

# On the stability of input-affine nonlinear systems with sampled-data control

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**Abstract**—This paper is dedicated to the stability analysis of nonlinear sampled-data systems, which are affine in the input. Assuming that a stabilizing continuous-time controller exists and it is implemented digitally, we intend to provide sufficient asymptotic/exponential stability conditions for the sampled-data system. This allows to find an estimate of the upper bound on the asynchronous sampling periods. The stability analysis problem is formulated both globally and locally. The main idea of the paper is to address the stability problem in the framework of dissipativity theory. Furthermore, the result is particularized for the class of polynomial input-affine sampled-data systems, where stability may be tested numerically using sum of squares decomposition and semidefinite programming.

**Index Terms**—Sampled-data control, control affine nonlinear systems, stability analysis, dissipativity.

## I. INTRODUCTION

The stability of nonlinear sampled-data systems is a challenging problem. It is of great interest since in applications practical controllers are often implemented digitally. For the case of nonlinear systems, the *emulation* approach is often considered [1]. In this approach, a continuous-time controller is designed, next it is implemented using a sample-and-hold device. Intuitively, the sampling period must be chosen small enough to ensure the stability. A quantitative estimation of the so-called *maximum allowable sampling period* MASP is very important from the practical point of view, and several works in the literature target this problem (see for example [1], [2] and [3]).

The case of linear sampled-data systems has been extensively studied. For the input delay approach, based on Lyapunov-Krasovskii functionals, see [4], [5] and [6]. The works in [7] and [8] use tools from robust control theory. A polytopic approximation of the discrete-time model is used in [9] and [10] to handle the sampling effect based on Lyapunov-Razumikhin functions.

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For the case of nonlinear systems, we cite as follows some recent works. In [1], the authors specialized the results on generic Networked Control Systems (NCSs), for the particular case of sampled-data systems; local and global stability conditions are presented based on the hybrid systems theory. In [11], asymptotic stability of NCSs is studied using the same hybrid systems formulation; the Lyapunov functions are constructed with a sum of squares (SOS) techniques. The input delay approach is explored in [2] for the nonlinear case. A method for finding a control strategy that guarantees robust global stabilization of nonlinear sampled-data systems is presented in [3], based on the notion of sequential reachability.

The notion of dissipativity was introduced by [12]. Since its introduction, the dissipativity theory has been attracting an increasing attention, it can be used to study stability, passivity, robustness and other analysis and design problems. It was motivated by passivity properties of electrical circuits, and it can be seen as a generalized notion of abstract energy for dynamical systems. Recently, local asymptotic stability of bilinear sampled-data systems controlled by a linear state feedback has been considered in [13], using the analysis of contractive invariant sets and the dissipativity theory. The obtained results are promising compared to the existing literature [14]. However, the extension for generic nonlinear systems does not seem to be trivial.

The purpose of this work is to extend our previous result in [13], concerning the analysis of bilinear sampled-data systems, to the case of input-affine nonlinear sampled-data systems. Conditions are presented for both asymptotic and exponential stability. Dissipativity based conditions are used to estimate the MASP. The robustness with respect to the sampling jitters is considered. The result is shown to be applicable for local and global analysis. Additionally, we study the particular case of polynomial systems, where SOS techniques are used to derive storage and supply functions. We apply the result to a benchmark example from the literature to show the usefulness of the proposed stability conditions.

The remainder of the paper is organized as follows: the problem under study is introduced in Section II; in Section III the system is represented by an equivalent model which is useful for our dissipativity analysis; sufficient conditions for the asymptotic/exponential stability of affine nonlinear sampled-data systems is given in Section IV; finally, an illustrative example is presented in Section V.

