of mass dm' displaced from $\mathbf{r}'$ by $\boldsymbol{\xi}(\mathbf{r}', t)$.

Its potential at some point $\mathbf{r}$ was

$$G\frac{dm'}{|\mathbf{r}-\mathbf{r}'|}$$

but it will now be

$$G\frac{dm'}{|\mathbf{r}-(\mathbf{r}'+\boldsymbol{\xi}')|}$$

The change in the potential at $\mathbf{r}$ due to the element is therefore

$$\boldsymbol{\xi}'.\nabla'\left(G\frac{dm'}{|\mathbf{r}-\mathbf{r}'|}\right)=G\,dm'\,\boldsymbol{\xi}'.\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right).$$

Summing over all elements $dm'=\rho_0'd^3r'$ the change in the gravitational potential at r is

$$\delta\psi(\mathbf{r})=G\int\boldsymbol{\xi}'.\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho_0'\,d^3r' \tag{20}$$

or by using Green's theorem

$$\delta\psi=G\int\frac{\rho_0'\boldsymbol{\xi}'}{|\mathbf{r}-\mathbf{r}'|}.d\mathbf{S}'-G\int\frac{\operatorname{div}'(\rho_0'\boldsymbol{\xi}')}{|\mathbf{r}-\mathbf{r}'|}d^3r', \tag{21}$$

where $d\mathbf{S}'$ is the outward normal vectorial surface element of the unperturbed fluid and the volume integral is over the volume of the unperturbed fluid. $\Delta\nabla\psi$ may now be derived from the formula

$$\Delta\nabla\psi=\nabla(\delta\psi)+(\boldsymbol{\xi}.\nabla)\nabla\psi_0$$

Defining the operator $\mathbf{V}$ by

$$-\mathbf{V}.(\boldsymbol{\xi})=\rho_0\nabla\left(G\int\frac{\rho_0'\boldsymbol{\xi}'}{|\mathbf{r}-\mathbf{r}'|}.d\mathbf{S}'-G\int\frac{\operatorname{div}'(\rho_0'\boldsymbol{\xi}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right)+\rho_0(\boldsymbol{\xi}.\nabla)\nabla\psi_0 \tag{22}$$

we have

$$\rho_0\Delta(\nabla\psi)=-\mathbf{V}.(\boldsymbol{\xi}) \tag{23}$$

Pressure. A convenient assumption to make when the basic interest is in dynamical instabilities as opposed to thermal instabilities, is that relative changes in pressure and density are proportional; that is

$$\frac{\Delta p}{p_0}=\gamma\frac{\Delta \rho}{\rho_0}=-\gamma \operatorname{div} \boldsymbol{\xi} \tag{24}$$

This includes the special cases of adiabatic changes ($\gamma=C_p/C_v$) and isothermal changes ($\gamma=1$). However in general γ can be a function of position. For the present we shall use this crude approximation to derive the simplest form of variational principle. We note that it is here that the equation of state and the equations of thermal transport should be used when the above approximation is inadequate. In equation (17) we need $\Delta[(1/\rho)\,\nabla p]$ which may be obtained as follows

$$\begin{aligned}\Delta\left(\frac{1}{\rho}\nabla p\right)&=-\frac{\Delta\rho}{\rho_0{}^2}\nabla p_0+\frac{1}{\rho_0}(\nabla\delta p+\boldsymbol{\xi}\,.\,\nabla(\nabla p_0))\\&=-\frac{\Delta\rho}{\rho_0{}^2}\nabla p_0+\frac{1}{\rho_0}[\nabla(\Delta p)-\nabla((\boldsymbol{\xi}\,.\,\nabla)p_0)+(\boldsymbol{\xi}\,.\,\nabla)\nabla p_0]\end{aligned}$$

Define the linear operator $\mathbf{P}$ by

$$\mathbf{P}\,.\,\boldsymbol{\xi}=\nabla[(1-\gamma)p_0 \operatorname{div}\boldsymbol{\xi}]-p_0\nabla(\operatorname{div}\boldsymbol{\xi})-\nabla[(\boldsymbol{\xi}\,.\,\nabla)p_0]+(\boldsymbol{\xi}\,.\,\nabla)\nabla p_0 \tag{25}$$

Then

$$\rho_0\Delta\left(\frac{1}{\rho}\nabla p\right)=\mathbf{P}\,.\,\boldsymbol{\xi} \tag{26}$$

Other forces. In all cases where the displacement vector suffices to define the deformation of a normal mode we find that the terms in $\mathbf{F}$ in equation (17) can be reduced to a linear operator acting on $\boldsymbol{\xi}$. For simplicity we shall now omit these terms having noted where we may insert them when they are needed. For the infinite conductivity magnetohydrodynamic case see Friemann & Rotenberg (1960).

In the linearized theory we note that because the initial flow was steady, equation (17) for $\boldsymbol{\xi}$ reduces to a linear differentio-integral equation with time independent coefficients. We may therefore Fourier transform in time and each Fourier component will be independent. A typical Fourier component or normal mode will have an $e^{i\omega t}$ dependence. The equations for this normal mode are found by writing

$$\frac{D_0}{Dt}=(i\omega+\mathbf{u}_0\,.\,\nabla) \tag{27}$$

and regarding $\boldsymbol{\xi}$ in the new notation as the complex Fourier coefficient of the mode with an $e^{i\omega t}$ dependence. We include this time dependence in $\boldsymbol{\xi}$. Equation (17) now becomes

$$\rho_0(i\omega+\mathbf{u}_0\,.\,\nabla)^2\boldsymbol{\xi}+2\rho_0\boldsymbol{\Omega}\times(i\omega+\mathbf{u}_0\,.\,\nabla)\boldsymbol{\xi}+\rho_0\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{\xi})=-\mathbf{V}.(\boldsymbol{\xi})-\mathbf{P}.(\boldsymbol{\xi}).$$

which may be reduced to the form

$$-\omega^2\mathbf{A}\,.\,\boldsymbol{\xi}+\omega\mathbf{B}\,.(\boldsymbol{\xi})+\mathbf{T}\,.(\boldsymbol{\xi})=-\mathbf{V}\,.(\boldsymbol{\xi})-\mathbf{P}\,.(\boldsymbol{\xi}) \tag{28}$$

where $\mathbf{A} = \rho_0 \mathbf{I}$ (unit tensor)

$$\mathbf{B}.\boldsymbol{\xi} = 2i\rho_0 \boldsymbol{\Omega} \times \boldsymbol{\xi} + 2i\rho_0(\mathbf{u}_0.\nabla)\boldsymbol{\xi} \tag{29}$$

$$\mathbf{T}.(\boldsymbol{\xi}) = \rho_0(\mathbf{u}_0.\nabla)(\mathbf{u}_0.\nabla)\boldsymbol{\xi} + 2\rho_0\boldsymbol{\Omega} \times [(\mathbf{u}_0.\nabla)\boldsymbol{\xi}] + \rho_0\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi}) \tag{30}$$

We shall show presently that the operators **A, B, T, V, P** are Hermitian and that all except **B** are real whereas **B** is pure imaginary. If we define $\mathbf{C} = \mathbf{T} + \mathbf{V} + \mathbf{P}$ we see that our problem reduces to the eigenvalue problem for ω

$$-\omega^2 \mathbf{A}.\boldsymbol{\xi} + \omega \mathbf{B}.(\boldsymbol{\xi}) + \mathbf{C}.(\boldsymbol{\xi}) = 0. \tag{31}$$

where **A** and **C** are real and Hermitian, **B** is pure imaginary and Hermitian and **A** is positive definite. Equation (31) is our basic equation.

