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On the Stability of Naked Singularities with Negative Mass

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We study the linearised stability of the nakedly singular negative mass Schwarzschild solution against gravitational perturbations. There is a one parameter family of possible boundary conditions at the singularity. We give a precise criterion for stability depending on the boundary condition. We show that only one particular boundary condition gives perturbations of finite energy and show that the spacetime is stable with this boundary condition.

§1. Introduction: phantoms and runaways

The positive mass theorems are generally taken as a triumphs of modern relativity theory. They establish that, under plausible physical assumptions, asymptotically flat solutions of the Einstein equations with physically acceptable matter sources cannot have negative total mass. There is no shortage of solutions with negative mass, for example the Schwarzschild solution with negative mass parameter, but they are typical nakedly singular and hence assumed to be physically unacceptable. They are not expected to arise from regular initial conditions. However, this begs the question of why exactly it is that we are reluctant to consider solutions with negative mass. The question acquires topicality from various suggestions by cosmologists that the observed acceleration of the scale factor of the universe may be due to 'phantom matter', that is some matter, typically a scalar field, with negative kinetic energy.[1),2)]

The idea of anti-gravity is, of course, a staple of science fiction writers. According to Mach[3)] it appears to have been Föppl[4)] who first, by analogy with electrostatics, explored the idea that gravitational masses could be both positive and negative. The discovery of dark energy may be said to establish anti-gravity as a serious subject for scientific discussion and the Randall-Sundrum scenario I,[5)] with its negative tension branes, has only reinforced the trend of considering matter with exotic energy momentum tensors.

One problem with negative masses was pointed out long ago in a beautiful analysis of Bondi.[6)] He drew attention to some special features of negative masses in general relativity.

Firstly, as a consequence of the weak equivalence principle, a particle of negative mass falling in a gravitational field should fall at the same rate and in the same direction as a particle of positive mass. At the level of the geodesic equations this is because the mass cancels out from the equations of motion. Thus, for example, a cloud of negative mass particles, let us call them ghosts or phantoms, should accrete

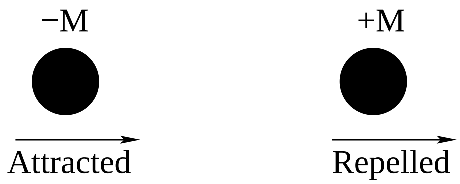

Fig. 1. A negative mass particle chasing a positive mass particle.

onto ordinary black hole of positive mass just like ordinary particles.

On the other hand, a body with negative mass should repel particles with either negative or positive mass. Again, at the level of geodesic equations this is easily verified for the negative mass Schwarzschild solution. Combining this fact with our first observation, we see that the interesting possibility of a runaway solution exists in which a positive mass particle is chased in some direction by a negative mass particle, the combined system going into a state of constant acceleration. The positive mass particle attracts the negative mass particle to itself, but at the same time the negative mass particle pushes the positive mass particle away. See Fig. 1.

Bondi showed that the runaway phenomenon actually arises by exhibiting exact runaway solutions of the Einstein equations. His paper was written before the development of black hole theory, but may readily be adapted to include black holes.[7) Bondi considered the axially symmetric static vacuum metrics first studied by Weyl

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[e^{2k}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]. \qquad (1\cdot1)$$

The vacuum equations are satisfied if the Newtonian potential $U = U(\rho, z)$ is harmonic with respect to the standard Laplacian on three-dimensional Euclidean space $\mathbb{E}^3$ with cylindrical polar coordinates (ρ, ϕ, z). The function $k(\rho, z)$ may then be obtained by quadratures. A physically acceptable solution must have $k = 0$ on any portion of the axis of symmetry to avoid conical singularities. This places restrictions on the possible solutions. The general formalism is reviewed in Ref. 8). The Schwarzschild solution with positive mass M is obtained by taking for U the Newtonian potential of a uniform rod of length $L = 2M$. See Fig. 2. The portion of the line $\rho = 0$ occupied by the rod is then a regular event horizon. The standard Schwarzschild coordinates are given by a system of confocal prolate ellipses in the ρ-z plane, with the rod a degenerate member of the family.

A rod of mass per unit length 1/2 and of infinite length, occupying the positive z axis for example, corresponds to a Rindler horizon. Superposing a positive mass rod along the negative z axis gives a solution with a conical defect somewhere on the axis of symmetry. This solution is the well known C-metric. Again, see Fig. 2. These same conical defects arose in the solutions considered by Bondi which had as sources a continuous matter distribution and which were meant to model stars. Bondi's observation,[6) which also applies to the black hole case,[7) was that by superposing a further source in between the positive mass Schwarzschild rod and the Rindler rod the conical defects may be eliminated as long as the new source has negative mass. In this way a uniformly accelerating runaway solution may be constructed. The simplest negative mass solution for $U(\rho, z)$ to take is that of a negative mass