*Basic definitions and notation:*  $\mathbb{R}^n$  is the  $n$ -dimensional euclidean space. The set of real matrices of dimension  $n \times m$

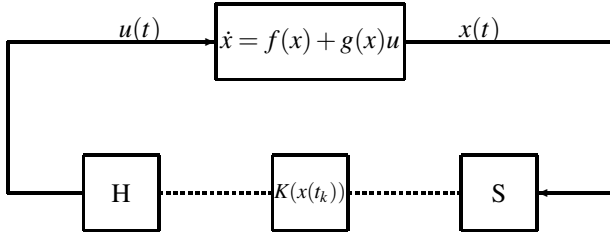


Fig. 1. The sampled-data feedback control of an affine nonlinear system.

is denoted by  $\mathbb{R}^{n \times m}$ . The euclidean norm is denoted by  $|\cdot|$ . For a signal  $x(\cdot)$ , we denote by  $\|x\|$  the  $L_2$  norm of  $x(\cdot)$ , and  $\|\Delta\|$  is the  $L_2$ -induced norm of the operator  $\Delta$ . The transpose of a vector or a matrix  $A$  is denoted by  $A^T$ . For  $P \in \mathbb{R}^{n \times n}$ ,  $P > 0$  means that it is a positive definite, and  $P \geq 0$  means that it is positive semi-definite matrix. The notation  $p(x) \in \mathbb{R}[x]$  with  $x \in \mathbb{R}^n$ , denotes that  $p(x)$  belongs to the set of polynomials in the variables  $\{x_1, x_2, \dots, x_n\}$  with coefficients in  $\mathbb{R}$ . For  $x_1, x_2 \in \mathbb{R}^n$ ,  $(x_1, x_2)$  denotes  $[x_1^T, x_2^T]^T$ . A function  $\beta: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{K}$  if it is continuous, zero at zero and strictly increasing. It is said to be of class  $\mathcal{K}_\infty$  if it is of class  $\mathcal{K}$ , and it is unbounded. A function  $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to be of class  $\mathcal{KL}$  if  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  for each  $t \geq 0$ , and  $\beta(s, \cdot)$  is non-increasing and satisfies  $\lim_{t \rightarrow \infty} \beta(s, t) = 0$  for each  $s \geq 0$ .

## II. PROBLEM FORMULATION

Consider the affine nonlinear control system given by

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad \forall t > 0, \quad x(0) = x_0. \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^m$  are the state and the input respectively. The functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $f(0) = 0$ , and  $g: \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$  are sufficiently smooth to make the system well defined, i.e. for any  $x(t_0)$  and any admissible  $u(\cdot)$ , the existence and uniqueness of a solution is ensured on  $[t_0, \infty)$ . We suppose that a continuous-time controller  $u(t) = K(x(t))$  stabilizes the equilibrium  $x = 0$  of the system, where  $K: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a continuously differentiable function. We consider the following notions of stability

*Definition 2.1:* The equilibrium point  $x = 0$  of (1) is *locally uniformly asymptotically stable* in a neighbourhood  $\mathcal{L}$  of the equilibrium, if there exists a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$ , such that

$$|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0, \quad \forall x(t_0) \in \mathcal{L}. \quad (2)$$

The equilibrium point  $x = 0$  is *globally uniformly asymptotically stable* if (2) is satisfied for any initial state  $x(t_0) \in \mathbb{R}^n$ .

*Definition 2.2:* the equilibrium point  $x = 0$  of (1) is *locally exponentially stable* in a neighbourhood  $\mathcal{L}$  of the equilibrium, if (2) is satisfied with

$$\beta(s, t) = cse^{-\lambda t}, \quad c > 0, \lambda > 0$$

and it is *globally exponentially stable* if this condition is satisfied for any initial state  $x(t_0) \in \mathbb{R}^n$ .

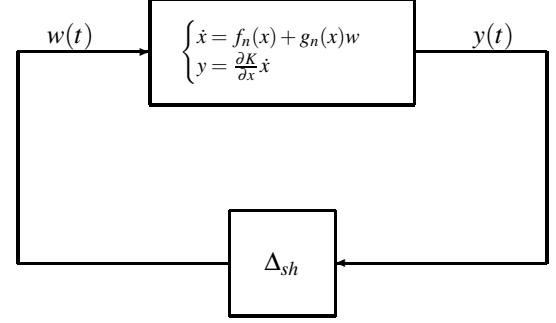


Fig. 2. The equivalent representation of the sampled-data system (3).