2.2. *Proof that the operators are Hermitian.* We have to show that given any non-singular functions $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ defined in the unperturbed volume of the body and having continuous first derivatives everywhere then for each of our operators **O** say

$$\int \boldsymbol{\eta}^*.\mathbf{O}.(\boldsymbol{\xi})d^3r = \left[\int \boldsymbol{\xi}^*.\mathbf{O}.\boldsymbol{\eta} d^3r\right]^*$$

where stars denote complex conjugates.

A. It is trivial that

$$\int \boldsymbol{\eta}^*.\mathbf{A}.\boldsymbol{\xi} d^3r = \left(\int \rho_0 \boldsymbol{\xi}^*.\boldsymbol{\eta} d^3r\right)^* = \left[\int \boldsymbol{\xi}^*.\mathbf{A}.\boldsymbol{\eta} d^3r\right]^*$$

Furthermore **A** is real since the same relation is true when all the stars are omitted and it is positive definite because

$$\int \boldsymbol{\xi}^*.\mathbf{A}.\boldsymbol{\xi} d^3r = \int \rho_0 \boldsymbol{\xi}^*.\boldsymbol{\xi} d^3r > 0 \quad \text{for } \boldsymbol{\xi} \neq 0.$$

The following proofs are more laborious and the rest of this section should be omitted on first reading.

B. We prove that $\int \boldsymbol{\eta}^*.\mathbf{B}.(\boldsymbol{\xi})d^3r$ is skew in $\boldsymbol{\eta}^*$ and $\boldsymbol{\xi}$ with an imaginary coefficient.

$$\int \boldsymbol{\eta}^*.\mathbf{B}.(\boldsymbol{\xi})d^3r = 2i\int \rho_0\boldsymbol{\eta}^*.(\mathbf{u}_0.\nabla\boldsymbol{\xi} + \boldsymbol{\Omega} \times \boldsymbol{\xi})d^3r$$

$$= 2i\int \{\text{div}\,[\rho_0\mathbf{u}_0(\boldsymbol{\eta}^*.\boldsymbol{\xi})] - \rho_0\boldsymbol{\xi}.(\mathbf{u}_0.\nabla)\boldsymbol{\eta}^* + \rho_0\boldsymbol{\Omega}.(\boldsymbol{\xi} \times \boldsymbol{\eta}^*)\}d^3r.$$

The last equation has been obtained by use of the unperturbed continuity equation. The first term converts into a surface integral which vanishes because the component of the unperturbed fluid velocity perpendicular to the unperturbed surface vanishes ($\mathbf{u}_0.d\mathbf{S} = 0$) so

$$\int \boldsymbol{\eta}^*.\mathbf{B}.\boldsymbol{\xi} d^3r = -2i\int [\boldsymbol{\xi}.(\mathbf{u}_0.\nabla)\boldsymbol{\eta}^* - \boldsymbol{\Omega}(\boldsymbol{\xi} \times \boldsymbol{\eta}^*)]\rho_0 d^3r$$

$$= i\int \{\boldsymbol{\eta}^*.(\mathbf{u}_0.\nabla)\boldsymbol{\xi} - \boldsymbol{\xi}.(\mathbf{u}_0.\nabla)\boldsymbol{\eta}^* + \boldsymbol{\Omega}.(\boldsymbol{\xi} \times \boldsymbol{\eta}^* - \boldsymbol{\eta}^* \times \boldsymbol{\xi})\}\rho_0 d^3r$$

$$= \left(\int \boldsymbol{\xi}^*.\mathbf{B}.(\boldsymbol{\eta})d^3r\right)^* \tag{32}$$

Expression (32) shows that **B** is skew and pure imaginary and therefore Hermitian

T. There are three terms in **T** corresponding to the coefficients of Ω^2, Ω^1 and Ω^0. Each of these is Hermitian so we taken them separately.

$$\int \rho_0 \boldsymbol{\eta}^* . [\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{\xi})] d^3r = -\int \rho_0 (\boldsymbol{\Omega} \times \boldsymbol{\eta}^*) . (\boldsymbol{\Omega} \times \boldsymbol{\xi})$$

hence the term in Ω^2 is Hermitian and symmetric.

$$\begin{aligned} \int 2\rho_0 \boldsymbol{\eta}^* . [\boldsymbol{\Omega} \times (\mathbf{u}_0 . \nabla) \boldsymbol{\xi}] d^3r &= -\int 2\rho_0 (\boldsymbol{\Omega} \times \boldsymbol{\eta}^*) . (\mathbf{u}_0 . \nabla) \boldsymbol{\xi} d^3r \\ &= -2 \int \operatorname{div} [\rho_0 \mathbf{u}_0 (\boldsymbol{\Omega} \times \boldsymbol{\eta}^*) . \boldsymbol{\xi}] d^3r \\ &\qquad + 2 \int \rho_0 \boldsymbol{\xi} . (\mathbf{u}_0 . \nabla)(\boldsymbol{\Omega} \times \boldsymbol{\eta}^*) d^3\tau \end{aligned}$$

Again the div term may be converted into a surface integral which vanishes because $\mathbf{u}_0 . d\mathbf{S} = 0$ so the expression reduces to

$$2 \int \rho_0 \boldsymbol{\xi} . [\boldsymbol{\Omega} \times (\mathbf{u}_0 . \nabla) \boldsymbol{\eta}^*] d^3r$$

demonstrating that the term in Ω is Hermitian.

Finally the term in Ω^0 is

$$\begin{aligned} &\int \rho_0 \boldsymbol{\eta}^* . (\mathbf{u}_0 . \nabla)(\mathbf{u}_0 . \nabla) \boldsymbol{\xi} d^3r \\ &\qquad = \int \rho_0 (\mathbf{u}_0 . \nabla) [\eta^* . (\mathbf{u}_0 . \nabla) \boldsymbol{\xi}] d^3r - \int \rho_0 [(\mathbf{u}_0 . \nabla) \boldsymbol{\eta}^*] . [(\mathbf{u}_0 . \nabla) \boldsymbol{\xi}] d^3r \end{aligned}$$

By the continuity equation for the unperturbed state $\operatorname{div} (\rho_0 \mathbf{u}_0) = 0$ so the first of these terms may be written

$$\int \operatorname{div} \{\rho_0 \mathbf{u}_0 [\boldsymbol{\eta}^* . (\mathbf{u}_0 . \nabla) \boldsymbol{\xi}]\} d^3r$$

which again converts into a vanishing surface integral.

Thus

$$\int \rho_0 \boldsymbol{\eta}^* . (\mathbf{u}_0 . \nabla)(\mathbf{u}_0 \nabla) \boldsymbol{\xi} d^3r = -\int \rho_0 (\mathbf{u}_0 . \nabla) \boldsymbol{\eta}^* . (\mathbf{u}_0 . \nabla) \boldsymbol{\xi} d^3r$$

showing that this term too is Hermitian and symmetric.