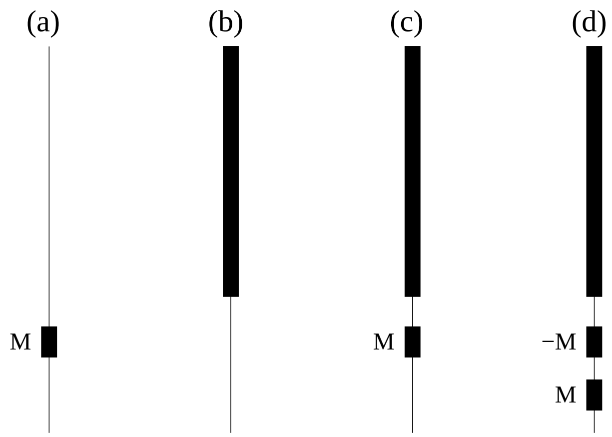

Fig. 2. Rods at $\rho = 0$ corresponding to: (a) A positive mass Schwarzschild black hole, (b) a Rindler horizon, (c) an accelerating black hole with a conical defect [the C-metric], (d) a negative mass black hole chasing a positive mass black hole in a nonsingular spacetime.

point particle which results in a singular solution called the Curzon solution, but there are many other possibilities. In particular, one could use the negative mass Schwarzschild solution. This would amount to taking the Newtonian potential of a uniform rod of mass per unit length $-1/2$. This configuration is shown in Fig. 2.

It seems clear from the discussion above that under suitable circumstances the negative mass Schwarzschild solution should be unstable. As derived above the instability is dynamical and rather nonlinear. However, it is an interesting question whether this instability, or indeed other possibly unrelated instabilities, appear in a linearised analysis of classical perturbations around the Schwarzschild phantom. As far as we are aware, a discussion of the linearised stability of the negative mass Schwarzschild solution against gravitational perturbations has not been given before, and the purpose of the present article is to provide one.

The crucial subtlety in our analysis will concern boundary conditions at the singularity. The linearised Einstein equations reduce to a set of Schrödinger-like equations with a time-independent 'Hamiltonian'. The choice of boundary conditions is constrained by the requirement of a self-adjoint Hamiltonian, and hence a unitary time evolution. For scalar metric perturbations there is not a unique boundary condition.[*)] We shall find a precise range of boundary conditions that guarantee linearised stability of naked singularity. The existence of boundary conditions giving linearised stablility of the negative mass singularity is somewhat counterintuitive. Our results have a very similar flavour to recent results on the stability of Anti-de Sitter space, which is also not globally hyperbolic and for which an ambiguity of boundary conditions exists at infinity.[12)]

In the Randall-Sundrum scenario I,[5)] the negative tension brane is stabilised against increasing its area by a $\mathbb{Z}_2$ quotient of spacetime that fixes the brane. A similar phenomenon stabilises orientifolds in string theory and their supergravity

[*)] The problem of boundary conditions in negative mass Schwarzschild spacetime has been studied for test scalar fields in Refs. 9)–11).

realisation in terms of the Atiyah-Hitchin metric. The linearised stability of negative mass black holes that we have found here does not depend on any such discrete quotient.

The organisation of this paper is as follows. We review the Schrödinger equations describing perturbations of the spacetime. There is seen to be a one dimensional family of possible boundary conditions at the singularity. We derive a critical boundary condition separating stable from unstable spacetimes. There is one particular boundary condition which results in perturbations with a more physical behaviour than the others. The spacetime with this boundary condition is stable. We end with a discussion of our results.

§2. Stability analysis

2.1. *Negative mass Schwarzschild spacetime*

We will consider linearised perturbations about the four dimensional Schwarzschild spacetime. We write the metric as

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\sigma^2_{(2)}\,, \tag{2·1}$$

where $d\sigma^2_{(2)}$ is the metric of a unit sphere and

$$f(r) = 1 + \frac{\mu}{r}\,. \tag{2·2}$$

We are interested in the negative mass case, so $\mu = -2M > 0$. The negative mass Schwarzschild spacetime is well known to have a naked singularity at $r = 0$. This is illustrated in Fig. 3, which shows the Penrose diagram for the spacetime.

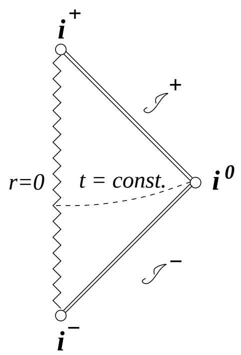

Fig. 3. Penrose diagram for the negative mass Schwarzschild: There is a timelike singularity at $r = 0$. The boundary at $r \to \infty$ consists of two null lines. Anywhere in this region the Killing field $\partial/\partial t$ is timelike.

As usual, it will be convenient to introduce the Regge-Wheeler coordinate

$$r_* \equiv \int \frac{dr}{f(r)} = r - \mu \log\left(1 + \frac{r}{\mu}\right)\,. \tag{2·3}$$

As $r \to 0$, $f(r)$ diverges. Therefore the range of the Regge-Wheeler coordinate is $0 < r_* < \infty$. The timelike singularity is located at $r_* = 0$ and infinity corresponds to $r_* \to \infty$.

2.2. *Gravitational perturbations*

Gravitational perturbations, $g_{ab} \to g_{ab} + h_{ab}$, of the four dimensional background metric (2·1) may be grouped into two types: those of axial (vector) and polar (scalar) type with respect to their parity transformation on the sphere. Different types of perturbation do not mix at the linearised level.