We consider the emulation of the controller  $u = K(x)$  with the following assumptions:

- the set of uncertain sampling instants  $\{0 = t_0 < t_1, \dots < t_k < \dots\}$  satisfies

$$0 < t_{k+1} - t_k \leq h_{max}, \quad \forall k \in \mathbb{N},$$

for a given upper bound on the sampling periods  $h_{max}$ , and

$$\lim_{k \rightarrow \infty} t_k = \infty;$$

- the control input is then calculated from the sampled-data state

$$u(t) = K(x(t_k)), \quad \forall t \in [t_k, t_{k+1}).$$

Under these assumptions, we obtain a closed-loop sampled-data system (Fig.1)

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))K(x(t_k)), \\ &\forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}. \end{aligned} \quad (3)$$

**Problem:** Find a criterion for the local and global asymptotic/exponential stability of the equilibrium point  $x = 0$  of the sampled-data system (3).

## III. ROBUSTNESS ANALYSIS REPRESENTATION

Note that the system (3) can be written

$$\dot{x}(t) = f_n(x(t)) + g_n(x(t))w(t), \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}, \quad (4)$$

where  $f_n(x) = f(x) + g(x)K(x)$ ,  $g_n(x) = g(x)$  and  $w(t) = K(x(t_k)) - K(x(t))$ . Note that  $f_n(x)$  represents the dynamics of the nominal continuous-time closed-loop system, i.e. the dynamics without the sampled-data implementation. This shows that the sampled-data system (3) can be represented by the equivalent feedback connection of

$$\begin{cases} \dot{x} = f_n(x) + g_n(x)w, \\ y = \frac{\partial K}{\partial x} x, \end{cases} \quad (5)$$

with the operator  $\Delta_{sh}: y \rightarrow w$

$$w(t) = (\Delta_{sh}y)(t) = - \int_{t_k}^t y(\tau) d\tau, \quad \forall t \in [t_k, t_{k+1}). \quad (6)$$

This representation is shown in Fig.2. Note that the effect of sampling and the variations of the sampling periods are modelled by the operator  $\Delta_{sh}$ . This approach has been considered in [7] and [8] to study the stability of linear sampled-data systems. In [7], the lifting technique is used to find a bound on the gain of the operator  $\Delta_{sh}$ . It is shown that  $\|\Delta_{sh}\| \leq \delta_0$  with  $\delta_0 = \frac{2}{\pi}h_{max}$ . This bound is attained ( $\|\Delta_{sh}\| = \delta_0$ ) when  $t_{k+1} - t_k = h_{max}$ . The scaled small gain theorem is then used to find linear matrix inequalities (LMI) stability conditions. In [8], the last boundedness of the operator is used along with a passivity type property, to find less conservative LMI stability conditions. The result is based on robust control theory, by using a frequency domain approach, Integral Quadratic Constraints (IQC), and the Kalman-Yakubovich-Popov lemma.

The methods in both [7] and [8] are developed for linear time invariant (LTI) sampled-data systems, and they cannot be applied to nonlinear systems. In [13], the operator's properties are used to develop a stability analysis for bilinear sampled-data systems, based on dissipativity theory. We recall as follows two important lemmas stated in [13], based on the work in [8].

*Lemma 3.1:* [13] Let  $\Delta_{sh}$  be the operator defined in (6). Then, for any  $y \in L_2[0, h_{max})$  and  $0 < X^T = X \in \mathbb{R}^{m \times m}$  we have the following inequality:

$$\mathcal{I}_1(t) = \int_{t_k}^t (\Delta_{sh}y)^T(\tau)X(\Delta_{sh}y)(\tau) - \delta_0^2 y^T(\tau)Xy(\tau) d\tau \leq 0, \quad (7)$$

for all  $t \in [t_k, t_{k+1})$ .

*Lemma 3.2:* [13] Let  $\Delta_{sh}$  be the operator defined in (6). Then, for any  $y \in L_2[0, h_{max})$  and  $0 \leq Y^T = Y \in \mathbb{R}^{m \times m}$ , we have the following inequality:

$$\mathcal{I}_2(t) = \int_{t_k}^t (\Delta_{sh}y)(\tau)^T Y y(\tau) + y^T(\tau)Y(\Delta_{sh}y)(\tau) d\tau \leq 0, \quad (8)$$

for all  $t \in [t_k, t_{k+1})$ .

In this note we propose to exploit the previous lemmas to develop a stability criterion for nonlinear sampled-data systems, which are affine in the control. The method is inspired by the so called *exponential dissipativity* [15].