Hence, since each of its terms if Hermitian and symmetric so is **T**.

V. From equation (22) we have

$$-\int \boldsymbol{\eta}^* . \mathbf{V} . (\boldsymbol{\xi}) d^3r = \left\{ \begin{aligned} &\int \rho_0 \boldsymbol{\eta}^* . \nabla \left[G \int \frac{\rho_0' \boldsymbol{\xi}'}{|\mathbf{r} - \mathbf{r}'|} . d\mathbf{S}' - G \int \frac{\operatorname{div}' (\rho_0' \boldsymbol{\xi}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' \right] d^3r \\ &\qquad + \int \rho_0 \boldsymbol{\eta}^* . (\boldsymbol{\xi} . \nabla) \nabla \psi_0 d^3r. \end{aligned} \right.$$

$$= \left\{ \begin{aligned} &\int \operatorname{div} \left\{ \rho_0 \boldsymbol{\eta}^* \left[G \int (\;) . d\mathbf{S}' - G \int (\;) d^3r' \right] \right\} d^3r \\ &\qquad - \int \operatorname{div} (\rho_0 \boldsymbol{\eta}^*) \left[G \int (\;) . d\mathbf{S}' - G \int (\;) d^3r' \right] d^3r \\ &\qquad\qquad + \int \rho_0 \eta_i^* \xi_j \partial_j \partial_i \psi_0 d^3r \end{aligned} \right.$$

where the summation convention is assumed and $\partial_i \equiv \partial/\partial x_i$ and the blanks correspond to the integrands of the equation above.

$$-\int \boldsymbol{\eta}^*.\mathbf{V}.\boldsymbol{\xi} d^3r = \begin{cases} G\int \rho_0\boldsymbol{\eta}^* \int \dfrac{\rho_0'\boldsymbol{\xi}'.d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|}.d\mathbf{S} \\ -G\Big[\int\int \operatorname{div} \rho_0\boldsymbol{\eta}^* \int \dfrac{\rho_0'\boldsymbol{\xi}'}{|\mathbf{r}-\mathbf{r}'|}.d\mathbf{S}'d^3r \\ \qquad\qquad + \int \operatorname{div}' \rho_0'\boldsymbol{\xi}' \int \dfrac{\rho_0\boldsymbol{\eta}^*}{|\mathbf{r}-\mathbf{r}'|}.d\mathbf{S}d^3r'\Big] \\ +G\int\int \dfrac{\operatorname{div}(\rho_0\boldsymbol{\eta}^*) \operatorname{div}(\rho_0'\boldsymbol{\xi}')}{|\mathbf{r}-\mathbf{r}'|} d^3r' d^3r \\ +\int \rho_0\eta_i^*\xi_j\partial_i\partial_j\psi_0 d^3r. \end{cases}$$

Thus

$$\int \boldsymbol{\eta}^*.\mathbf{V}.(\boldsymbol{\xi})d^3r = \int \boldsymbol{\xi}.\mathbf{V}.(\eta^*)d^3r = \left[\int \boldsymbol{\xi}^*.\mathbf{V}.(\boldsymbol{\eta})d^3r\right]^*$$

so $\mathbf{V}$ is Hermitian and symmetric.

P. From equation (25) we have

$$\int \boldsymbol{\eta}^*.\mathbf{P}.(\boldsymbol{\xi})d^3r = \begin{cases} \int \{\operatorname{div}[(1-\gamma)p_0\boldsymbol{\eta}^* \operatorname{div}\boldsymbol{\xi}] - (1-\gamma)p_0 \operatorname{div}\boldsymbol{\eta}^* \operatorname{div}\boldsymbol{\xi}\}d^3r \\ -\int \{p_0(\boldsymbol{\eta}^*.\nabla)\operatorname{div}\boldsymbol{\xi} + \boldsymbol{\eta}^*.\nabla(\boldsymbol{\xi}.\nabla)p_0 - \boldsymbol{\eta}^*.(\boldsymbol{\xi}.\nabla)(\nabla p_0)\}d^3r. \end{cases}$$

p_0 vanishes at the surface so the first term vanishes and the last two terms may be simplified by using the formulae

$$\begin{aligned} \operatorname{div}[p_0(\boldsymbol{\eta}^*.\nabla)\boldsymbol{\xi}] &= p_0(\boldsymbol{\eta}^*.\nabla)\operatorname{div}\boldsymbol{\xi} + (\nabla p_0).(\boldsymbol{\eta}^*.\nabla)\boldsymbol{\xi} + p_0(\partial_i\eta^*_j)(\partial_j\xi_i^*) \\ &= p_0(\boldsymbol{\eta}^*.\nabla)\operatorname{div}\boldsymbol{\xi} + (\boldsymbol{\eta}^*.\nabla)(\boldsymbol{\xi}.\nabla p_0) - \boldsymbol{\eta}^*.(\boldsymbol{\xi}.\nabla)\nabla p_0 \\ &\qquad + p_0(\partial_i\eta^*_j)(\partial_j\xi_i). \end{aligned}$$

Thus we obtain throwing away another surface term because p_0 is zero there

$$\int \boldsymbol{\eta}^*.\mathbf{P}.(\xi) = \int (\gamma-1)p_0 \operatorname{div}\boldsymbol{\eta}^* \operatorname{div}\boldsymbol{\xi} + p_0(\partial_i\eta_j^*)(\partial_j\xi_i)d^3r.$$

So $\mathbf{P}$ is symmetric and Hermitian. Note that we have not assumed that γ is independent of position.

2.3. *Properties of eigenfunctions and eigenfrequencies.* Our basic equation (31) is

$$-\omega^2\mathbf{A}.\boldsymbol{\xi} + \omega\mathbf{B}.\boldsymbol{\xi} + \mathbf{C}.\boldsymbol{\xi} = 0. \tag{31}$$

The only boundary condition is $\Delta p = 0$ at the surface. By equations (19) and (24) this is satisfied already (since p_0 is zero there) provided that div $\boldsymbol{\xi}$ is non-singular. Taking complex conjugates and remembering the properties of **A**, **B** and **C** we have

$$-\omega^{*2}\mathbf{A}.\boldsymbol{\xi}^* - \omega^*\mathbf{B}.\boldsymbol{\xi}^* + \mathbf{C}.\boldsymbol{\xi}^* = 0 \tag{33}$$

Comparing equations (31) and (32) we see that if $\boldsymbol{\xi}$ is a right-eigenvector with eigenvalue ω then $\boldsymbol{\xi}^*$ is a right-eigenvector with eigenvalue $-\omega^*$. Thus even when ω is real the eigenvalues come in positive and negative pairs. We now show that this property is extended when ω is complex so that eigenvalues come in quartets. Taking the transpose of equation (31)

$$-\omega^2 \boldsymbol{\xi}.\mathbf{A} - \omega \boldsymbol{\xi}.\mathbf{B} + \boldsymbol{\xi}.\mathbf{C} = 0 \tag{34}$$

thus $-\omega$ is an eigenvalue with left-eigenvector $\boldsymbol{\xi}$. Similarly by taking the complex conjugate of equation (34) ω^* is an eigenvalue with left eigenvector $\boldsymbol{\xi}^*$. So ω, $-\omega$, ω^*, $-\omega^*$ are all eigenvalues together. These properties depend not only on time reversal invariance but also on the Lagrangian nature of the fundamental equations. They do not all hold when dissipation is introduced.