In order to solve the linearised Einstein equations, there is a standard technique for separating the angular coordinates and constructing gauge invariant scalar variables which we will denote universally by Φ.[13),14)] The quantities Φ are linear combinations of perturbed metric components and their derivatives: $h, \partial_r h, \partial_{rr} h$. For each type of perturbation, one obtains an equation of motion for Φ of the form

$$\frac{\partial^2}{\partial t^2}\Phi = \left(\frac{\partial^2}{\partial r_*^2} - V\right)\Phi\,. \tag{2·4}$$

The potentials appearing in such equations were first derived in a fixed gauged by Regge-Wheeler[15)] and by Zerilli.[16)] A unified treatment is given by Chandrasekhar.[17),18)] The gauge invariant approach has recently allowed an extension to higher dimensional black holes,[13),14)] including the tensor modes that only appear in higher dimensions.[19)]

The potential function V for the vector/axial/Regge-Wheeler perturbation[13),15),17)] is

$$V_V = \frac{f(r)}{r^2}\left[\frac{3\mu}{r} + l(l+1)\right]\,, \tag{2·5}$$

and for the scalar/polar/Zerilli perturbation the potential is[13),16),17)]

$$V_S = \frac{f(r)}{(mr-3\mu)^2}\left[-9\frac{\mu^3}{r^3} + 9m\frac{\mu^2}{r^2} - 3m^2\frac{\mu}{r} + 2m^2 + m^3\right]\,, \tag{2·6}$$

with $m = (l-1)(l+2)$ and $l = 2, 3, 4, \cdots$ is the angular momentum. Note that the $l = 0, 1$ modes are special. The former is spherically symmetric and hence, by Birkhoff's theorem, does not describe gravitational radiation. Rather it corresponds to a perturbation changing the mass parameter M. The $l = 1$ mode is in fact pure gauge and arises from a translation. We discuss this further in the conclusion below.

2.3. *Boundary conditions and self-adjoint extensions*

Because of the naked singularity the spacetime fails to be globally hyperbolic. In general it is far from obvious how to define the dynamics of any field in such a non-globally hyperbolic spacetime. However, since the background spacetime we consider is static we can in fact define sensible dynamics for linear perturbations. A general prescription for doing so is as follows.[9),10),20),21)]

Let A be the spatial derivative part of (2·4) in a non-globally hyperbolic, static spacetime

$$A = -\frac{d^2}{dr_*^2} + V\,, \qquad 0 < r_* < \infty\,. \tag{2·7}$$

One may view A as an operator acting on the Hilbert space $\mathcal{H} = L^2(r_*, dr_*)$ on a timeslice Σ_t orthogonal to the static Killing field. In order to define unitary dynamics, we need a self-adjoint extension A_E of A. We will see shortly that choosing a self-adjoint extension corresponds in our case to choosing boundary conditions at the naked singularity $r_* = 0$. Given a self-adjoint extension, then the time evolution of Φ with normalisable initial data $(\Phi_0, \dot{\Phi}_0)$ on Σ_0 is given by

$$\Phi_t = \cos(A_E^{1/2} t)\Phi_0 + A_E^{-1/2} \sin(A_E^{1/2} t)\dot{\Phi}_0 \,, \tag{2·8}$$

where $\dot{\Phi}_0 = \partial\Phi/\partial t|_{\Sigma_0}$. One rigorous result that may be proven[20] is that whenever the initial conditions are smooth and with compact support, $(\Phi_0, \dot{\Phi}_0) \in C_0^\infty(\Sigma_0) \times C_0^\infty(\Sigma_0)$, then Φ_t is smooth everywhere in the spacetime and furthermore within the domain of dependence of the initial surface Σ_0, Φ_t agrees with the solution to Eq. (2·4) determined from the initial data $(\Phi_0, \dot{\Phi}_0)$. Furthermore, it was shown in Ref. 21) that under certain reasonable conditions the prescription (2·8) is the unique way of defining the dynamics.

Our main interest is in the positivity of A_E or lack thereof. If A_E is positive then the dynamics is classically stable since $\cos(A_E^{1/2} t)$ and $A_E^{-1/2} \sin(A_E^{1/2} t)$ in (2·8) become bounded operators. Therefore the time evolution of Φ remains bounded for all time and the naked singularity is stable at the linearised level. On the other hand, if all possible self-adjoint extensions are not positive, then the spacetime is unavoidably unstable.

Given a self-adjoint extension A_E, one knows that any vector in the Hilbert space may be expressed in terms of a basis of eigenvectors of A_E. In particular we could prepare smooth initial data with compact support if we like, even though the individual eigenvectors of A_E will satisfy neither of these properties. We therefore turn to a mode analysis of the stability. A rigorous proof of stability follows from establishing positivity of A_E.[22]

More concretely, consider a mode

$$\Phi_t = e^{-i\omega t}\Phi(r_*) \,. \tag{2·9}$$

The spatial part then satisfies the Schrödinger equation

$$A\Phi \equiv -\frac{d^2\Phi}{dr_*^2} + V\Phi = \omega^2\Phi \,. \tag{2·10}$$

Once we fix the boundary conditions at the singularity, the corresponding self-adjoint operator A_E will be positive if all normalisable solutions to (2·10) have $\omega^2 \geqslant 0$, and hence ω real.