#### IV. MAIN RESULTS

##### A. Stability analysis

In the following we provide the main results of this note.

*Theorem 4.1:* Consider the sampled-data system (3), and the equivalent representation (5), (6). Consider the quadratic form

$$\mathcal{S}(y(t), w(t)) = \begin{bmatrix} y(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y(t) \\ w(t) \end{bmatrix} \quad (9)$$

with  $\delta_0 = \frac{2}{\pi}h_{max}$ ,  $0 < X^T = X \in \mathbb{R}^{m \times m}$ , and  $0 \leq Y^T = Y \in \mathbb{R}^{m \times m}$ . Consider a neighbourhood  $\mathcal{D} \subset \mathbb{R}^n$  of the equilibrium point  $x = 0$ , and suppose that there exist a differentiable positive definite function  $V : \mathcal{D} \rightarrow \mathbb{R}^+$ , such that there exist class  $\mathcal{K}$  functions  $\beta_1$  and  $\beta_2$ , with

$$\beta_1(|x|) \leq V(x) \leq \beta_2(|x|), \quad \forall x \in \mathcal{D}, \quad (10)$$

and the following inequalities are satisfied:

$$\dot{V}(x(t)) + \alpha V(x(t)) \leq \mathcal{S}(y(t), w(t)), \quad (11)$$

$$\dot{V}(x(t)) + \alpha V(x(t)) \leq \mathcal{S}(y(t), w(t))e^{-\alpha h_{max}}, \quad (12)$$

with  $\alpha > 0$ , for and any  $x(t) \in \mathcal{D}$ . Then, the equilibrium  $x = 0$  of the system (3) is locally uniformly asymptotically stable. Moreover, consider the sub-level set defined by  $V(\cdot)$  and a scalar  $c > 0$

$$\mathcal{L}_c := \{x \in \mathbb{R}^n : V(x) \leq c\} \quad (13)$$

Then the set  $\mathcal{L}_{c^*}$  defined by the maximal sub-level set of  $V$  contained in  $\mathcal{D}$

$$c^* = \max_{\mathcal{L}_c \subset \mathcal{D}} c \quad (14)$$

is an estimate of the domain of attraction. Finally, if all the conditions are satisfied globally, with class  $\mathcal{K}_\infty$  functions  $\beta_1$  and  $\beta_2$ , then the equilibrium  $x = 0$  is globally uniformly asymptotically stable.

*Proof:* To show the stability of the sampled-data system, we define first the following function

$$W(t) = V(x(t))e^{\alpha(t-t_k)} - \int_{t_k}^t \mathcal{S}(y(\tau), w(\tau)),$$

for any  $t \in [t_k, t_{k+1})$ . The conditions (11) and (12) are sufficient to have

$$\dot{W}(t) \leq 0, \quad \forall t \in [t_k, t_{k+1}), \quad \forall x(t) \in \mathcal{D}. \quad (15)$$

The last equation yields

$$V(x(t))e^{\alpha(t-t_k)} - \int_{t_k}^t \mathcal{S}(y(\tau), w(\tau)) \leq V(x(t_k)). \quad (16)$$

From Lemma 3.1 and Lemma 3.2, it is easy to see that

$$V(x(t)) \leq e^{-\alpha(t-t_k)}V(x(t_k)), \quad \forall t \in [t_k, t_{k+1}), \quad \forall x(t) \in \mathcal{D}. \quad (17)$$

Clearly, the set  $\mathcal{L}_{c^*}$  is positively invariant [16], and it is the largest sub-level set contained in  $\mathcal{D}$ . Consider an initial condition  $x_0 \in \mathcal{L}_{c^*}$ . From the continuity of the solution  $x(t)$ , (17) leads to

$$V(x(t)) \leq e^{-\alpha(t-t_0)}V(x(t_0)), \quad \forall t \geq t_0, \quad \forall x_0 \in \mathcal{L}_{c^*}. \quad (18)$$

From (10) and (18), we see that for any solution with  $x(t_0) \in \mathcal{L}_{c^*}$

$$\begin{aligned} |x(t)| &\leq \beta_1^{-1}(V(x(t_0))e^{-\alpha(t-t_0)}) \\ &\leq \beta_1^{-1}(\beta_2(|x(t_0)|)e^{-\alpha(t-t_0)}) \\ &:= \beta(|x(t_0)|, t-t_0), \quad \forall t \geq t_0, \quad \forall x(t_0) \in \mathcal{L}_{c^*}. \end{aligned}$$