2.4. *Sufficient conditions for stability.* If we multiply equation (31) by $\boldsymbol{\xi}^*$. and integrate over the volume occupied by the unperturbed fluid we obtain the equation

$$-\omega^2 a + \omega b + c = 0 \tag{35}$$

where a, b, c are the real numbers

$$c = \int \boldsymbol{\xi}^*.\mathbf{C}.\boldsymbol{\xi} d^3r \text{ etc.}$$

Solving equation (35) we have

$$\omega = +\frac{b}{2a} \pm \left(\frac{b^2}{4a^2} + \frac{c}{a}\right)^{1/2} \tag{36}$$

By the properties proved in Section 2.2 a is positive while b and c are real so equation (36) shows that the system is stable if c is positive for each eigen $\boldsymbol{\xi}$. This is assured if $\mathbf{C}$ is positive definite for then c is positive for any $\boldsymbol{\xi}$. Thus:

A sufficient condition for stability is that $\mathbf{C}$ is positive definite.

This is *the* condition for secular stability. (See Section 4.)

From the form of equation (36) it is clear that a necessary and sufficient condition for stability is that for each eigen $\boldsymbol{\xi}$,

$$\left[\left(\int \boldsymbol{\xi}^*.\mathbf{B}.\boldsymbol{\xi} d^3r\right)^2 + 4\int \boldsymbol{\xi}^*.\mathbf{A}.\boldsymbol{\xi} d^3r \int \boldsymbol{\xi}^*.\mathbf{C}.\boldsymbol{\xi} d^3r\right] \geqslant 0 \tag{37}$$

This criterion cannot yet be stated in a form that does not involve the unknown eigen $-\boldsymbol{\xi}$s; however, a sufficient condition may be derived by demanding that this expression be positive semi-definite not just for the eigen-$\boldsymbol{\xi}$ but for any non-singular $\boldsymbol{\xi}$ whatsoever. In practice this condition is neither as simple nor as useful than the more restrictive secular stability condition that $\mathbf{C}$ should be positive definite.

2.5. *The variational principle.* We have just shown that the eigenfrequencies ω are often real. We now show that when an ω is real it is the stationary value of a variational expression. In order to show how this proof breaks down when ω is in fact complex we shall not assume that ω is real from the outset but only where it becomes necessary.

Following Clement (Clement 1964) we consider the expression

$$-\omega^2 a + \omega b + c = 0 \tag{38}$$

where

$$c = \int \boldsymbol{\xi}^*.\mathbf{C}.\boldsymbol{\xi} d^3r \text{ etc.}$$

and $\boldsymbol{\xi}$ is a trial function which is finite and has finite derivatives in the volume occupied by the unperturbed fluid. We now show that those ω which are the solutions of equation (38) and which are stationary for arbitrary small variations of the $\boldsymbol{\xi}$ appearing in a, b, c are the real eigenfrequencies. Taking δ of equation (38) and putting $\delta\omega = 0$ we have

$$\int(-\omega^2\boldsymbol{\xi}^*.\mathbf{A}+\omega\boldsymbol{\xi}^*.\mathbf{B}+\boldsymbol{\xi}^*.\mathbf{C}).\delta\boldsymbol{\xi}+\delta\boldsymbol{\xi}^*.(-\omega^2\mathbf{A}.\boldsymbol{\xi}+\omega\mathbf{B}.\boldsymbol{\xi}+\mathbf{C}.\boldsymbol{\xi})d^3r=0$$

Since $\delta\xi$ and $\delta\xi^*$ are algebraicly independent and arbitrary each coefficient must vanish, that is

$$-\omega^2\mathbf{A}.\boldsymbol{\xi}+\omega\mathbf{B}.\boldsymbol{\xi}+\mathbf{C}.\boldsymbol{\xi}=0 \tag{39}$$

$$-\omega^2\boldsymbol{\xi}^*.\mathbf{A}+\omega\boldsymbol{\xi}^*.\mathbf{B}+\boldsymbol{\xi}^*.\mathbf{C}=0 \tag{40}$$

When ω is real equation (40) is the Hermitian conjugate of equation (39) so they are both equivalent to our basic equation (31). Thus the real eigenfrequencies are stationary and the stationary real frequencies are eigen-frequencies. We now come to the strange situation when an eigenfrequency is not real. Then equation (39) is our desired equation but equation (40) far from being its Hermitian conjugate actually conflicts with it. In this case there will normally be no stationary complex frequencies obeying (39) and (40). In spite of this difficulty we show in the next sections that it is still possible to find approximations to the eigenfrequencies.

2.6. *Practical use of the variational principle.* Consider the variation to be performed by varying the coefficients a_i in the trial function

$$\boldsymbol{\xi}=\sum_{i=1}^{N} a_i\boldsymbol{\xi}^{(i)} \tag{41}$$

where the $\boldsymbol{\xi}^{(i)}$ are any fixed functions (e.g. polynomial functions of position). Define matrices A_{ij}, B_{ij}, C_{ij} by

$$A_{ij}=\int\boldsymbol{\xi}^{(i)*}.\mathbf{A}.\boldsymbol{\xi}^{(j)}d^3r \text{ etc.} \tag{42}$$

Our variational principle now reads

$$\delta(a_i^*a_j(-\omega^2A_{ij}+\omega B_{ij}+C_{ij}))=0$$

where the summation convention is assumed. Varying the a_i^* and the a_j we obtain

$$(-\omega^2A_{ij}+\omega B_{ij}+C_{ij})a_j=0 \tag{43}$$

$$a_i^*(-\omega^2A_{ij}+\omega B_{ij}+C_{ij})=0 \tag{44}$$

Once again these are Hermitian conjugates of each other if ω is real and are in conflict if ω is complex. However both of them lead to the secular determinant

$$|-\omega^2A_{ij}+\omega B_{ij}+C_{ij}|=0 \tag{45}$$

for the determination of the variationally best eigenfrequencies ω.

2.7. *Complex frequencies.* Write $(-\omega^2\mathbf{A}+\omega\mathbf{B}+\mathbf{C}).\boldsymbol{\xi}=\mathbf{Z}(\omega).\boldsymbol{\xi}$.
Consider the variational principle

$$\delta\left(\int\boldsymbol{\eta}^*.\mathbf{Z}(\omega).\boldsymbol{\xi}d\tau\right)=0 \tag{46}$$

for $\delta\omega = 0$ and all small variations in two trial functions $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. This principle leads to

$$\boldsymbol{\eta}^* . \mathbf{Z}(\omega) = 0$$

and

$$\mathbf{Z}(\omega) . \boldsymbol{\xi} = 0$$

which are not in conflict. In particular the properties of $\mathbf{Z}$ allow us to take the Hermitian conjugate of the first equation and write it

$$\mathbf{Z}(\omega^*) . \boldsymbol{\eta} = 0.$$

$\boldsymbol{\eta}$ is thus the eigenvector with eigenvalue ω^*.