Let us determine the possible boundary conditions. Near the singularity, the operator for the vector perturbations takes the form

$$A_V \sim -\frac{d^2}{dr_*^2} + \frac{3}{4r_*^2} + \cdots \,, \quad \text{as} \quad r_* \to 0 \,. \tag{2·11}$$

Therefore, the general solution to (2·10) behaves as

$$\Phi \sim a_1(r_*^{-1/2} + \cdots) + b_1(r_*^{3/2} + \cdots) \,, \quad \text{as} \quad r_* \to 0 \,. \tag{2·12}$$

It is immediate to see that both normalisability and self-adjointness require $a_1 = 0$. Thus there is a unique self-adjoint extension A_E of A in this case, which is defined on the restricted set of functions that satisfy $r_*^{1/2}\Phi|_{r_*=0} = 0$.

The operator for scalar perturbations takes the form

$$A_S \sim -\frac{d^2}{dr_*^2} - \frac{1}{4r_*^2} + \cdots, \quad \text{as} \quad r_* \to 0\,. \tag{2·13}$$

In this case the general solution to (2·10) near the singularity is

$$\Phi \sim a_0 \left(r_*^{1/2} \log \frac{r_*}{\mu} + \cdots \right) + b_0 (r_*^{1/2} + \cdots)\,, \quad \text{as} \quad r_* \to 0\,. \tag{2·14}$$

In this case any choice of $(a_0, b_0) \neq (0,0)$ gives normalisable functions and the corresponding extension of A is always self-adjoint. From the linearity of the equations involved it is clear that (a_0, b_0) defines the same self-adjoint extension as $\lambda(a_0, b_0)$. This equivalence relation implies that the family of self-adjoint operators is parameterised by

$$q \equiv \frac{b_0}{a_0} \quad \in \quad \mathbb{R}P^1\,. \tag{2·15}$$

The positivity of A_E will depend on this parameter. A result of this work will be to show for which values of q the singularity is stable.

2.4. *A critical boundary condition for stability*

We immediately see that the potential V_V for vector perturbations (2·5) is positive definite. Therefore the lowest eigenvalue of A_V in (2·10) is positive and the spacetime is stable under vector perturbations.

For scalar perturbations the situation is more complicated. Our strategy is as follows. Firstly we will show that there is a unique boundary condition, $q = q_C$, such that the minimum eigenvalue of A_S is precisely zero, $\omega^2 = 0$. This critical boundary condition separates positive and non-positive self-adjoint extensions. We will then exhibit boundary conditions with positive and non-positive spectra on either side of q_C.

The most dangerous mode is the $l = 2$ mode, as modes with higher angular momentum are more positive. Figure 4 shows a plot of the $l = 2$ potential. For simplicity we will only consider the $l = 2$ case in this subsection.

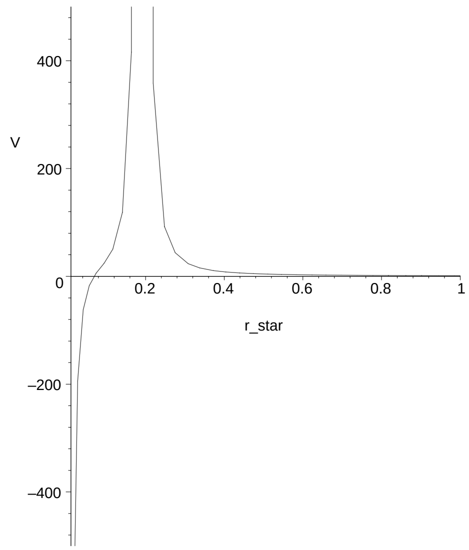

Fig. 4. The potential $V(r_*)$ for the $l = 2$ scalar mode and $\mu = 1$. The potential is unbounded below at the singularity and blows up at an interior point.

A curious feature of the scalar potential is that the infinite centrifugal barrier appears to have been shifted away from the origin to a finite radius

$r_C = 3\mu/4$. This divergence is not a physical effect, but rather an artifact of the variables used to put the perturbation equations in Schrödinger form. We will give explicit transformations below to curvature variables. The divergence will have the effect of localising the ground state, and hence any potential instabilities, near the singularity. This is reasonable given that the perturbations are required to be normalisable and that μ is the only length scale in the system. One can check that near the singular point the solutions behave as

$$\Phi \sim C_1[(r-r_C)^2+\cdots]+C_2[(r-r_C)^{-1}+\cdots] \quad \text{as} \quad r \to r_C\,, \tag{2·16}$$

where C_1 and C_2 are constants. The solution that diverges as $r \to r_C$ is not normalisable, so one must impose that $C_2 = 0$, that is

$$\Phi \sim (r-r_C)^2 \quad \text{as} \quad r \to r_C\,. \tag{2·17}$$

Note that the point $r = r_C$ is not a boundary and does not correspond to a singular radius in the spacetime. The imposition of the regularity condition above is analogous to what one does in ordinary quantum mechanics when analysing the wave function of a free particle in flat three-dimensional space, where a centrifugal potential diverges at the origin. The condition (2·17) implies that $\Phi = \Phi' = 0$ at $r = r_C$. Therefore we may truncate the solution at $r = r_C$ and match it onto any other solution for $r > r_C$ that also satisfies $\Phi = \Phi' = 0$ at this point. In particular, we may take $\Phi = 0$ for $r > r_C$. This will be the ground state because V_S is positive for $r > r_C$.