The function  $\beta(\cdot, \cdot)$  can be easily seen to be a class  $\mathcal{KL}$  function. This shows that  $x = 0$  is locally uniformly asymptotically stable. Finally, it is trivial to see that if all the conditions are satisfied globally, with a class  $\mathcal{K}_\infty$  functions  $\beta_1$  and  $\beta_2$ , then  $x = 0$  is globally uniformly asymptotically stable. This completes the proof. ■

*Corollary 4.1:* Suppose that all the conditions of Theorem 4.1 are satisfied with

$$\beta_1(|x|) \geq k_1|x|^q, \beta_2(|x|) \leq k_2|x|^q, \quad \text{for some } k_1, k_2, q > 0. \quad (19)$$

Then, the equilibrium  $x = 0$  is locally exponentially stable. Moreover, the sub-level set  $\mathcal{L}_{c^*}$  defined in (14) and (13), is an estimate of the domain of attraction. If the conditions hold globally, then  $x = 0$  is globally exponentially stable.

*Proof:* Following the same steps as in the proof of Theorem 4.1, we get

$$V(x(t)) \leq e^{-\alpha(t-t_0)}V(x(t_0)), \quad \forall t \geq t_0, \quad \forall x_0 \in \mathcal{L}_{c^*}.$$

Thus, from (10) and (19)

$$\begin{aligned} |x(t)| &\leq \left( \frac{V(x(t_0))e^{-\alpha(t-t_0)}}{k_1} \right)^{1/q} \leq \left( \frac{k_2|x(t_0)|^q e^{-\alpha(t-t_0)}}{k_1} \right)^{1/q} \\ &= \left( \frac{k_2}{k_1} \right)^{1/q} |x(t_0)| e^{-(\alpha/q)(t-t_0)}, \quad \forall t \geq t_0, \quad \forall x(t_0) \in \mathcal{L}_{c^*}. \end{aligned}$$

This shows that  $x = 0$  locally exponentially stable. If the conditions hold globally, global exponential stability is trivial. ■

*Remark 4.1:* Considering the storage function  $V(x(t))$ , the inequalities (11) and (12) show that (5) is exponentially dissipative with respect to the supply rates  $\mathcal{S}(y, w)$  and  $e^{-\alpha h_{max}}\mathcal{S}(y, w)$  respectively, with  $\mathcal{S}$  defined in (9). This can be seen from the Remark 2.8 in [15].

### B. Sum of squares stability conditions for the class of polynomial systems

When the linear approximation fails, the dynamics of many physical phenomena can be modelled by polynomial differential equations. They are frequently found in several domains like process control, biology, robotics, and electrical systems. For this class of systems, SOS decomposition and semidefinite programming [17], are shown to be a useful tool. It has been used in several analysis and synthesis control problems [18].

In this section we specialize the previous result for the class of affine polynomial sampled-data systems, using SOS decomposition and semidefinite programming techniques. We formulate a constructive method to find a storage function and a supply rate, which satisfy the asymptotic/exponential stability conditions proposed in the previous section.

Let us consider the stability problem defined in Section II for the particular case where the  $f(x)$ ,  $g(x)$  and  $K(x)$  are polynomial functions. The system (5) will be defined by polynomial functions  $F(x, w) := f_n(x) + g_n(x)w$  and  $G(x, w) := \frac{\partial K}{\partial x}F(x, w)$ :

$$\begin{cases} \dot{x} = F(x, w), \\ y = G(x, w). \end{cases} \quad (20)$$

When looking for a polynomial storage function  $V(x)$ , checking the dissipativity inequalities in Theorem 4.1 is a problem of checking the non negativity of polynomials. This can be

seen from (9) and (20), as for the polynomial case (11) and (12) are (respectively) equivalent to

$$0 \leq -\frac{\partial V}{\partial x}F(x, w) - \alpha V(x) + [-\delta_0^2 G^T(x, w)XG(x, w) + 2G^T(x, w)Yw + w^T Yw],$$

and

$$0 \leq -\frac{\partial V}{\partial x}F(x, w) - \alpha V(x) + [-\delta_0^2 G^T(x, w)XG(x, w) + 2G^T(x, w)Yw + w^T Yw]e^{-\alpha h_{max}},$$

for any  $x(t) \in \mathcal{D}$ . In fact, the right terms in the last inequalities can be written as polynomials of the form  $p(\xi) \geq 0$ , with  $p(\xi) \in \mathbb{R}[\xi]$ , and  $\xi = (x, w)$ .