Once again we take trial functions

$$\left.\begin{aligned} \boldsymbol{\xi} &= \sum_i^N a_j \boldsymbol{\xi}^{(j)} \\ \boldsymbol{\eta}^* &= \sum_i^N b_i \boldsymbol{\xi}^{(i)*} \end{aligned}\right\} \tag{47}$$

where the a_i and the b_i are to be varied.

$$\delta\left(\int \boldsymbol{\eta}^* . \mathbf{Z}(\omega) . \boldsymbol{\xi} d\tau\right) = 0$$

leads to the equations

$$\left.\begin{aligned} Z_{ij}(\omega) a_i &= 0 \\ b_i Z_{ij}(\omega) &= 0 \end{aligned}\right\} \tag{48}$$

where

$$Z_{ij} = \int \boldsymbol{\xi}^{(i)*} \mathbf{Z}(\omega) \boldsymbol{\xi}^{(j)} d\tau.$$

Equations (47) and (48) lead to equations for the best eigenfrequencies of the form

$$|Z_{ij}(\omega)| = 0 \tag{49}$$

which is just equation (45) written in our more compact notation. Owing to the properties of the matrices **A**, **B**, **C** it will be found that this determinant always leads to an Nth degree equation in ω^2 which has real coefficients. This demonstrates that in approximations at any level ω, ω^*, $-\omega$, $-\omega^*$ are all eigenvalues.

Before leaving the general case it should be remarked that there is a class of displacements all of which have eigenfrequency zero; the momenta corresponding to the displacement coordinates will be constant, and an application of Routh's procedure might lead to a great simplification of the problem. If $\boldsymbol{\xi}$ is zero except on a particular stream line of the unperturbed flow and is of the form $\epsilon \mathbf{u}_0(\mathbf{r})$ for all points on that stream line (where ϵ is a small parameter) then the density distribution etc. do not change $\mathbf{C} . \boldsymbol{\xi} = 0$ and thus $\omega = 0$.

We should also remark that where the unperturbed system possesses symmetry it is possible to apply group theory and reduce the problem to sets of modes which transform as the irreducible representations of the symmetry group. The method can then be used to determine the low lying eigenfrequencies of modes of each symmetry type *separately*. More details of this method applied to axially symmetrical steady states are given below.

3. *Systems with axial symmetry.* For axially symmetrical systems the oscillation eigenfunctions can be taken to transform like the irreducible representations of the rotation group. Thus $\boldsymbol{\xi}$ may be taken to have an exp (im ϕ) dependence and the different positive, negative and zero integers m give independent sets of eigenfunctions. However before discussing such modes for general m it is interesting to show that there is a far simpler variational principle for the axially symmetrical ($m=0$) oscillations of an axially symmetrical system.

3.1. *Axially symmetrical oscillations.* For such oscillations each element of fluid preserves not only its mass but also its angular momentum about the axis. Thus each element preserves its angular momentum per unit mass. That is

$$\Delta h=0$$

where

$$h=R\mathbf{u}\,.\,\hat{\boldsymbol{\phi}}$$

and R is the distance from the axis about which ϕ is measured. $\hat{\boldsymbol{\phi}}$ is the unit vector in the toroidal direction if increasing ϕ.

$$\Delta h=\Delta(R\mathbf{u}\,.\,\hat{\boldsymbol{\phi}})=\Delta R\mathbf{u}_0\,.\,\hat{\boldsymbol{\phi}}+R(\Delta\mathbf{u})\,.\,\hat{\boldsymbol{\phi}}+R\mathbf{u}_0\,.(\Delta\hat{\boldsymbol{\phi}})=0 \tag{50}$$

hence

$$\xi_R u_{0\phi}+R\,\frac{D_0\boldsymbol{\xi}}{Dt}\,.\,\hat{\boldsymbol{\phi}}+R\mathbf{u}_0\,.\left(\frac{\xi_\phi(-\hat{\mathbf{R}})}{R}\right)=0 \tag{51}$$

that is

$$2u_{0\phi}\xi_R+i\omega R\xi_\phi+R(\mathbf{u}_{0M}\,.\,\nabla)\xi_\phi-u_{0R}\xi_\phi=0 \tag{52}$$

where $\mathbf{u}_{0M}$ is the vector obtained from $\mathbf{u}_0$ by setting the ϕ component equal to zero. The above relationship may be obtained from the ϕ component of the equation of motion to which it is equivalent. We now use it to eliminate ξ_ϕ from the other components of the equation of motion and thence we obtain a variational principle independent of ξ_ϕ.

We take the equations of motion in the form of equation (17) with $\boldsymbol{\Omega}$ the angular velocity of the axes set equal to zero. Examining the R component

$$\frac{D_0{}^2(\boldsymbol{\xi})}{Dt^2}\,.\,\hat{\mathbf{R}}=\frac{D_0}{Dt}\left(\hat{\mathbf{R}}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}\right)-\frac{D_0\hat{\mathbf{R}}}{Dt}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}=\frac{D_0}{Dt}\left(\hat{\mathbf{R}}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}\right)-\frac{u_{0\phi}}{R}\hat{\boldsymbol{\phi}}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}$$

$$=\frac{D_0{}^2}{Dt^2}(\xi_R)-\frac{D_0}{Dt}\left(\frac{u_{0\phi}}{R}\hat{\boldsymbol{\phi}}\,.\,\boldsymbol{\xi}\right)-\frac{u_{0\phi}}{R}\hat{\boldsymbol{\phi}}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}$$

$$=\frac{D_0{}^2}{Dt^2}(\xi_R)-\frac{2u_{0\phi}}{R}\hat{\boldsymbol{\phi}}\,.\,\frac{D_0\boldsymbol{\xi}}{Dt}-\boldsymbol{\xi}\,.\,\frac{D_0}{Dt}\left(\frac{u_{0\phi}}{R}\hat{\boldsymbol{\phi}}\right)$$

substituting for $\hat{\boldsymbol{\phi}}\,.\,\dfrac{D_0\boldsymbol{\xi}}{Dt}$ from equation (51) we obtain

$$\frac{D_0{}^2(\boldsymbol{\xi})}{Dt^2}\,.\,\hat{\mathbf{R}}=\frac{D_0{}^2}{Dt^2}(\xi_R)+\frac{2u_{0\phi}}{R^2}(u_{0\phi}\xi_R-u_{0R}\xi_\phi)-\frac{\xi_\phi}{R}\frac{D_0}{Dt}(u_{0\phi})+\frac{\xi_\phi u_{0\phi}}{R^2}u_{0R}+\frac{u_{0\phi}{}^2}{R^2}\xi_R$$

$$=\frac{D_0{}^2}{Dt^2}(\xi_R)+\frac{3u_{0\phi}{}^2}{R^2}\xi_R-\frac{\xi_\phi}{R}\left(\frac{Du_{0\phi}}{Dt}+\frac{u_{0\phi}u_{0R}}{R}\right)$$

The last term is
$$\frac{\xi_\phi}{R}\frac{D_0(\mathbf{u}_0)}{Dt}\cdot\hat{\boldsymbol{\Phi}}$$

which is zero by virtue of the equation of motion of the unperturbed axially symmetrical state.