Remarkably, following observations in Ref. 14), one may explicitly solve the scalar perturbation Schrödinger equation in the case when $\omega^2 = 0$. One finds

$$\begin{aligned}\Phi &= C_3\frac{r(3\mu^3-6r^2\mu+4r^3)}{\mu^3(4r-3\mu)}\\ &\quad + C_4\left[\frac{r(13\mu^3-24\mu^2 r+12\mu r^2)}{3\mu^3(4r-3\mu)} - \frac{r(3\mu^3-6r^2\mu+4r^3)}{\mu^3(4r-3\mu)}\log\frac{r+\mu}{r}\right],\end{aligned} \tag{2·18}$$

where C_3 and C_4 are constants. We must take the linear combination that does not diverge at $r = r_C = 3\mu/4$. The well-behaved solution has

$$C_3 = C_4\left[\log\frac{7}{3}-\frac{4}{9}\right]. \tag{2·19}$$

This solution is seen to have no nodes for $r < r_C$. The expression (2·18) diverges as $r \to \infty$. However, as indicated above we can take $\Phi = 0$ beyond $r = r_C$. Thus we have a normalisable $\omega^2 = 0$ ground state. The limit as $r \to 0$ of this solution therefore gives us the critical boundary condition. Comparing with (2·14) we can read off

$$q_C = 2+\log\frac{98}{9}\,. \tag{2·20}$$

We have at present no conceptual understanding of why there is a marginal mode and why it occurs at this value of q. The existence of a boundary condition giving $\omega^2 = 0$

suggests that $q < q_C$ will give ground states with $\omega^2 > 0$ and hence stable spacetimes and that $q > q_C$ will give unstable spacetimes. To show this it is sufficient to exhibit one boundary condition with $q < q_C$ that gives a positive self-adjoint extension and one negative self-adjoint extension with $q > q_C$.

In fact the situation is a little more complicated because q takes values in $\mathbb{R}P^1 \cong S^1$ rather than $\mathbb{R}$. There must therefore be at least one other critical boundary condition at which the minimum eigenvalue changes discontinuously. Figure 5 illustrates what happens in the simplest case of one discontinous point. This case occurs for perturbations in Anti-de Sitter space.[12]

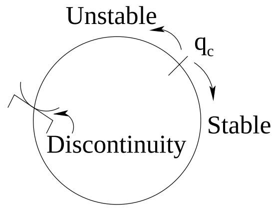

Fig. 5. Stable and unstable boundary conditions as a function of q.

2.5. *A boundary condition for stability*

In this section we show that the boundary condition $a_0 = 0$ in (2·14), corresponding to $q = \pm\infty$, gives a positive self-adjoint extension. The method will be to find a first order operator of the form

$$\tilde{D} \equiv \frac{d}{dr_*} + S\,, \tag{2·21}$$

with S being some smooth function of r. In terms of this S-deformed operator, we formally obtain

$$(\Phi, A\Phi)_{L^2} = \left[-\Phi^* \tilde{D}\Phi\right]_{r_*=0}^{r_*=\infty} + \int dr_* \left(|\tilde{D}\Phi|^2 + \tilde{V}|\Phi|^2\right), \tag{2·22}$$

where

$$\tilde{V} \equiv V + f\frac{dS}{dr} - S^2, \tag{2·23}$$

where f is the metric function (2·2) and V is the potential in (2·7). The expression is formal because we should check that the boundary term and integral are not both infinite.

For any smooth function Φ of compact support the boundary term vanishes and integration in the second term is finite. If $\tilde{V}$ is now shown to be positive for some appropriately chosen S, then the symmetric operator A_S with domain $C_0^\infty(r_*)$ is positive definite. Indeed we can find a function S that makes manifest the positivity of the symmetric operator A for scalar perturbations. We take S to be

$$S = -\frac{f}{r}\,. \tag{2·24}$$

Then we have

$$\tilde{V}_S = \frac{f(r)\tilde{U}(r)}{16(mr - 3\mu)^2}\,, \tag{2·25}$$

where

$$\tilde{U}(r) = 16m\left[\frac{(mr-3\mu)^2}{3r^2} + \frac{2}{3}m^2 + 2m\right], \qquad (2\cdot26)$$

and where as before $m = (l-1)(l+2)$. It is clear that $\tilde{V}_S$ is positive definite.

Note that $\tilde{V}_S$ is unbounded above at $r = 3\mu/m$. For $l = 2$ we have $m = 4$ and hence this is the same divergence in the potential as we found before. As we saw in the previous subsection, states Φ must vanish at this point in order to be square integrable. In this way, we see that A_S is a positive symmetric operator with domain consisting of smooth functions of compact support satisfying the regularity condition (2·17) as $r \to r_C$.

Next we need to extend the domain so that A_S becomes self-adjoint. This is possible because a positive symmetric operator always has at least one positive self-adjoint extension, known as the Friedrichs extension.[20] We can be completely explicit in our case: Any self-adjoint extension corresponding to boundary conditions satisfying

$$\Phi^* \tilde{D}\Phi\Big|_{r_*=0} \geqslant 0, \qquad (2\cdot27)$$

will be positive. From (2·14) we can see that the boundary term is zero if $a_0 = 0$ and diverges otherwise. Thus we have shown that the boundary condition $a_0 = 0$ results in a stable spacetime. This boundary condition corresponds to the Friedrichs extension, and may be viewed as the simultaneous imposition of generalised Neumann and Dirichlet boundary conditions in the sense of Ref. 12).