Checking the non negativity of a polynomial is known to be a hard problem. Recent methods relaxed this problem using semidefinite programming and the SOS decomposition [17]. The relaxation is based on checking whether a polynomial is a SOS, which is sufficient to ensure the semidefinite positivity.

*Definition 4.1:* [18] A multivariate polynomial  $p(x) \in \mathbb{R}[x]$  is said to be a sum of squares (SOS), if there exist some polynomials  $p_i(x) \in \mathbb{R}[x]$ ,  $i \in \{1, \dots, M\}$ , such that  $p(x) = \sum_{i=1}^M p_i^2(x)$ .

The relaxation is only sufficient, but there are suggestions in the literature which indicate that it is not too conservative (see [18] and the references therein). However, it must be noted that the computational complexity for testing whether a polynomial  $p(x)$  is an SOS increases rapidly with the degree of  $p(x)$ .

SOS techniques are shown to be very useful in systems analysis [18]. In the following, we reformalize Theorem 4.1 and Corollary 4.1 using the SOS method. The local applicability of the dissipativity inequalities inside a region  $\mathcal{D}$  is ensured using a technique similar to the S-procedure [19]. Note that when looking for a Lyapunov or a storage function, we need to ensure that its positive definiteness. Thus, guaranteeing that it is an SOS is not sufficient, as it only guarantees its non negativity. To overcome this problem, we use the following proposition

*Proposition 4.1:* [18] Given a polynomial  $V(x) \in \mathbb{R}[x]$  of degree  $2d$ , let

$$\varphi(x) = \sum_{i=1}^n \sum_{j=1}^d \varepsilon_{ij} x_i^{2j}, \quad \text{such that} \quad \sum_{j=1}^d \varepsilon_{ij} > \gamma, \quad \forall i = 1, \dots, n \quad (21)$$

with  $\gamma$  a positive number, and  $\varepsilon_{ij} \geq 0$  for all  $i$  and  $j$ . Then the condition

$$V(x) - \varphi(x) \quad \text{is SOS} \quad (22)$$

guarantees the positive definiteness of  $V(x)$ .

*Corollary 4.2:* Consider the sampled-data system (3) in the case where  $f(x)$ ,  $g(x)$  and  $K(x)$  are polynomial functions, or the equivalent representation (20) and (6). Let  $\mathcal{D} = \{x \in \mathbb{R}^n : \mu_l(x) \geq 0, l = 1, 2, \dots, s\}$  be a neighbourhood of the origin  $x = 0$ . Suppose that there exist a polynomial function  $V(x) \in \mathbb{R}[x]$ , and sums of squares  $\sigma_l(\xi)$  and  $\zeta_l(\xi)$ , with  $l \in$

$\{1, \dots, s\}$  and  $\xi = (x, w)$ , such that the following polynomials are SOS

$$\hat{V}(x) = V(x) - \varphi(x), \quad (23)$$

$$\begin{aligned} \rho_1(\xi) = & - \sum_{l=1}^s \sigma_l(\xi) \mu_l(x) - \frac{\partial V}{\partial x} F(x, w) - \alpha V(x) \\ & + [-\delta_0^2 G^T(x, w) X G(x, w) + 2G^T(x, w) Y w + w^T Y w], \end{aligned} \quad (24)$$

$$\begin{aligned} \rho_2(\xi) = & - \sum_{l=1}^s \varsigma_l(\xi) \mu_l(x) - \frac{\partial V}{\partial x} F(x, w) - \alpha V(x) \\ & + [-\delta_0^2 G^T(x, w) X G(x, w) + 2G^T(x, w) Y w + w^T Y w] e^{-\alpha h_{max}}. \end{aligned} \quad (25)$$

with  $\delta_0 = \frac{2}{\pi} h_{max}$ ,  $0 < X^T = X \in \mathbb{R}^{m \times m}$ ,  $0 \leq Y^T = Y \in \mathbb{R}^{m \times m}$ , and  $\varphi(x)$  a positive definite polynomial defined in (21). Then, the equilibrium  $x = 0$  of the system (3) is locally uniformly asymptotically stable. Moreover, the sub-level set  $\mathcal{L}_{c^*}$  defined in (14) and (13), is an estimate of the domain of attraction. Finally, if (24) and (25) are SOS while  $\mu_l(x) = 0$ , for all  $l \in \{1, 2, \dots, s\}$ , then the equilibrium is globally uniformly asymptotically stable.