Thus
$$\frac{D_0{}^2(\boldsymbol{\xi})}{Dt^2}\cdot\hat{\mathbf{R}}=\frac{D_0{}^2}{Dt^2}\,\xi_R+\frac{3u_{0\phi}{}^2}{R^2}\,\xi_R \tag{53}$$

Since ξ_R is independent of ϕ the $\hat{\mathbf{R}}$ component of the equations of motion may be written, using the notations

$$\frac{D_M}{Dt}=\frac{\partial}{\partial t}+(u_{0M}\cdot\nabla), \tag{54}$$

$$\mathbf{u}_{0M}=(u_{0R},\ 0,\ u_{0z}). \tag{55}$$

$$\rho_0\left(\frac{D_M{}^2}{Dt^2}\,\xi_R+\frac{3u_{0\phi}{}^2}{R^2}\,\xi_R\right)=\left\{\begin{array}{l}\rho_0\delta\left(\dfrac{\partial\psi}{\partial R}-\dfrac{1}{\rho}\dfrac{\partial p}{\partial R}\right)\\ +\rho_0\left[\hat{\mathbf{R}}\cdot(\boldsymbol{\xi}\cdot\nabla)\left(\nabla\psi_0-\dfrac{1}{\rho_0}\,\nabla p_0\right)\right]\end{array}\right\}.$$

The last square bracket simplifies to

$$\boldsymbol{\xi}\cdot\nabla\left(\frac{\partial\psi_0}{\partial R}-\frac{1}{\rho_0}\frac{\partial p_0}{\partial R}\right)-\frac{\xi_\phi}{R}\left[\hat{\boldsymbol{\Phi}}\cdot\left(\nabla\psi_0-\frac{1}{\rho_0}\,\nabla p_0\right)\right]$$

of which the last bracket vanishes by virtue of the axial symmetry of the equilibrium state.

We thus have

$$\rho_0\left(\frac{D_M{}^2}{Dt^2}\,\xi_R+\frac{3u_{0\phi}{}^2}{R^2}\,\xi_R\right)=\rho_0\delta\left(\frac{\partial\psi}{\partial R}-\frac{1}{\rho}\frac{\partial p}{\partial R}\right)+\boldsymbol{\xi}_M\cdot\nabla\left(\frac{\partial\psi_0}{\partial R}-\frac{1}{\rho_0}\frac{\partial p_0}{\partial R}\right) \tag{56}$$

where $\boldsymbol{\xi}_M=(\xi_R,\ 0,\ \xi_z)$ that is $\boldsymbol{\xi}_M$ is $\boldsymbol{\xi}$ but with the toroidal component suppressed and this reduction is again possible by virtue of the symmetry. The suffix M stands for meridional components. The z component of equation (17) reduces directly to a form similar to equation (56) with which we combine it to obtain the vector equation

$$\rho_0\left(\frac{D_M{}^2}{Dt^2}\,\boldsymbol{\xi}_M+\frac{3u_{0\phi}{}^2}{R^2}\,\hat{\mathbf{R}}\xi_R\right)=\rho_0\nabla\delta\psi+\frac{\delta\rho}{\rho_0}\,\nabla p-\nabla\delta p+(\boldsymbol{\xi}_M\cdot\nabla)\left(\nabla\psi_0-\frac{1}{\rho_0}\,\nabla p_0\right) \tag{57}$$

It should be emphasized that δp, $\delta\rho$ and $\delta\psi$ are independent of ξ_ϕ owing to th axial symmetry of the perturbation. Thus equation (57) only involves $\boldsymbol{\xi}_M$ an its components. We note that if $\mathbf{u}_{0M}$ is written for $\mathbf{u}$ and $\boldsymbol{\xi}_M$ is written for $\boldsymbol{\xi}$ the equation (17) ($\boldsymbol{\Omega}=0$) becomes equation (57) except for the term

$$\rho_0\,\frac{3u_{0\phi}{}^2}{R^2}\,\xi_R\hat{\mathbf{R}}.$$

Following our earlier work we may reduce equation (57) to the form

$$-\omega^2\mathbf{A}_M\,.\,\boldsymbol{\xi}_M+\omega\mathbf{B}_M\,.\,\boldsymbol{\xi}_M+\mathbf{C}_M\,.\,\boldsymbol{\xi}_M=0 \tag{58}$$

where

$$\left.\begin{aligned}\mathbf{A}_M&=\rho_0\mathbf{I}\\ \mathbf{B}_M\,.\,\boldsymbol{\xi}_M&=2i\rho_0(\mathbf{u}_{0M}\,.\,\nabla)\boldsymbol{\xi}_M\\ \mathbf{C}_M&=\mathbf{T}_M+\mathbf{V}+\mathbf{P}\\ \mathbf{T}_M\,.\,\boldsymbol{\xi}_M&=\rho_0(\mathbf{u}_{0M}\nabla)(\mathbf{u}_{0M}\,.\,\nabla)\boldsymbol{\xi}_M+\rho_0\frac{3u_{0\phi}{}^2}{R^2}\hat{\mathbf{R}}\xi_R\end{aligned}\right\} \tag{59}$$

and $\mathbf{V}$ and $\mathbf{P}$ are the same operators as before defined by equations (22) and (26). (Note however that they now operate on $\boldsymbol{\xi}_M$ rather than $\boldsymbol{\xi}$.) It is noticeable that when the motions are purely toroidal in the unperturbed state $\mathbf{u}_{0M}\equiv 0$ and $\mathbf{B}_M$ vanishes. In that case we obtain a simple eigenvalue problem for ω^2 and the system will be stable or unstable to axially symmetric modes according as $\mathbf{C}_M$ is positive semi-definite* or not. Thus in this situation the secular and ordinary stability criteria for axially symmetric displacements coincide.

3.2. *Non-axially-symmetric modes.* These modes split into groups according to the value of the positive integer m where the angular dependence of $\boldsymbol{\xi}$ is $\exp\,(im\,\phi)$. One wants a variational principle for the modes of a definite m. This is easily obtained as follows: consider the general variational principle with $\boldsymbol{\Omega}=0$ but wherever $(\mathbf{u}_0\,.\,\nabla)$ occurs write it in the form

$$(\mathbf{u}_0\,.\,\nabla)\mathbf{Q}=(\mathbf{u}_{0M}\,.\,\nabla)\mathbf{Q}+\frac{im\,u_{0\phi}}{R}\mathbf{Q}+\frac{u_{0\phi}}{R}\hat{\mathbf{z}}\times\mathbf{Q} \tag{60}$$

when it operates on vectors $\mathbf{Q}$ and omit the last term when it operates on scalars Q. This together with the rule that the third component of grad Q is $(im/R)Q$ gives us the required variational principle.