2.6. *Some boundary conditions for instability*

We may use the Rayleigh-Ritz trial function method to exhibit explicitly a range of values for q resulting in unstable spacetimes. Adapted to our context, one has that for any trial function $\chi \in \mathcal{H}$ we have

$$\omega^2_{\min} \leqslant \frac{(\chi, A_S\chi)}{(\chi,\chi)}, \qquad (2\cdot28)$$

where (,) denotes the inner product of $L^2(r_*, dr_*)$. We consider the following set of functions, depending on three parameters P, Q, S

$$\chi = \left(r - \frac{3}{4\mu}\right)^2 \left[(r+Sr^2)\log\frac{r}{\mu} + Pr + Qr^2\right] \quad \text{for} \quad r < \frac{3}{4\mu}, \qquad (2\cdot29)$$

and vanishing for $r \geqslant 3/4\mu$. In this ansatz, the boundary condition at the singularity is specified by P. Substituting the ansatz into (2·28) we minimise the functional with respect to Q and S at fixed P. Using these minimum values we see that $(\chi, A_S\chi)$ is negative for P in the range $[1.850, 5.087]$. Translating into values of a_0 using the relation (2·3) between r and r_* it follows that

$$q \in [4.393, 10.867] \quad \Rightarrow \quad \text{Unstable}. \qquad (2\cdot30)$$

Note that the critical value we found previously has numerical value $q_C = 4.389$, so the trial functions give a good approximation to the exact critical value.

2.7. *Energetics*

In this subsection and the following we will relate the variable Φ for the scalar perturbations to curvature perturbations and to an energy integral. We hope in this manner to clarify the presence of an unphysical divergence in the Schrödinger potential and to see if there is any sense in which the $a_0 = 0$ boundary condition is preferred.

One may immediately write down an energy which is always finite and conserved and furthermore always positive for positive self-adjoint extensions:

$$E_0 = (\dot{\Phi}, \dot{\Phi}) + (\Phi, A_E \Phi) \,, \tag{2·31}$$

where we use (,) to denote the inner product of $L^2(r_*, dr_*)$ and the dot denotes a time derivative. However, in order to connect this expression to the usual energies that are considered in physics one should integrate by parts to obtain a term quadratic in $\partial_{r_*}\Phi$ rather than $\Phi\partial_{r_* r_*}\Phi$. It is at this point that the different boundary conditions produce different behaviours. Let us see what happens if we consider energies with terms quadratic in $\partial_{r_*}\Phi$.

There are at least two notions of energy that we could consider. The first is the energy associated with the Schrödinger equation

$$E_1 = \int dr_* \left[\left| \frac{\partial \Phi}{\partial t} \right|^2 + \left| \frac{\partial \Phi}{\partial r_*} \right|^2 + V_S \left|\Phi\right|^2 \right] . \tag{2·32}$$

Use $\mathcal{E}_1$ to denote the integrand of the previous expression

$$E_1 = \int \mathcal{E}_1 dr_* \,. \tag{2·33}$$

By finding a series expansion for the solutions of the Schrödinger equation (2·10) one finds that if $a_0 = 0$ the energy is finite whilst if $a_0 \neq 0$ then $\mathcal{E}_1$ diverges at the singularity

$$\mathcal{E}_1 \sim \frac{|a_0|^2}{r_*} \log \frac{r_*}{\mu} + \cdots \quad \text{as} \quad r_* \to 0 \,. \tag{2·34}$$

The corresponding total energy is therefore infinite. This result might be taken to suggest that $a_0 = 0$ is the more physical boundary condition. The difference between E_0 and E_1 may be understood as arising upon an integration by parts

$$\mathcal{E}_0 = \mathcal{E}_1 - \frac{\partial}{\partial r_*} \Phi^* \tilde{D} \Phi \,, \tag{2·35}$$

where $\tilde{D}$ is given in (2·21) and $\mathcal{E}_0$ is defined analogously to (2·33). The boundary term is the same that arose in (2·22) and diverges unless $a_0 = 0$.

So far the energies we have considered have been somewhat abstract. However, we can clarify the relation of (2·32) to a more commonly considered definition of energy in general relativity. An expression for the energy of a perturbation about a background spacetime is given by an integral over a spatial hypersurface

$$E_2 = -\frac{1}{8\pi} \int G^{(2)}_{ab} n^a \xi^b d\Sigma_{(3)} \,, \tag{2·36}$$

where $G^{(2)}_{ab}$ is the quadratic variation of the Einstein tensor under the perturbation, n is a unit vector orthogonal to the hypersurface, $n^t = 1/f^{1/2}$, and ξ is the timelike Killing vector $\xi^t = 1$. The linear variation will vanish because the perturbation satisfies the equation of motion.