*Proof:* First, note that from (23) and Proposition 4.1, the function  $V(x)$  is ensured to be definite positive and radially unbounded ( $V(x) \rightarrow \infty$  when  $x \rightarrow \infty$ ). Therefore, using Lemma 4.3 from [16], there exist class  $\mathcal{K}$  functions  $\beta_1$  and  $\beta_2$ , such that

$$\beta_1(|x|) \leq V(x) \leq \beta_2(|x|), \quad \forall x \in \mathbb{R}^n.$$

Moreover, when  $x(t) \in \mathcal{D}$ , i.e.  $\mu_l(x) \geq 0$  for all  $l \in \{1, 2, \dots, s\}$ , then from the non negativity of the SOS polynomials  $\sigma_l(\xi)$  and  $\varsigma_l(\xi)$ , we can see that  $\rho_1(\xi) \geq 0$  ( resp.  $\rho_2(\xi) \geq 0$ ). The later implies that the dissipativity condition (11) ( resp. (12)) is satisfied. Thus all the local stability conditions of Theorem 4.1 are satisfied. The case where (24) and (25) are SOS for  $\mu_l(x) = 0 \forall l \in \{1, 2, \dots, s\}$  satisfies obviously the global stability conditions in Theorem 4.1. ■

*Corollary 4.3:* Suppose that all the conditions of Corollary 4.2 are satisfied, and that the storage function  $V(x)$  satisfies

$$k_1 |x|^q \leq V(x) \leq k_2 |x|^q, \quad \forall x \in \mathbb{R}^n. \quad (26)$$

Then, the equilibrium  $x = 0$  is locally exponentially stable. Moreover, the sub-level set  $\mathcal{L}_{c^*}$  defined in (14) and (13), is an estimate of the domain of attraction. If the conditions hold globally, then  $x = 0$  is globally exponentially stable.

*Proof:* The proof follows the same steps of Corollary 4.2, and it is a direct result of Corollary 4.1. ■

## V. ILLUSTRATIVE EXAMPLE

In the following, we revisit the example in [1]. We find the MASP which guarantees the global uniform asymptotic stability of the sampled-data system.

### A. Example 1

Consider the following system from [1]

$$\dot{x} = dx^2 - x^3 + u,$$

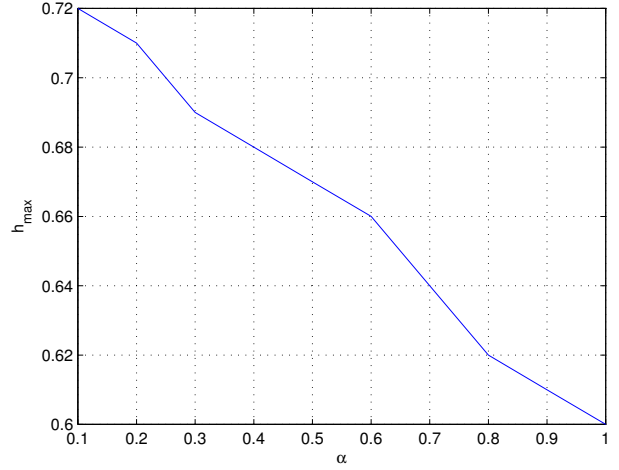


Fig. 3. Tradeoff between  $\alpha$  (the exponential decay rate of the storage function), and the estimation of the MASP.

with a bounded time-varying  $|d| \leq 1$ , and a stabilizing control  $u = K(x) = -2x$ . Emulating this controller results in a sampled-data system that can be represented by the operator  $\Delta_{sh}$  in (6), and a system (5) described by

$$\begin{cases} \dot{x} = dx^2 - x^3 - 2x + w, \\ y = -2(dx^2 - x^3 - 2x + w). \end{cases}$$