3.3. *Equilibrium systems with no meridional circulation.* When $\mathbf{u}_{0M}\equiv 0$ some further interesting results follow. By use of equation (60), equation (28) may be written

$$\begin{pmatrix}-\omega^2\rho_0\xi_R+2\omega\rho_0\dfrac{u_0}{R}(-m\xi_R-i\xi_\phi)-\rho_0\dfrac{u_0{}^2}{R^2}[(m^2+1)\xi_R+2im\xi_\phi]\\[2ex] -\omega^2\rho_0\xi_\phi+2\omega\rho_0\dfrac{u_0}{R}(-m\xi_\phi+i\xi_R)-\rho_0\dfrac{u_0{}^2}{R^2}[(m^2+1)\xi_\phi-2im\xi_R]\\[2ex] -\omega^2\rho_0\xi_z+2\omega\dfrac{u_0}{R}(-m\xi_z)-\rho_0\dfrac{u_0{}^2}{R^2}m^2\xi_z\end{pmatrix}=-(\mathbf{V}+\mathbf{P})\,.\,\boldsymbol{\xi}$$

We have written the left hand side as a column vector to show explicitly that when the second row is multiplied by i and added to the first only the combination $\xi^+=\xi_R+i\xi_\phi$ appears. The same thing happens on the R.H.S. Similarly when the second row is multiplied by $-i$ only $\xi^-=\xi_R-i\xi_\phi$ occurs. We therefore take the

* Instability can sometimes occur when $\mathbf{C}_M$ is positive semi-definite but these are case of marginal instability which are normally easy to detect by proceding further along sequence of equilibria and seeing if the mode immediately becomes unstable.

complex vectorial components and write

$$\left.\begin{matrix}\left[-\omega^2\rho_0-2\omega\rho_0\dfrac{u_0}{R}(m+1)-\rho_0\dfrac{u_0^2}{R^2}(m+1)^2\right]\xi^+ \\ \left[-\omega^2\rho_0-2\omega\rho_0\dfrac{u_0}{R}(m-1)-\rho_0\dfrac{u_0^2}{R^2}(m-1)^2\right]\xi^- \\ \left[-\omega^2\rho_0-2\omega\rho_0\dfrac{u_0}{R}m-\rho_0\dfrac{u_0^2}{R^2}m^2\right]\xi_z\end{matrix}\right\}=-(\mathbf{V}_m+\mathbf{P}_m).\begin{pmatrix}\xi^+\\ \xi^-\\ \xi_z\end{pmatrix} \quad (61)$$

Here $\mathbf{V}_m$ and $\mathbf{P}_m$ are obtained directly from $\mathbf{V}$ and $\mathbf{P}$ and they will now be shown to be real. The weary reader should take this on trust as they are somewhat tedious to calculate. Using the relations

$$\operatorname{div}\boldsymbol{\xi}=\frac{1}{R}\frac{\partial}{\partial R}\left(R\left(\frac{\xi^++\xi^-}{2}\right)\right)+\frac{m}{R}\left(\frac{\xi^+-\xi^-}{2}\right)+\frac{\partial\xi_z}{\partial z}$$

$$(\boldsymbol{\xi}.\nabla)p_0=\left(\frac{\xi^++\xi^-}{2}\right)\frac{\partial p_0}{\partial R}+\xi_z\frac{\partial p_0}{\partial z}$$

$$(\boldsymbol{\xi}.\nabla)\nabla p_0=\frac{\xi^++\xi^-}{2}\frac{\partial}{\partial R}\begin{pmatrix}\dfrac{\partial p_0}{\partial R}\\ 0\\ \dfrac{\partial p_0}{\partial z}\end{pmatrix}+\frac{\xi^+-\xi^-}{2iR}\begin{pmatrix}0\\ \dfrac{\partial p_0}{\partial R}\\ 0\end{pmatrix}+\xi_z\frac{\partial}{\partial z}(\nabla p_0)$$

We find that $\mathbf{P}_m$ is a purely real operator on

$$\begin{pmatrix}\xi^+\\ \xi^-\\ \xi_z\end{pmatrix}$$

We write

$$\mathbf{P}_m.\begin{pmatrix}\xi^+\\ \xi^-\\ \xi_z\end{pmatrix}=\begin{pmatrix}P_m^{++} & P_m^{+-} & P_m^{+z}\\ P_m^{-+} & P_m^{--} & P_m^{-z}\\ P_m^{z+} & P_m^{z-} & P_m^{zz}\end{pmatrix}\begin{pmatrix}\xi^+\\ \xi^-\\ \xi_z\end{pmatrix}$$

and use the notation $P_m^{\pm+}$ for the component P_m^{++} if upper signs are taken and P_m^{-+} if lower signs are taken.

For convenience we define

$$\operatorname{div}^+\xi^+=\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{R}{2}\xi^+\right)+\frac{m}{2R}\xi^+$$

and $\operatorname{div}^-\xi^-$ as the same expression with $-m$ written for m and ξ^- written for ξ^+. With this notation we find

$$P_m^{\pm+}\xi^+=\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\left\{(1-\gamma)p_0\operatorname{div}^+\xi^+-\frac{\partial p_0}{\partial R}\frac{\xi^+}{2}\right\}$$
$$-p_0\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\operatorname{div}^+\xi^++\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial p_0}{\partial R}\right)\frac{\xi^+}{2}$$

$P_m{}^{\pm-}\xi^-$ follows by replacing the superscripts + (superscripts *only*) by superscripts −.

$$P_M{}^{\pm z}\,\xi_z = \left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\left\{(1-\gamma)p_0\,\frac{\partial\xi_z}{\partial z}-\frac{\partial p_0}{\partial z}\,\xi_z\right\}-p_0\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\frac{\partial\xi_z}{\partial z}+\xi_z\,\frac{\partial}{\partial z}\left(\frac{\partial p_0}{\partial R}\right)$$

$$P_m{}^{zz}\xi_z = \frac{\partial}{\partial z}\left((1-\gamma)p_0\,\frac{\partial\xi_z}{\partial z}-\frac{\partial p_0}{\partial z}\,\xi_z\right)-p_0\,\frac{\partial^2\xi_z}{\partial z^2}+\xi_z\,\frac{\partial^2}{\partial z^2}\,p_0.$$

Similarly we obtain for $\mathbf{V}_m$

$$V_m{}^{\pm+}\xi^+ = -\rho_0\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\left[G\int\frac{\rho_0'\xi^{+\prime}}{2\,|\mathbf{r}-\mathbf{r}'|}\,\hat{\mathbf{R}}'.d\mathbf{S}'-G\int\frac{\mathrm{div}^{+\prime}(\rho_0'\xi^{+\prime})}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'\right]$$

where $\hat{\mathbf{R}}'$ is the unit vector outwards from the axis and dashes denote functions of $\mathbf{r}'$.

$$V_m{}^{\pm z}\xi_z = -\rho_0\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\left[G\int\frac{\rho_0'\xi_z'}{|\mathbf{r}-\mathbf{r}'|}\,\hat{\mathbf{z}}'.d\mathbf{S}'-G\int\frac{\frac{\partial}{\partial z'}(\rho_0'\xi_z')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'\right] + \xi_z\left(\frac{\partial}{\partial R}\mp\frac{m}{R}\right)\frac{\partial\psi_0}{\partial z}$$

$$V_m{}^{zz}\xi_z = -\rho_0\,\frac{\partial}{\partial z}\left[G\int\frac{\rho_0'\xi_z'}{|\mathbf{r}-\mathbf{r}'|}\,\hat{\mathbf{z}}'.d\mathbf{S}'-G\int\frac{\frac{\partial}{\partial z'}(\rho_0'\xi_z')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'\right]+\rho_0\xi_z\,\frac{\partial^2}{\partial z^2}\,\psi_0$$