For the energy to be conserved, there must be no flux of energy into or out of the singularity. Assuming that the perturbation dies off sufficiently rapidly at infinity, the change in the energy with time is seen to be

$$\dot{E}_2 = \lim_{r\to 0} \frac{1}{8\pi} \int_{S^2_r} r^2 f(r)^{1/2} G^{(2)}_{rt}\, d\sigma_{(2)} \,. \tag{2·37}$$

Using, for example, the formalism developed by Chandrasekhar,[17),18)] one can calculate $G^{(2)}_{ab}$ in terms of the scalar metric perturbations. The linearised equations of motion for the perturbations are equivalent to the scalar Schrödinger equation. We will not review Chandrasekhar's formalism here, although we have used it extensively to obtain the results in this subsection and the next. Considering the $r \to 0$ limit one finds two possible behaviours corresponding to the two possibilities in (2·14). The boundary term (2·37) may be shown to vanish in both cases, so that

$$\dot{E}_2 = 0 \,. \tag{2·38}$$

Thus (2·36) provides a conserved energy for the perturbations. If we evaluate (2·36) we again find that the different boundary conditions at the singularity give qualitatively different results. The $a_0 = 0$ boundary condition has finite energy, whilst the other cases may be shown to give

$$\mathcal{E}_2 \sim |a_0|^2 \frac{1}{r_*} + \cdots \quad \text{as} \quad r_* \to 0 \,. \tag{2·39}$$

Here $\mathcal{E}_2$ is defined analogously to (2·33). Again, the total energy diverges unless $a_0 = 0$. Thus we see that finiteness of the energy is independent of whether we use E_1 or E_2. However, the degree of divergence itself is slightly different. We might therefore ask whether there is any relation between these two expressions for the energy. Indeed there is. Using, for instance, Chandrasekhar's formalism[17),18)] the two expressions may be shown to be identical up to total derivative terms

$$\begin{aligned} \mathcal{E}_2 &\propto \mathcal{E}_1 + \frac{\partial}{\partial r_*} X \\ \Rightarrow E_1 &\propto E_2 + [\,X\,]_0^\infty \,. \end{aligned} \tag{2·40}$$

If $a_0 \neq 0$ then

$$X \sim |a_0|^2 \log^2 \frac{r_*}{\mu} + \cdots \quad \text{as} \quad r_* \to 0 \,. \tag{2·41}$$

Therefore the two energies are related by a divergent boundary term.

2.8. *Curvature*

2.8.1. Curvature scalars

Given that the background is Ricci flat, the natural curvature scalar to consider is the Weyl tensor squared. The background has

$$C_{abcd}C^{abcd} = \frac{12\mu^2}{r^6}\,. \tag{2·42}$$

It is possible to derive an expression for the linearised perturbation to this curvature scalar in terms of the metric perturbation variable Φ using, for example, Refs. 17) and 18). One finds

$$\delta\left(C_{abcd}C^{abcd}\right)(r) =$$
$$-12\mu m\,(m+2)\left[\frac{\Phi(r)}{r^6} + \frac{(6r\mu+9\mu^2)f(r)^{1/2}}{(m+2)r^8}\int^r \frac{\Phi(r')dr'}{f(r')^{1/2}(mr'-3\mu)}\right]e^{-i\omega t}P_l(\cos\theta)\,, \tag{2·43}$$

where we have restored the angular and time dependence and without loss of generality we have restricted to axisymmetric perturbations for simplicity.

There are at least two observations to make about Eq. (2·43). Firstly, we can see that there is no physical divergence at the point $r = 3\mu/m$, at which the Schrödinger potential diverges. Given that Φ must go to zero at this point, the integral in (2·43) is finite. Secondly, taking the limit $r \to 0$ we find

$$\delta\left(C_{abcd}C^{abcd}\right) \sim a_0\left(\frac{1}{r^6}\log\frac{r}{\mu}+\cdots\right) + b_0\left(\frac{1}{r^6}+\cdots\right) \quad \text{as} \quad r \to 0\,. \tag{2·44}$$

Therefore we see that if $a_0 = 0$ the perturbed curvature has the same degree of divergence as the background, whilst if $a_0 \neq 0$ then the divergence of the perturbed curvature is greater than that of the background.

2.8.2. Weyl scalars

Perturbations about black hole spacetime are often considered using the formalism of Weyl scalars. This is particularly useful for the case of rotating black holes where the Newman-Penrose formalism allows a separation of variables of the perturbation equations.[24),25)] In this subsection we briefly recast our analysis in terms of a Weyl scalar.

An explicit transformation is known[17),18)] between the variable Φ satisfying the scalar Schrödinger equation (the Zerilli equation) and the perturbed Weyl scalar

$$\delta\Psi_0 = -\delta C_{pqrs}\, l^p m^q l^r m^s\,, \tag{2·45}$$

where the null vectors are given by

$$l = \frac{1}{f(r)}\frac{\partial}{\partial t} + \frac{\partial}{\partial r}\,,$$

$$n = \frac{1}{2}\frac{\partial}{\partial t} - \frac{f(r)}{2}\frac{\partial}{\partial r}\,,$$
$$m = \frac{1}{\sqrt{2}}\left[\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{i}{r\sin\theta}\frac{\partial}{\partial\phi}\right]. \qquad (2\cdot46)$$

In the Schwarzschild background the only nonvanishing Weyl scalar is

$$\Psi_2 = -C_{pqrs}\, l^p m^q \bar{m}^r n^s = \frac{\mu}{2}\frac{1}{r^3}\,. \qquad (2\cdot47)$$