We apply the Corollary 4.2 in order to find a storage function of the form  $V(x) = ax^2 + bx^4$ , such that (23), (24) and (25) are SOS. We choose  $\varphi(x) = 10^{-3}x^2$ ,  $\alpha = 0.1$  and  $h_{max} = 0.72$ . We intend to test the global stability. In this case, the polynomials (24) and (25) are

$$\begin{aligned} \rho_1(\xi) = & -(2ax + 4bx^3)(dx^2 - x^3 - 2x + w) - \alpha(ax^2 + ax^4) \\ & + [-4\delta_0^2 X(dx^2 - x^3 - 2x + w)^2 \\ & - 4Y(dx^2 - x^3 - 2x + w)w + Yw^2], \end{aligned} \quad (27)$$

$$\begin{aligned} \rho_2(\xi) = & -(2ax + 4bx^3)(dx^2 - x^3 - 2x + w) - \alpha(ax^2 + ax^4) \\ & + [-4\delta_0^2 X(dx^2 - x^3 - 2x + w)^2 \\ & - 4Y(dx^2 - x^3 - 2x + w)w + Yw^2] e^{-\alpha h_{max}}, \end{aligned} \quad (28)$$

where  $a, b, X, Y$  are decision variables. Note that the time-varying terms  $d$  and  $d^2$  appear in the polynomial expressions. However, if both (27) and (28) are ensured to be SOS for all the values of  $(d, d^2) \in \{(1, 0), (1, 1), (-1, 0), (-1, 1)\}$ , then they will be SOS for any time-varying  $|d| \leq 1$ . This is found to be satisfied using the SOSTOOLS software [20], for the storage function  $V(x) = 0.77402x^2 + .19911x^4$ , and the supply function (9) defined by  $X = 0.47522$  and  $Y = 0.6230210^{-3}$ . By Corollary 4.2, we obtain the global uniform asymptotic stability of the equilibrium  $x = 0$ , of the sampled-data system. This result cannot be obtained when trying a quadratic storage function. Increasing  $\alpha$  (the exponential decay rate of the storage function), results in the decrement of the maximum value of  $h_{max}$  for which the problem is feasible. This can be seen in Fig 3. Previous works considered this example in the literature for estimating the

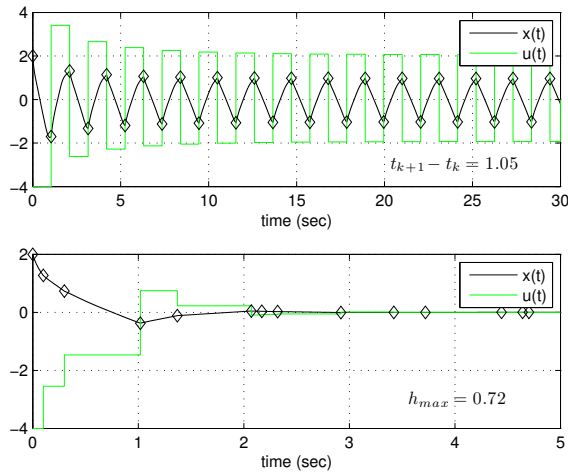


Fig. 4. State trajectory evolution for two sequences of sampling periods.

MASP. In [1], a bound of  $h_{max} = 0.368$  is found. In [2], the proposed upper bound is  $h_{max} = 0.1428$ . The conditions proposed in this paper are found feasible for  $h_{max} = 0.72$ . State trajectory evolutions are shown in Fig 4. It can be seen that the state trajectory is asymptotically stable when the sampling periods are inferior to the bound  $h_{max} = 0.72$ . Also, note that for a uniform sampling period of  $t_{k+1} - t_k = 1.05$ , asymptotic stability is no longer guaranteed.

## VI. CONCLUSION

In this paper we have provided sufficient conditions for the stability of nonlinear sampled-data systems, which are affine in the control. The main idea of the paper is to use the dissipativity theory to provide an estimate of the maximum allowable sampling period that guarantees asymptotic/exponential stability. The results are numerically illustrated for the case of polynomial systems, with the use of SOS decomposition and semidefinite programming.

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