$V_m{}^{\pm-}\xi^-$ is obtained from $V^{\pm+}\xi^+$ by replacing superscripts + by superscripts −. It is important to note that the operators in equation (61) are all real and symmetric. We may therefore write

$$(-\omega^2\mathbf{A}_m+\omega\mathbf{B}_m+\mathbf{C}_m).\zeta = 0 \qquad (62)$$

where $\boldsymbol{\zeta}$ stands for the column $\begin{pmatrix}\xi^+\\ \xi^-\\ \xi_z\end{pmatrix}\exp\left[-(im\phi+i\omega t)\right]$

Notice the $\boldsymbol{\zeta}$ is defined to have neither ϕ dependence nor t dependence.

$$\mathbf{A}_m = \rho_0\mathbf{I}\ \text{(unit tensor)}$$

$$\mathbf{B}_m = -2\rho_0\,\frac{u_0}{R}\begin{pmatrix} m+1 & 0 & 0\\ 0 & m-1 & 0\\ 0 & 0 & m\end{pmatrix}$$

$$\mathbf{C}_m = -\rho_0\,\frac{u_0{}^2}{R^2}\begin{pmatrix}(m+1)^2 & 0 & 0\\ 0 & (m-1)^2 & 0\\ 0 & 0 & m^2\end{pmatrix}+\mathbf{V}_m+\mathbf{P}_m.$$

Note that $\mathbf{A}_m$, $\mathbf{B}_m$ and $\mathbf{C}_m$ are real and symmetric over the space of trial functions $\boldsymbol{\zeta}$. Since $\mathbf{B}_m$ is now real as contrasted with the $\mathbf{B}$ of the general case that was pure imaginary we can no longer deduce that if ω_m is an eigenvalue then $-\omega_m$ is. Rather if ω_m is an eigenvalue belonging to class m then $-\omega_m$ is an eigenvalue belonging to the class $-m$. However it is now *obvious* that if ω_m is an eigenvalue of class m then so is $\omega_m{}^*$ whereas previously it was true but not overtly obvious. The fact that $\mathbf{A}_m$, $\mathbf{B}_m$ and $\mathbf{C}_m$ are real enables us to prove an important theorem relevant to theories of spiral structure both in the galaxy and in bath plugs.

3.4. *Anti-spiral theorem.* This theorem applies to the normal modes of steady flows of inviscid gas in which the velocity is in circles about the z axis and magnetic forces are absent. Further it is assumed that small relative changes in pressure and density are proportional (equation (24)). Our theorem is that under these conditions we may choose a complete set of normal modes such that no stable normal mode has a spiral structure. Further it is only possible to have stable normal modes with spiral structure if stable normal modes of the same symmetry are degenerate. Such degenerate spiral modes occur in conjugate pairs, one leading wherever the other is trailing.

To prove this theorem we first establish a connection between the phase of the complex function ζ and the occurrence of spiral structure.

From the continuity equation

$$\delta\rho = -\operatorname{div}(\rho_0 \boldsymbol{\xi}) = -\frac{1}{R}\frac{\partial}{\partial R}(R\rho_0\xi_R) - \frac{im\rho_0}{R}\,\xi_\phi - \frac{\partial}{\partial z}(\rho_0\xi_z),$$

$$= -\left[\frac{1}{R}\frac{\partial}{\partial R}(R\rho_0\zeta_R) + \frac{m\rho_0}{R}\,\zeta_\phi + \frac{\partial}{\partial z}(\rho_0\zeta_z)\right]e^{i(m\phi+\omega t)} \tag{63}$$

If the phase of ζ, χ say, is constant then $\delta\rho$ has phase $m\phi+\omega t+\chi$ and is of the form $\delta\rho = f(R, z)\exp[i(m\phi+\omega t+\chi)]$ with $f(R, z)$ real. Along any radius vector $\delta\rho$ has the same phase. Whenever $f(R, z)$ has a zero, $\delta\rho$ will be zero for that whole ring. Thus the density perturbations given by (63) have nodes which are planes $\phi = \text{const.}$, circles $R = \text{const.}$, $z = \text{const.}$ etc. Evidently such a system has no spiral structure. Spiral structure can only occur if the phase χ of ζ is R dependent. Furthermore if the twist of the spiral has the same sense at all radii then χ must vary monotonically with R.

We now investigate the phase of $\boldsymbol{\zeta}$ in our normal modes. For a stable mode ω is real so $-\omega^2\mathbf{A}_m + \omega\mathbf{B}_m + \mathbf{C}_m$ is real. Hence not only

$$(-\omega^2\mathbf{A}_m + \omega\mathbf{B}_m + \mathbf{C}_m)\,.\,\zeta = 0,$$

but also

$$(-\omega^2\mathbf{A}_m + \omega\mathbf{B}_m + \mathbf{C}_m)\,.\,\zeta^* = 0.$$

$\frac{1}{2}(\boldsymbol{\zeta}+\boldsymbol{\zeta}^*)$ and $(1/2i)(\boldsymbol{\zeta}-\boldsymbol{\zeta}^*)$ are both real eigenvectors. If these are linearly independent, we have two normal modes of the same symmetry type, m, which are degenerate since they have the same eigenvalue ω. Each mode is real by construction and therefore non-spiral but there is no reason why $\boldsymbol{\zeta}$ itself should be real. If the phase of $\boldsymbol{\zeta}$ increases with R then the phase of $\boldsymbol{\zeta}^*$ will decrease with R so if $\boldsymbol{\zeta}$ corresponds to a trailing spiral $\boldsymbol{\zeta}^*$ will correspond to a leading one.

However if there is no such degeneracy then the vectors are linearly dependent so $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}^*$ are dependent, i.e.

$$\boldsymbol{\zeta} = \alpha\boldsymbol{\zeta}^*$$

where α is a numerical (complex) constant.

Taking moduli $|\alpha| = 1$ so we write $\alpha = e^{2i\chi}$ with χ constant. Taking arguments

$$\operatorname{Arg}\boldsymbol{\zeta} = \chi$$

which shows that ζ is real but for a *constant* phase factor which could be removed by a trivial transformation.

Thus in either case we may take our stable normal modes to be real and in the non-degenerate case we are forced to take ζ of constant phase (i.e. non-spiral).

4. *Outlook*

(i) Application of the general variational principle to the stability of differentially rotating bodies is in principle straight-forward. In practice it may prove so tedious that very considerable ingenuity is required in the more complicated problems to which it is applicable. We feel the method will have been vindicated when we produce results that have not been obtained by other methods. This we hope to do.

(ii) The equivalence between the positive definiteness of the energy operator **C** and the stability of all normal modes in the presence of friction has been established by one of us. In particular if $\int \boldsymbol{\xi}^*.\mathbf{C}.\boldsymbol{\xi} d^3r$ is negative for some $\boldsymbol{\xi}$ then in the presence of dissipation the system will not be stable. The condition that **C** is non-negative is thus a secular stability criterion applicable to general flows and of considerable importance.

(iii) The Anti-spiral theorem shows that something more is needed in the theory of spiral structure than steady, non-dissipative, non-magnetic, linear stability analyses of gaseous systems and (ii) above focuses attention on the dissipation. One might speculate that the antispiral theorem could be extended to include the magnetic case and conceivably the unshocked non-linear regime also.

(iv) All the results of this paper are incomplete because modes of the type (Polynomial in t) $\times$ exp $(i\omega t)$ have not been considered. They only occur when there is degeneracy of modes of the same symmetry.

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