The relationship between the perturbed Weyl scalar and the scalar perturbation, including the angular and time dependence, is[17),18)]

$$\delta\Psi_0 = \frac{r}{2(r+\mu)^2} e^{-i\omega t}\left(\frac{d^2}{d\theta^2}P_l(\cos\theta) - \cot\theta\frac{d}{d\theta}P_l(\cos\theta)\right)$$
$$\times\left[\left(\frac{d}{dr_*} - i\omega\right) + \frac{2mr^2+3m\mu r-3\mu^2}{r^2(mr-3\mu)}\right]\left(\frac{d}{dr_*} - i\omega\right)\Phi(r)\,. \qquad (2\cdot48)$$

This equation was derived in Refs. 17) and 18) by working in a specific gauge, but given that $\delta\Psi_0$ is gauge invariant for the Schwarzschild background, the relation should also hold for our gauge invariant Φ.

In (2·48) we can see the appearance of a divergence at the finite radius $r = 3\mu/m$. For the Weyl scalar to remain bounded at this radius, Φ must vanish at $r = 3\mu/m$. This is precisely the condition that we found previously due to a divergence in the scalar potential. Therefore, we see explicitly that the unphysical divergence in the potential arises in the change of variables relating the metric perturbation variable to physical curvatures.

If we consider the $r \to 0$ limit of Φ (2·14) then we find that the Weyl scalar perturbation behaves as

$$\delta\Psi_0 \sim a_0\left(\frac{1}{r_*} + \cdots\right) + b_0\,(1+\cdots) \quad \text{as} \quad r_* \to 0\,. \qquad (2\cdot49)$$

§3. Discussion

The linearised dynamics of gravitational perturbations about the negative mass Schwarzschild spacetime are given a well defined dynamics by specifying boundary conditions at the singularity. There is a one parameter choice of possible boundary conditions corresponding to self-adjoint extensions of the Hamiltonian for the scalar/polar perturbation. We considered the stability of the spacetime as a function of the boundary condition. We have shown that there is a critical boundary condition separating stable and unstable spacetimes. For the vector/axial perturbation, we have shown that the boundary conditions are unique and the spacetime is stable.

Amongst the possible boundary conditions for the scalar/polar perturbation, we have seen that there is one particular choice, $a_0 = 0$ in (2·14), which gives perturbations with finite physical energies E_1 and E_2. Furthermore, these perturbations induce curvature perturbations with the same degree of divergence at the singularity as the background curvature. All the other boundary conditions result in perturbations with infinite energy which have an enhanced curvature divergence at the singularity. Therefore, the $a_0 = 0$ boundary condition seems to be more physical than the others, provided the linear perturbation analysis is valid for all the possible boundary conditions. We saw that the $a_0 = 0$ boundary condition gave a stable spacetime.

The main conclusion following from these results is that the negative mass Schwarzschild phantom can be perturbatively stable and in particular is stable with what appears to be the most physical choice of boundary conditions. This is perhaps counter to intuition, given the existence of nonlinear instabilities that we reviewed in the introduction.

There are two immediate limitations of our calculations. Firstly, very near to the singularity it is likely that a linearised analysis of perturbations is not valid. This is because higher order terms in the equations of motion will contain curvature tensors that diverge as $r_* \to 0$. The divergence that occurs for the $a_0 \neq 0$ boundary conditions might be viewed an indication of the breakdown of the linear analysis as mentioned below (2·44). By making the perturbations sufficiently small one might hope that the linearised approximation will be valid down to very small radii. Perhaps in this case the nonlinear effects near the singularity could be absorbed into an effective boundary condition. Indeed it would be interesting if a preferred boundary condition is selected in this way.

Secondly, near the singularity the curvatures become large and it seems likely that quantum corrections to the spacetime will be important. Again, such corrections could perhaps be renormalised into an effective boundary condition. In this connection, it is interesting to note that there are arguments suggesting that the negative mass Schwarzschild singularity should not be resolved by quantum corrections.[23)]

We would now like to return to the issue of the $l = 1$ scalar mode. As mentioned above, this is a 'translational zero mode' which could be interpreted as allowing the singularity to move from its initial position. Because the mass of the singularity is negative, one expects such a motion to carry negative kinetic energy. This could clearly lead to various nonlinear instabilities including that envisioned by Bondi, which we reviewed in the introduction.

There are various possible extensions of our calculations. Adapting to higher dimensional spacetimes should be straightforward, the necessary formalism may be found in Refs. 13), 14) and 19). Back in four dimensions it may be possible to extend the calculations to the case of rotating Kerr black holes. Remarkably, the linearised stability of positive mass Kerr black holes is a tractable problem and stability has been proven.[18),24)–26)] Negative mass rotating black holes would presumably also have a negative moment of inertia and therefore potentially suffer from additional 'spin up' instabilities.

More speculatively, there is an intriguing similarity between the behaviour of the perturbations we have considered near the singularity and the asymptotic behaviour of perturbations about Anti-de Sitter space.[12)] Perhaps there is a version of 'holographic renormalisation'[27)–29)] which applies to the naked singularity and allows one to absorb some divergences arising from integrations near the singularity?

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